WORKSHOP ON MATHEMATICAL CONTROL THEORY IN KOBE

Dedicated to Professor Shin-ichi Nakagiri and Professor Takao Nambu on the occasions of their 60th birthdays

Date: January 8–10, 2010
Site: Takikawa Memorial Hall at Kobe University, Japan
Preface

This booklet is dedicated to Professor Shin-ichi Nakagiri and Professor Takao Nambu, Kobe University, on the occasions of their 60th birthdays. As everyone knows, the both professors have been making challenging contributions in intersection areas of applied mathematics and control theory.

To celebrate these professors’ 60th birthdays a conference, titled as “Workshop on Mathematical Control Theory in Kobe”

was held on January 8–10, 2010 at Takikawa Memorial Hall, Kobe University with more than 30 participants, including scholars from foreign countries. Papers presented in this conference are included in this booklet.

Professor Nakagiri and Professor Nambu have kept active and strong vitalities in their spirits. We look forward to their long and healthy life and to perpetual contributions. On behalf of the contributors, we heartily dedicate this booklet to our great colleagues Professor Shin-ichi Nakagiri and Professor Takao Nambu.

Yuki Naito (Ehime Univ., Japan)
Ken Shirakawa (Kobe Univ., Japan)

Professor Shin-ichi Nakagiri (left) and Professor Takao Nambu (right) in banquet of the conference
Workshop on Mathematical Control Theory in Kobe

Takikawa Memorial Hall, Kobe University, Kobe, Japan
8–10 January 2010

8 January (Friday)

13:30~14:15  Fumitoshi Matsuno (Kyoto University, Japan)
Control of Bio-inspired Snake Robots -Constrained and Redundant System-

14:20~15:05  Kenji Maruo (Kobe University, Japan)
Solutions to Semilinear Degenerate Elliptic Equations with Radially Symmetric Coefficients in the Plane

15:10~15:55  Hideki Sano (Kobe University, Japan)
Stabilization of a coupled transport-diffusion system: a case with boundary control and boundary observation

16:10~16:55  Hiroyuki Ukai (Nagoya Institute of Technology, Japan)
Mathematical models and control issues in power system

9 January (Saturday)

9:15~9:50  Young-Chel Kwun (Dong-A University, Korea)
Jin Han Park (Pukyong National University, Korea)
Controllability for the fuzzy integrodifferential equations in n-dimensional fuzzy vector space

9:55~10:30  Junhong Ha (Korean University of Technology, Korea)
Inverse problem for a heat equation in the bar with piecewise constant thermal conductivity

10:45~11:30  Kimiaki Narukawa (Naruto University of Education, Japan)
Positive solutions of quasilinear elliptic equations involving indefinite lower term
11:35~12:20 Masahiro Yamamoto (University of Tokyo, Japan)
Carleman estimates for parabolic equations and Applications

14:00~14:45 Hiroki Tanabe (Emeritus Professor of Osaka University, Japan)
Construction of fundamental solution of degenerate parabolic differential equations

14:50~15:35 Atsushi Yagi (Osaka University, Japan)
Exponential attractors for non autonomous dynamical systems

15:50~16:35 Fumio Kojima (Kobe University, Japan)
Structural Health Monitoring of Nuclear Power Plants using Inverse Analysis in Measurements

16:40~17:25 Shin-ichi Nakagiri (Kobe University, Japan)
Boundary Feedback Stabilization of Parallel-Flow Heat Exchanger Process Using a Forwardstepping Method

19:00 ~ Banquet

10 January (Sunday)

9:15~9:50 Jito Vanualailai, Bibhya Sharma (University of South Pacific, Fiji)
A Lagrangian Swarm Model

9:55~10:30 Bibhya Sharma, Jito Vanualailai (University of South Pacific, Fiji)
Formation Navigation: Tunnel Passing maneuvers

10:45~11:20 Ken Shirakawa (Kobe University, Japan)
Continuous dependence among isotropic-anisotropic total variation flows associated with phase transitions

11:25~12:00 Yuki Naito (Ehime University, Japan)
Non-homogeneous semilinear elliptic equations involving critical Sobolev exponent

Organizing Comitee: Ken Shirakawa (Kobe University) Yuki Naito (Ehime University)

Contact: Ken Shirakawa  Graduate school of Engineering, Kobe University
Nada Kobe 657-8501 Japan  E-mail: control@boar.kobe-u.ac.jp
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Solutions to Semilinear Degenerate Elliptic Equations with Radially Symmetric Coefficients in the Plane

Kenji Maruo

1章。序
次の半線形退化楕円型方程式を $R^2$ で考える。

(1) \[ L(u(x)) = -g(|x|)\Delta u(x) + u(x)|u(x)|^{p-1} - f(|x|) = 0 \]

ここで、$p > 1, g : [0, \infty) \to [0, \infty), g, f \in C^\infty$ を満たすものとする。

係数関数 $g, f$ とも球対称である関係で、(1) の方程式の解は、非球対称な解を持つかどうかは興味ある問題である。事実、$g, f$ の無限遠方における振る舞いに関する関係式がある一定の関係不等式を満たすと連続な粘性解は半径対称解が存在し、他の関係不等式を満たすならば半径対称解の多様性が多様存在する。そのうえに、半径非対称解が存在する関係不等式の存在もある。ゆえに、$g, f$ の無限大の近傍における振る舞いと連続な粘性解の構造がどのようになっているか調べることは大切なことと考える。

非線形のところ monotone であるため、この方程式は monotone タイプの方程式になり粘性解の構成には非常に取り扱いやすい方程式である。このため粘性解の範疇で方程式の解を取り扱うこととする。また、$g(|x|)$ が正の領域では楕円型方程式の解の滑らかさに関する定理と、連続な粘性解の一意性定理を使用することにより粘性解が古典解になることがあることが示せる。故に、$g(|x|)$ が正の領域では古典解のみ考察すればよい。また、$g(|x|) = 0$ の領域近傍では $g$ の零点での零に行くオーダーは2次となること、解の連続性を用いると解 $u(x)$ は $g(|x|)$ の零点において、$u(x)|u(x)|^{p-1} - f(|x|) = 0$ を満たすことが判る。このことを考慮すると、領域 $g(|x|) > 0$ は有界または非有界の円環領域であるので、(1) をこの領域に制限すれば境界で退化した半線形楕円型方程式の Dirichlet 問題となる。$R^2$ での問題に関しては、$g(|x|) > 0$ での各円環領域での解と $g(|x|) = 0$ で $u(x)|u(x)|^{p-1} - f(|x|) = 0$ をみたす関数とを自然に連続した $R^2$ での新しい関数を考えれば、この関数が粘性解の性質により $R^2$ での (1) の解であることがわかる。それゆえに円環領域での解のみの構造が分析されればすべての解がわかることになる。

この問題は故富田義人教授（神戸商船大学）との共同研究で始めものであるが、初期段階で故人となりその意思を引き継ぎ一応の決着を見たものである。また、半径非対称解に関することは山田直記教授（福岡大学）との共同研究である。その総合的結果の報告をするのがこの報告書の内容である。

2章。仮定と定理
この章で定理とそれに必要な仮定を述べる。
定理 0
有界な円環領域で (1) の方程式を考える。このとき (1) の連続な粘性解は半径対称解のみで唯一つ存在する。

では非有界の円環領域においてはどうであろうか。この場合は無限遠点での境界条件が無いため $g, f$ の無限遠点近傍の状態によっては無限個の解が存在する可能性がある。この解析のため解の解析に必要な $g, f$ の仮定を述べそのときどのような解構造になっているかを述べる。
以下、非有界円環領域での話となる。

定理 1
$g(|x|)$ の零点が無限遠方まで存在するとする。そのとき、(1) の連続な粘性解は半径対称解のみでただ一つである。

次に定理 1 より、$g(|x|)$ の零点は有限に留まる場合を考察すればよい。即ち、つぎを満たす $T_0$ が存在するすることを以後仮定する。（$g$ の連続性を考慮すれば、$T_0$ の存在を仮定してよい。）

$$g(T_0) = 0, \ T_0 > 0, \ g(t) > 0 \ for \ t > T_0.$$ ここで、$x = (t, \theta) x \in R^2$ と極座標表示する。

仮定 1
$g(t)$ は無限遠点の近傍で次のような振る舞いをするものとする。

$$g(t) = t^\ell \left(1 + \sum_{i=1}^{\ell} \sigma_i t^{-i}\right) + O(t^{-\epsilon})$$ ここで、$\ell > 0, \sigma_i (i = 1, 2, \cdots)$ and $\epsilon > 0$ である。

以後、仮定 1 は成立しているものとする。

定理 2
$\ell \leq 2$ あるいは $\ell > 2$ でかつ $\lim_{t \to \infty} \frac{f(t)}{t^{ap}} = \infty$ を満たすとする。
そのとき (1) の連続な粘性解は半径対称解のみでただ一つである。

次に $\alpha$ は $\alpha = \frac{\ell - 2}{p - 1}$ と決める。また、$f$ に次の仮定を入れる。
仮定 2
$f(t)$ は無限遠点の近傍で次のような振る舞いをするものとする。

$$f(t) = t^{\alpha p} \left(\kappa^{\alpha p} + \sum_{i=1}^{\alpha p} \tau_i t^{-i}\right) + O(t^{-\epsilon})$$
ここで、\( \alpha = \frac{\ell - 2}{p - 1} \)、\( \kappa > 0 \) で、\( \tau_i (i = 1, 2, \cdots) \) はある定数とする。

次の代数方程式を考える。

(2) \[ X |X|^{p-1} - \kappa^p - \alpha^x = 0. \]

(2) の方程式の実根は次の 3 通り存在する。
(C1) 正の単根のみ \( (\omega_+) \).
(C2) 正の単根 \( (\omega_+) \) と負の重根 \( (\omega_0) \).
(C3) 正の単根 \( (\omega_+) \) と 2 つの負の単根 \( (\omega_0 \text{ and } \omega_-) \).

定理 3
(C1) \( \Rightarrow \) (1) の解が唯一つで半径対称解が存在 \( \text{s.t. } \lim_{|x| \to \infty} \frac{u(|x|)}{|x|^\alpha} = \omega_+ \).
(C2) \( \Rightarrow \) (1) は次を満たす半径対称解が存在する。もし、\( p > 2 \) を満たせば解は半径対称解のみとなる。

\[ \lim_{|x| \to \infty} \frac{u(|x|)}{|x|^\alpha} = \omega_+ \text{ (Only one) or } \omega_0 \text{ (Many)}. \]

(C3) \( \Rightarrow \) (1) は多くの半径対称解を持つ。このとき、\( p > 2 \) と仮定すると
(a) \( \alpha^2 - p|\omega_0|^p - 1 \) なければ半径対称解のみである。
(b) \( \alpha^2 - p|\omega_0|^p - 1 \) なければ半径対称解が多数存在している。半径非対称解の存在は否定できない

注意 1
(1-1) (C3) のときの半径対称解は \( p > 1 \) で存在し、次を満たすものが存在する。

\[ \lim_{|x| \to \infty} \frac{u(|x|)}{|x|^\alpha} = \omega_+ \text{ (Only one) or } \omega_0 \text{ (Many) or } \omega_- \text{ (Only one)}. \]

なお、\( |x| \to \infty \) の時 \( \omega_0 \text{ (Many) に収束する半径対称解の漸近展開公式がわかりある展開オーダーの係数と} \) が \( 1 \text{ 対 } 1 \text{ に対応し係数は実数全体を埋め尽くしていることがわかる。}\]
(1-2) (C2) のとき \( p > 2 \) とすると半径対称解のみであるが、\( |x| \to \infty \) の時 \( \omega_0 = \omega_- \text{ (Many) に収束する半径対称解の漸近展開公式が (C3) と同様に}} \) 判りある展開オーダーの係数と \( R^1 \) は同相であることもわかっている。
(1-3) \( \kappa > 0 \) であるので \( \omega_0 \) は正数であることが要求される。
(1-4) \( p > 2 \) の条件について。\( 1 < p < 2 \) の場合でも半径非対称解は存在する場合が分かっているがまだその構造は、未解決であるのでここでは省略する。
定理 3 において (C3) の (b) の半径非対称解について詳しく述べよう。

定理 4
\[ p \geq 2 \text{ で, } m^2 < \alpha^2 - p|\omega_0|^{p-1} \leq (m + 1)^2 \text{ (} m \text{ は自然数) を満たす。ただし, } f, g \text{ は本質的に主部のみとする。このとき, } \]
\[ \lim_{t \to \infty} \sup_{\theta} u(t, \theta) - \inf_{\theta} u(t, \theta) \geq 0 \]
を満たす (1) の連続な粘性解が少なくとも \( m \) 個存在。

注意 2
定理 4 は角度により変化する解の存在であり半径非対称解となっている。なお、この解は無限大の近傍における (b) の半径対称解の漸近挙動のトップのオーダーと同じオーダーで大きく振動する解である。また、定理 4 はそれぞれ周期 \( \frac{2\pi}{n}, (n = 1, 2, \cdots, m) \) を持つ \( m \) 個の解の存在を保証するものである。

\( f, g \) が本質的主部のみならば
\[ f(t) = t^{\alpha \omega_0} + O(t^{-\alpha + \sqrt{p|\omega_0|^{p-1}}}) \quad g(t) = t^{\ell \omega_0} + O(t^{-\alpha + \sqrt{p|\omega_0|^{p-1}}}) \]
を満たすことを言う。条件 0 と 1 を満たす一般的な関数 \( f, g \) に対しては低階項が解に及ぼす影響をまた処理できず未解決である。

1 < \( p \) の場合は \( \alpha^2 - p|\omega_0|^{p-1} < 1 \) のときでも周期 2\( \pi \) の半径非対称解の存在する場合がある。ただし、半径非対称解の存在する \( \alpha^2 - p|\omega_0|^{p-1} \) の下限は未解決である。

次に小さく振動する解について述べよう。

定理 5
\[ m^2 < \alpha^2 - p|\omega_0|^{p-1} \leq (m + 1)^2 \text{ (} m \text{ は自然数) を満たす。このとき, } \]
\[ \lim_{|t| \to \infty} \frac{u_n(t, \theta) - y(t)}{|t|^{p|\omega_0|^{p-1}+n^2}} = C_n \cos n\theta \quad (n = 1, 2, \cdots, m) \]
を満たす (1) の半径非対称で連続な粘正解が少なくとも \( m \) 個存在する。\( C_n \) は \( u_n \) に付随する小さな定数である。また、\( y(t) \) は \( y(t) \approx t^p \omega_0 \text{ as } t \to \infty \) を満たす連続な粘性解で (1) 半径対称解である。

逆にある小さな数 \( \delta \) があって |\( C_n \)| < \( \delta \) を満たす \( C_n \) に対し上式を満たす連続な粘正解が存在する。

注意 3
定理 5 はそれぞれ周期が \( \frac{2\pi}{n}, (n = 1, 2, \cdots, m) \) の \( m \) 種類の連続な粘性解の存在を保証し、
かつそれぞれの周期においては連続な濃度を持つ解の存在を保証している。この解は半径対称解 $g(t)$ からの分岐と考えることができる。では、定理 4 の半径非対称解からの分岐はないかという問題が出るが存在する雰囲気はあるがまだ解決した問題である。

注意 1 で述べたように (1) の半径対称解についての構造も調べてあるので報告する。
(1) の半径対称解を $u(t)$ と書き、$u(t) = t^\alpha (\omega_0 + w(t))$ 置くとき $w(t)$ が満たす方程式は (1) の方程式を書き換えで

$$
\ddot{w}(t) + \frac{(2\alpha + 1)}{t} \dot{w}(t) + \frac{\alpha^2 - p|\omega_0|^{p-1}}{t^2} w(t)
= \frac{1}{t^2} \{ F(w) + g_2(t) H(w) + q_0(t) \}
\lim_{t \to \infty} w(t) = 0
$$

ここで、

$$
F(z) = (z + \omega_0) |z + \omega_0|^{p-1} - \omega_0 |\omega_0|^{p-1} - p|\omega_0|^{p-1} z,
$$

$$
g_1(t) = \frac{t \alpha}{g(t)}, \quad g_2(t) = g_1(t) - 1,
$$

$$
q_0(t) = g_1(t) \left( \frac{f(t)}{t \alpha} - \kappa^p \right) + g_2(t) \alpha^2 \omega_0,
$$

$$
H(z) = (z + \omega_0) |z + \omega_0|^{p-1} - \omega_0 |\omega_0|^{p-1} z.
$$

である。このとき $w(t)$ の方程式の線形化部分の決定方程式を考えると

$$
\lambda^2 + 2\alpha \lambda + (\alpha^2 - p|\omega_0|^{p-1}) = 0.
$$

となるが、この方程式の解を $-\lambda_1, -\lambda_2$ とおく。

注意 2 の (2) に記した内容は次のものである。

定理 6

$\alpha^2 - p|\omega_0|^{p-1} > 0$ のとき (1) の半径対称解で $|x| \to \infty$ での次の漸近展開を持つ。

(a-1) $\lambda_1 \neq $ 自然数のとき

$$
\lim_{|x| \to \infty} \frac{u(|x|) - |x|^\alpha (\omega_0 + \sum_{j=1}^{\lambda_1} d_j |x|^{-j})}{|x|^{\alpha-\lambda_1}} = C. \tag{1}
$$

(a-2) $\lambda_1 = $ 自然数のとき

$$
\lim_{|x| \to \infty} \frac{u(|x|) - |x|^\alpha (\omega_0 + \sum_{j=1}^{\lambda_1-1} d_j |x|^{-j}) - d_{\lambda_1} |x|^{\alpha-\lambda_1} \log |x|}{|x|^{\alpha-\lambda_1}} = C. \tag{2}
$$

逆に任意の実数 $C$ に対して (a-1) または (a-2) の時上の漸近展開を満たす (1) の半径対
称解がそれぞれ一意に存在する。

定理 7
\[ \alpha^2 - p|\omega_0|^{p-1} = 0 \] を満たすとする。\((1)\) の連続な粘性解は半径対称解のみで \(|x| \to \infty\) の時、次を満たす \((1)\) の解が唯一存在するか、
\[
\lim_{|x| \to \infty} (u(x) - |x|^{\alpha} \omega_0) \frac{\log |x|}{|x|^\alpha} = 0
\]
次の漸近展開を持つ。
\[
u(x) = |x|^\alpha \left( \omega_0 + \frac{a}{\log t} + \frac{(a^2 a_3 - 2a) \log \log t}{2\alpha} \right) + |x|^\alpha \frac{C}{(\log t)^2} + O\left( \frac{|x|^\alpha}{(\log t)^{2.5}} \right).
\]
逆に任意の数 \(C\) に対して上式を満たす \((1)\) の半径対称解が一意に存在する。
ここで、
\[
\beta = \frac{p(p-1)|\omega_0|^{p-2}}{2}, \quad a = \frac{2\alpha}{\beta} \text{ and } a_3 = \frac{p(p-1)(p-2)}{3!}|\omega_1|^{p-3}.
\]

注意 4
半径対称解の構造は定理 6, 7 に出てくる定数 \(C\) と半径対称解を対応させることにより \(R^1\) と半径対称解全体は同相であることがわかり半径対称解の解構造は解明されたことになる。ただし、\((1)\) の連続な粘性解に関しては定理 4, 5 以外の半径非対称解の非存在は解明されていないのでできるだけ多くの半径非対称解の存在を保証するのにとどまっている。
また、高次元の場合においては半径対称解の構造は 2 次元とよく似た結果がわかり話は済むのであるが、半径非対称解に関しては高次元の非線形ラプラスベルトラミの方程式の解構造が今一つはっきりせず存在もまだ未解決になっている。

3 章。定理の概証
この章では簡単な定理の証明の概要を述べる。ただし、これらの証明はすでに発刊済みのこの雑誌に掲載されているので詳しくはそれらを見ていたくとしてここではそれぞれの定理において大切だと思われる事柄を記述することとする。

注意 5
前の章においても述べているが、\(g(x) > 0\) の内部領域において楕円型の Dirichlet 問題の解の存在定理と内部正則性の議論により滑らかな解が存在することと連続な粘性解の最大値（最小値）原理を使うことにより連続な粘性解は \(g(x) > 0\) の内部領域において古典解になることが示される。ゆえに \(g(x) > 0\) の領域では古典解で話をすればよいことを注意しておく。
注意6
次に，\( g(x) > 0 \)の領域に含まれる任意の有界な円環領域: \( a \leq |x| \leq b \)において \( y(a) = c_1, y(b) = c_2 \)を満たす (1)の半径対称解 \( y(t) \)は存在して一意である。ただし，\( c_1, c_2 \)は任意の実数である。これは常微分方程式の理論から示される。

注意7
(1)の連続な粘性解 \( u(x) \)において
\[
\overline{U}(x) = \sup_{y:|y|=|x|} u(y), \quad \underline{U}(x) = \inf_{y:|y|=|x|} u(y)
\]
と定義すると \( \overline{U}(x), \underline{U}(x) \)はそれぞれ (1)の連続な subsolution, supersolution となる。この証明は O.J.M [1]を挙げている。

注意8
(1)の連続な粘性解が存在したとすると，\( g(x) = 0 \)の点において解は自動的に \( u(x) = f(|x|)^{1/p} \)を満たしていることが \( g, f \in C^2 \)より示される。
これは，半径対称解が常微分方程式の理論によりこの主張が示されることと，注意6と注意7を組み合わせて粘性解の理論により示される。

注意9
(1)の解で半径対称解が考えている領域でただ一つ存在すると仮定する。そのときその領域における連続な粘性解は半径非対称解は存在しなくて，半径対称解がただ一つ存在するのみである。
これは注意8と同じく注意5と注意7を組み合わせて粘性解の理論により示される。

定理0について
この報告書のすべての基になった定理であり，O.J.M [1]に詳しく書かれている。概略は領域が有界な円環領域であるので，注意6と8を使用して (1)の半径対称解を構成しその解は一意性を粘性解の最大値原理から示す。その上で，注意9より半径対称解のみを示す。

定理1について
定理0から各 \( g(x) > 0 \)の環状領域では連続的な粘性解は唯一つ存在する。全体領域の連続な粘性解は各環状領域の解を注意8を考慮して単純につなぐと構成できて一意性も示される。

定理2について
この証明は Japonicae[2] に掲載された。その大雑把な方針方針のみを記載する。
まず，\( \ell \leq 2 \)であるとしよう。区間 \([T_0, \infty)\)での (1)の半径対称解 \( y_{a,P} \)は \( y_{a,P}(P) = \alpha, y_{a,P}(T_0) = f^{1/p}(T_0), (P > T_0) \)を満たすものとする。またここで、\( \alpha \)は任意の実数。
また，$T_{a,P}$は解 $y_{a,P}$ のlife span とする。このこのとき $S^{++}$，$S^{--}$ と $S$ を次の様に定義する。

$$
S^{++} = \{ \beta : \lim_{t \to T_{a,P}} y_{b,P}(t) = \infty, \ T_{a,P} < \infty \}
S^{--} = \{ \beta : \lim_{t \to T_{a,P}} y_{b,P}(t) = -\infty, \ T_{a,P} < \infty \}
S = \{ \beta : T_{a,P} = \infty \}
$$

このとき，$S^{++} = (b, \infty)$，$S = [a, b]$，$S^{--} = (-\infty, a)$ となる空でない区間で表わされることを証明する。このとき定理の仮定で (1) の解の積分表示と背理法を使用して解の差（正としてよい）の微分不等式をとり，有限区間で爆発することを示し解は無限区間で存在しているので矛盾より $S$ は唯一であることがわかる。それゆえに注意 9 より，半径対称解のみであることが示される。

つぎに，$\ell > 2$ であるとしよう。このときは (1) の半径対称解 $y(t)$ を $y(t) = t^\alpha v(t)$ と
そして $v(t)$ の方程式をつくりこの解に対する $S^{++}$，$S^{--}$ と $S$ を定義し，前と同様に $S^{++} = [b, \infty)$，$S^{--} = (-\infty, a]$，$S = [a, b]$ をしめす。ちるに $v$ の方程式は

$$
\begin{cases}
\ddot{v}(t) + \frac{2\theta + (N-1)}{t} \dot{v}(t) = \frac{1}{t^2}(g(t)^{-1}t^\ell (v|v|^{p-1} - \kappa(t)) - \alpha^2 v(t)), \\
v(T_0) = -\kappa(T_0).
\end{cases}
$$

ここで，$\kappa(t) = \frac{f(t)^{1/p}}{t^\alpha}$. 次に $S$ に関係する (3) の解 $v(t)$ は $\lim_{t \to \infty} v(t) = \infty$ を
$
\lim_{t \to \infty} \kappa(t) = \infty$ と (3) から導かれる微分不等式を使用して背理法により示す。これ
を使用して $\ell \leq 2$ と同じ手法により解的一意性を示す。

定理 3 について

(3) の方程式において $t = \infty$ の近傍では $g(t)^{-1}t^\ell$ は 1 と見なせ，その定常解は (2) を満
たす解である。また，定常解における線形方程式から $\omega_+$ と $\omega_-$ は不安定点であり，$\omega_0$
の不安定であること分かる。それゆえ (1) の方程式は解を半径対称解に限れば (3) になる
ことより (C1) を示すことができる。(C2)，(C3) の場合でも半径対称解が多数あるので
半径非対称解のことを除けば証明は終わっている。これらの事は Japonicae[2] に掲載
されている。

$\omega_0$ が存在すると，必ず $\alpha^2 - p|\omega_0|^{p-1} \geq 0$ となっている。また，$\alpha^2 - p|\omega_0|^{p-1} = 0$ と (C2)
は同等である。(C2) の半径非対称解に関する事は (C3) の (a) に帰着される。さて，(1)
の解 $u(t, \theta) = t^\alpha v(t, \theta) = w(t, \theta) - \omega_0$ として $w$ の方程式に (1) の方程式を書き換える。 $t$
が充分大きいとき ($t > t_0$) は次のようになる。

$$
\frac{\partial^2 w}{\partial t^2} + (2\alpha + 1) \frac{\partial w}{\partial t} + \frac{1}{t^2} \frac{\partial^2 w}{\partial \theta^2} = \frac{1}{t^2} F(w) + \frac{h(t, \theta, v)}{t^3}
$$

ここで，$F(v) = (v + \omega_0)|v + \omega_0|^{p-1} + |\omega_0|^p - \alpha^2 v$ で，$h(t, \theta, v) = h_1(t, \theta, v + \omega_0) t^3$. また，$h_1 \in B^1([t_0, \infty))$。

これより，充分大きい $t$ においては (4) における $h$ の項は無視できる。無視した方程式の解を $w$ であらわしその方程式を (5) と名づける。このとき,

$$
\lim_{t \to \infty} \frac{\partial w}{\partial t} = 0 \quad \text{and} \quad \int_{t_0}^{\infty} \int_0^{2\pi} s^{1+2i} |\frac{\partial w}{\partial s}|^2 d\theta ds < \infty, (i = 1, 2)
$$
が示せてこれより (5) の方程式は \[ \frac{\partial^2 w}{\partial \theta^2} - F(w) = \epsilon(t, \theta) \] となる。ここで、\( \lim_{t \to \infty} \epsilon(t, \theta) = 0 \)。
それ故に、次の方程式の解析をすればよい。

\[(6)\quad \frac{\partial^2 w}{\partial \theta^2} - F(w) = 0\]

(6) の方程式の相平面解析をすることにより振幅が大きい周期解が存在することが分かる。この中で周波が 2π となるものを探すと、\( \alpha^2 - p|\omega|^{p-1} = 1 \) を境にして非存在、存在が示される。ゆえに振幅が大きい半径非対称解の非存在が示せる。次に (6) が振幅が小さな解を持つときの周期解の非存在は (6) の線形化方程式の第一固有値が \( \alpha^2 - p|\omega|^{p-1} \) であるところよりこれが 1 より小であれば存在しないことが分かる。これらを基に定理 3 は示される。


定理 4 について
\[ \alpha^2 - p|\omega|^{p-1} = \lambda^2 \ (\lambda > 0) \] とする。(6) の方程式の解の周期を楕円関数で表わし周期が \( \lambda \) と振幅をパラメーターとして周期の解に関する相平面解析を行い周期が \( 2\pi/n \ (n = 1, 2, \cdots, m) \) なるものが唯一つ存在していることを示した。存在に関しては、周期が \( \lambda \) と振幅に関して連続であることと中間値の定理より証明できるが、一意性は周期の \( \lambda \) と振幅に関する偏微分が必要になる。この計算は厄介であるが、Loud[8] の論文を参考に計算した。この関数に \( t^\alpha \) をかけた関数を無限大の近傍での関数とし \( |x| = T_0 \) の近くでは (1) の半径対称解を使いこれらをうまく結ぶことにより \( C^2 \) に属する super, sub solution を構成して解の存在を示した。この詳細は JMAA[5] に掲載されている。

定理 5 について
(1) の方程式を \( u(t, \theta) = t^\alpha(\omega_0 + w(t, \theta)) \) とおき \( w(t, \theta) \) に関する \( t \) に関するオーダー方程式を作ると、

\[
\left( \frac{\partial^2}{\partial t^2} + \frac{2\alpha + 1}{t} \frac{\partial}{\partial t} + \frac{\alpha^2 - p|\omega_0|^{p-1}}{t^2} + \frac{1}{t^2} \frac{\partial^2}{\partial \theta^2} \right) w(t, \theta) \\
= \frac{1}{t^2} \left\{ \sum_{i=0}^{N} \sigma_i t^{-i} \right\} \{(\omega_0 + w)^p - \omega_0^p - p|\omega_0|^{p-1} w \} \\
+ \frac{1}{t^2} \sum_{i=1}^{N} \sigma_i \left( p|\omega_0|^{p-1} \right) t^{-i} w \\
+ \frac{1}{t^2} \left\{ \sum_{i=1}^{N} \sigma_i (\omega_0^p - f_i) t^{-i} \right\} \\
+ O(t^{-\alpha+\sqrt{p|\omega_0|^{p-1}+1}})
\]

ここで、\( \sigma_i, f_i \) は \( g(t), f(t) \) の係数によってきまる定数である。また \( r = \alpha - \sqrt{p|\omega_0|^{p-1}+n^2} \), \( r_0 = \)
\( \alpha - \sqrt{\beta_0} \) とすると仮定から \( 0 < r < 1 \) となる。\( h_s(t) \) を
(7)
\[ h_{-1}(t) = \sum_{k=1}^{N} c_{-1,k} t^{-k} h_0(t) = \sum_{k=2}^{M} \sum_{j=0}^{[r_0-rk]} c_{0,k,j} t^{-(kr+j)}, \quad h_q(t) = \sum_{k=2}^{M} \sum_{j=0}^{[r_0-rk]} c_{q,k,j} t^{-(kr+j)} \]
とおく、
\[ w(t, \theta) = h_{-1}(t) + \sum_{q=0}^{M} h_q(t) \cos(q\theta) \]
とおく。\( h_{-1} \) は半径対称解のオーダー展開の項である。係数 \( c_{-1,k} \) 決め方は次の定理 6 の
g説に任せるとしてここでは決まったとする。\( w(t, \theta) \) を \( w \) のオーダー方程式に代入し (7)
の係数 \( \{ c_{s,k,j} \} \) がオーダー方程式を満たしつつ、\( c_{1,0,0} \neq 0 \) を満たすようにうまく決める
ことができたとすると、\( u_{\infty}(t, \theta) = t^\omega w(t, \theta) \) とおくとにより \( u_{\infty} \) はオーダー的に (1) の
方程式を満たすものになっている。\( \alpha \) の関数を無限遠方の関数として定理 4 と同様に sub, super solution を構成して解の存在を示す。それゆえに
(7) の係数 \( \{ c_{s,k,j} \} \) を決定することが問題である。\( w \) を \( h_s \) で表わし、オーダー方程式に代入し、ディナー展開を用い、\( t \) のべき乗 と \( \cos \theta \) の積の同一項の係数をまとめ、まとめた各係数を零とした \( \{ c_{s,k,j} \} \) 連立方程式をつくると、未知関数 \( \{ c_{s,k,j} \} \) の方が連立方程式より一つ少ないので陰関数定理から \( c_{1,0,0} \) 以外の変数 \( \{ c_{s,k,j} \} \) を \( c_{1,0,0} \) で表わせるので条件
を満たす係数の存在が示せる。以下、定理 4 と同じ方法で super, sub solution を作り
条件に合う解を構成する。詳しい証明は DEF[6] に掲載予定である。

定理 6 について
定理 6 に関しては Ad.M.S.A[4] に掲載すみである。方法は定理 5 の概説に記した \( h_{-1}(t) \)
の係数 \( \{ c_{-1,k} \} \) を \( \theta \) 方向をなくしたオーダー方程式の解になるように決めばよい。や
り方は定理 5 の概説と同じ方法で、オーダー方程式に代入しディナー展開し \( \{ c_{-1,k} \} \) に関
する連立方程式を作ると漸化式になっていることが分かり係数を决定することができる。
以下定理 5 の概説と同じである。

定理 7 について
これはプレプリントの段階であるので、方針のみであるが少し詳しく説明する。(1) の半
径対称解 \( u(t) \) に対して \( u(t) = t^\alpha (\omega_0 + w(t)) \) として \( \lim_{t \to \infty} w(t) = 0 \) として \( w(t) \) を見
つければよい。\( w(t) \) は (1) に代入することにより次のオーダー方程式を満たすことがわ
かる。
(8)
\[ \frac{d^2 w}{dt^2}(t) + \frac{2\alpha + 1}{t} \frac{dw}{dt}(t) = \frac{1}{t^2} F(w(t)) + O(t^{-3}). \]
ここで、\( F(w(t)) = (\omega_0 + w(t))|\omega_0 + w(t)|^{p-1} + |\omega_0|^p - p|\omega_0|^{p-1} w(t) \)。
また、\( F(w(t)) \) は
\( \lim_{t \to \infty} w(t) = 0 \) を考慮しディナー展開を使用すると、\( F(w(t)) \leq -k_0 w^2(t) \) を得る。
\( w(t) \) に関しては \( t \) が充分大きければ、
(9)
\[ (1) \, w(t) \geq C_1 t^{-2\delta_0}; \quad (2) \, w(t) \leq C_2 t^{-2\delta_0} \]
と/orかになっていることがわかる。ここで、\( C_i \ (i = 1, 2) \) はある正の定数で、\( \delta_0 \) は \( 0 <
2\delta_0 < \min(1, 2\alpha) \) を満たすある任意の定数である。

\[ \Box \]
一方、(8) の方程式に \( F(w(t)) \leq -k_0w^2(t) \) を考慮した微分不等式から、解 \( w(t) \) の積分不等式をつくり、\( \lim_{t \to \infty} w(t) = 0 \) から,

\[
w(t) > w(t_1) \left( \frac{t_1}{t} \right)^{2\alpha} - C(t_1) \frac{1}{t^{2\delta_0}}
\]
をえる。このことより \( t \) が無限大の近傍にあるときは \( |w(t)| \leq Ct^{-2\delta_0} \) としてよい。9 の (2) の場合に対応する元の方程式 (1) の解が 2 つ以上あるとして差を取り、非線形項にテーラー展開を持てて、微分不等式をつくり解の差がオーダーとして \( t\sqrt{\|u_0\|^p} \) 以上ある事示す。いまま、\( |w(t)| \leq C_2 t^{-2\delta_0} \) より差はオーダーとして \( t^{\alpha - 2\delta_0} = t\sqrt{\|u_0\|^p} \) 以下であるのでお互いに矛盾することより一般性が示せる。また、(9) の (2) の場合に対応する元の方程式 (1) の解の存在は、\( y_{\pm}(t) = t^\alpha (w_0 \pm Ct^{-2\delta_0}) \) とけば \( y_{+}(t) \) は super solution, \( y_{-}(t) \) は sub solution より定理 5 の方法により無限大の近傍では \( y_{\pm}(t) \) になる super,sub solution が \( [T_0, \infty) \) で作成することより存在はいえる。これが、定理 7 の前半である。

(9) の (1) の場合を考察する。\( x(t) = (\log t)w(t) \) とおき (8) に代入すると \( x(t) \) の満たす方程式は

\[
\frac{d^2 x}{dt^2}(t) + \left( \frac{2\alpha + 1}{t} - \frac{2}{t\log t} \right) \frac{dx}{dt}(t) = (2\alpha - \frac{2}{\log t} - a_2 x(t)) \frac{x(t)}{t^2 \log t} + \frac{\log t}{t^2} F_3(x(t)) + O\left( \frac{\log t}{t^3} \right)
\]

ここで,

\[
a_2 = \frac{p(p-1)}{2!} |w_0|^{p-2},
\]

\[
F_3(x(t)) = -\frac{p(p-1)(p-2)}{3!} |w_0| + \theta \left( \frac{x(t)}{\log t} \right)^{p-3} \frac{x(t)}{\log t},
\]

\( \theta : \) constant sch と \( 0 < \theta < 1. \)

である。このうえに,

\[
\lim_{t \to \infty} \frac{x(t)}{\log t} = 0 \text{ または } \lim_{t \to \infty} \frac{x(t)}{t^{\delta_1}} > 0 \quad (0 < \delta_1 < 2\delta_0)
\]

を満たす。\( x(t) \) の解析は複雑であるがその結果、

\[
\lim_{t \to \infty} x(t) = \begin{cases} 0 & 2\alpha \leq a_2 \end{cases}
\]

をえる。その上、\( \lim_{t \to \infty} t^2 \dot{z}(t)^2 < Const \) と \( (t \log t) \dot{z}(t)^2 \in L^1(t_1, \infty) \) も得る。さて、\( \lim_{t \to \infty} x(t) = 0 \) の場合 \( Ct^{-\delta_1} \leq x(t) < \epsilon \) として解析すると \( x(t) \leq C_3 t^{-2\delta_0} \) を得る。この場合は少し修正はいるがこれを満たす \( w(t) \) の解析で終わっている。ゆえに \( a = 2\alpha / a_2 \) とし、\( x(t) = a + z(t) / \log t \) とおくと (10) と \( \lim_{t \to \infty} x(t) = a \) により \( \lim_{t \to \infty} z(t) = 0 \) で、

\[
\ddot{z}(t) + \frac{1}{t} (2\alpha + 1 - \frac{2}{\log t}) \dot{z}(t)
\]
$$+ \frac{2}{t^2(\log t)^2}(a + z(t))$$

$$= \frac{-1}{t^2 \log t}(2\alpha + a_2z(t))z(t)$$

$$+ \frac{a_3(a + z(t))^3}{t^2(\log t)^2} + O(\frac{1}{t^2(\log t)^3}).$$

の方程式を満たすことがわかる。ここで、この $z(t)$ が満たす不等式を導くと複雑な計算の結果

$$| \frac{\dot{z}(t)}{t \log t} | \leq \frac{M}{t^2(\log t)^{2.5-\epsilon}}$$

$$| \frac{z(t)^2}{t^2 \log t} | \leq \frac{M}{t^2(\log t)^{3-\epsilon}}$$

と得る。これを考慮しると $z(t)$ は次の方程式をみたすように書き換えられる。

$$\ddot{z}(t) + \frac{(2\alpha + 1)}{t} \dot{z}(t) + \frac{(2\alpha)}{t^2 \log t} z(t)$$

$$= -\frac{(2a - a_3a^3)}{t^2(\log t)^2} + O(\frac{1}{t^2(\log t)^{2.5-\epsilon}}).$$

ここで, $z(t) = v(t)/\log t$ とおき上記の不等式を考慮して $v(t)$ の方程式をみると

$$\ddot{v}(t) + \frac{(2\alpha + 1)}{t} \dot{v}(t) = -\frac{(2a - a_3a^3)}{t^2(\log t)^2} + O(\frac{1}{t^2(\log t)^{1.5-\epsilon}}).$$

となる。この方程式を解析すると

$$v(t) = \frac{(a_3a^3 - 2a)}{2\alpha} \log \log t + C + O(\frac{1}{(\log t)^{1.5-\epsilon}}).$$

を得ることになる。そこで $C$ は $v(t)$ によってきまる定数である。

これをすべて戻せば、(1) の半径対称解 $y(t)$ から

$$y(t) = t^a(w_0 + \frac{a}{t} + \frac{a_3a^3 - 2a \log \log t}{2\alpha} \log (\frac{t}{\log t})^2) + O(\frac{1}{(\log t)^{2.5-\epsilon}})$$

を満たすことになる。また、一意性に関しては 2 つ以上の解が存在するとして、差の微分不等式を導き、この不等式から解の差はオーダーとして $t^\alpha/(\log t)^2$ より大きいとの評価を導くことにより矛盾を導く。次に存在は

$$y_{\pm}(t) = t^a(w_0 + \frac{a}{\log t} + \frac{b \log \log t}{(\log t)^2} + \frac{C}{(\log t)^2} \pm \frac{1}{(\log t)^{2.5-\epsilon}})$$

と置くとそれぞれが super, sub solution となり解の存在が言える。この証明は Preprint[7] に書いてある。
Reference


住所
丸尾健二
〒636-0093
奈良県北葛城郡河合町大輪田1673-1

[0]
移流拡散方程式系の安定化 — 境界制御・境界観測を伴う場合 —
佐野 英樹（神戸大学大学院工学研究科）

Stabilization of a coupled transport-diffusion system: a case with boundary control and boundary observation
Hideki Sano (Graduate School of Engineering, Kobe University)

1 はじめに

1980 年代のはじめより、無限次元動的システムに対する有限次元安定化コントローラの構成法に関する研究が、多くの研究者によってなされてきた（例えば、文献 [9], [12], [3], [7], [4], [1], [10], [2], [6], [8], [11]）。一般に、無限次元システムから導出された有限次元モデルを用いて有限次元コントローラを構成し、それを無限次元システムに取付けるとき、モデル化されていないモードの影響により、スピルオーバ現象が生じる可能性がある。坂和は単独の方程式で記述される線形拡散系を取り上げ、有限次元コントローラを取り付けた閉ループ系に対してモデル化されていないモードの影響を弱めるために、二種類の有限次元オブザーバをはじめて導入した [9]。後に Balas はその片方を “residual mode filter” と呼び、residual mode filter (RMF) が有限次元安定化コントローラの構成において本質的な役割を果たしていることを明らかにした [1]。一方、著者らは最近、坂和の手法を以下のような化学反応プロセスに関連した境界入力をもつ移流拡散系に拡張した [11]。

\[
\begin{align*}
\frac{\partial z_1}{\partial t}(t, x) &= \frac{\partial^2 z_1}{\partial x^2}(t, x) - a_1 \frac{\partial z_1}{\partial x}(t, x) - a_1 z_1(t, x), \\
\frac{\partial z_2}{\partial t}(t, x) &= \frac{\partial^2 z_2}{\partial x^2}(t, x) - a_2 \frac{\partial z_2}{\partial x}(t, x) + a_2 z_1(t, x), \quad (t, x) \in (0, \infty) \times (0, 1) \\
- \frac{\partial z_1}{\partial x}(t, 0) &= u(t), \quad \frac{\partial z_1}{\partial x}(t, 1) = 0, \quad \frac{\partial z_2}{\partial x}(t, 0) = \frac{\partial z_2}{\partial x}(t, 1) = 0, \quad t > 0 \\
z_1(0, x) &= z_{10}(x), \quad z_2(0, x) = z_{20}(x), \quad x \in [0, 1]
\end{align*}
\]

\[
y(t) = [y_1(t), y_2(t)]^T = \left[ \int_0^1 c_1(x)z_1(t, x)dx, \int_0^1 c_2(x)z_2(t, x)dx \right]^T \quad \text{(分布観測)}
\]

ここで、\(a, a_1, a_2\) は正の物理定数、\(c_i(x) (i = 1, 2)\) は領域内の内部に配置されたセンサの影響関数、\(u(t) \in \mathbb{R}\) は制御入力、\(y(t) \in \mathbb{R}^2\) は観測出力である。そこで、はじめにシステム (1), (2) をあるヘルベルト空間における非有界出力作用素を有する発展方程式

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= -A_1 x_1(t) + B_1 u(t), \quad x_1(0) = x_{10} \\
\frac{dx_2(t)}{dt} &= (-A_1 + a_1) x_2(t) + a_2 x_1(t), \quad x_2(0) = x_{20} \\
y(t) &= \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} C_1 A_1^T x_1(t) \\ C_2 A_1^T x_2(t) \end{bmatrix}
\end{align*}
\]

ただし

\[
\gamma = \frac{1}{4} + \epsilon \in (1/4, 1/2)
\]
として定式化し、つぎにシステム (3) に対して有限次元モデルを導出し、その有限次元モデルに対して効果的に働く安定化コントローラを構成した。そして、その有限次元コントローラに RMF を付加したコントローラが、RMF の数が十分大きいときに、無限次元のシステム (3) に対して有限次元安定化コントローラとなり得ることを証明した。本研究の目的は、出力方程式を (2) から

\[ y(t) = [y_1(t), y_2(t)]^T = [z_1(t, 1), z_2(t, 1)]^T \quad (\text{境界観測}) \] に変更しても同様に、RMF を用いた方法で有限次元安定化コントローラが構成できることを示すことである。

### 2 境界制御・境界観測を伴う移流拡散系

境界入力をもつ移流拡散系 (1) と出力方程式 (4) からなるシステム

\[
\begin{align*}
\frac{\partial z_1}{\partial t}(t, x) &= \frac{\partial^2 z_1}{\partial x^2}(t, x) - \alpha \frac{\partial z_1}{\partial x}(t, x) - a_1 z_1(t, x), \\
\frac{\partial z_2}{\partial t}(t, x) &= \frac{\partial^2 z_2}{\partial x^2}(t, x) - \alpha \frac{\partial z_2}{\partial x}(t, x) + a_2 z_1(t, x), \\
\frac{\partial z_1}{\partial x}(t, 0) &= u(t), \quad \frac{\partial z_1}{\partial x}(t, 1) = 0, \\
& \quad \frac{\partial z_2}{\partial x}(t, 0) = \frac{\partial z_2}{\partial x}(t, 1) = 0, \quad t > 0 \\
z_1(0, x) &= z_{10}(x), \quad z_2(0, x) = z_{20}(x), \quad x \in [0, 1] \\
y(t) &= [y_1(t), y_2(t)]^T = [z_1(t, 1), z_2(t, 1)]^T
\end{align*}
\]  

(5)

を考える。\( \mathcal{L} \) を

\[
\mathcal{L} \varphi = -\frac{d^2 \varphi}{dx^2} + \alpha \frac{d \varphi}{dx} + a_1 \varphi
\]

で定義すると、システム (5) は以下のよう表せる。

\[
\begin{align*}
\frac{\partial z_1}{\partial t}(t, x) &= -\mathcal{L} z_1(t, x), \\
\frac{\partial z_2}{\partial t}(t, x) &= -\mathcal{L} z_2(t, x) + a_1 z_2(t, x) + a_2 z_1(t, x), \\
\frac{\partial z_1}{\partial n}(t, \xi) &= g(\xi) u(t), \quad \frac{\partial z_2}{\partial n}(t, \xi) = 0, \quad (t, \xi) \in (0, \infty) \times \{0, 1\} \\
z_1(0, x) &= z_{10}(x), \quad z_2(0, x) = z_{20}(x), \quad x \in [0, 1] \\
y(t) &= [y_1(t), y_2(t)]^T = [z_1(t, 1), z_2(t, 1)]^T
\end{align*}
\]  

(6)

ここで \( \partial/\partial n \) は点 \( \xi \in \{0, 1\} \) における外向き法線微分を示し、\( g : \{0, 1\} \rightarrow \mathbb{R} \) は

\[
g(\xi) = \begin{cases} 
1, & \text{if } \xi = 0 \\
0, & \text{if } \xi = 1
\end{cases}
\]

によって定義される関数である。非有界作用素 \( A_1 \) を

\[
D(A_1) = \{ \varphi \in H^2(0, 1); \varphi'(0) = \varphi'(1) = 0 \}, \quad A_1 \varphi = \mathcal{L} \varphi, \quad \varphi \in D(A_1)
\]

で定義する。このとき、\( A_1 \) はスクリュルム・リュヴィル型の作用素

\[
(A_1 \varphi)(x) = \frac{1}{w(x)} \left( -\frac{d}{dx} \left( p(x) \frac{d \varphi(x)}{dx} \right) + q(x) \varphi(x) \right), \quad w(x) = p(x) = e^{-\alpha x}, \quad q(x) = a_1 e^{-\alpha x}
\]
とともに表され、作用素 $A_1$ は内積

$$
(\varphi, \psi)_\alpha = \int_0^1 \varphi(x) \psi(x) e^{-\alpha x} \, dx \quad \text{for } \varphi, \psi \in L^2_\alpha(0, 1)
$$

を有する重み付けられた $L^2$ 空間、すなわち $L^2_\alpha(0, 1)$ において自己共役になる。$A_1$ は $L^2_\alpha(0, 1)$ において固有値、固有関数 $(\lambda_i, \varphi_i)_{i=0}^\infty$ をもつ、${(\varphi_i)}_{i=0}^\infty$ は $L^2_\alpha(0, 1)$ において完全正規直交系を形成する。よって、任意の $f \in L^2_\alpha(0, 1)$ は以下のとおり展開できる。

$$
f = \sum_{i=0}^\infty (f, \varphi_i)\varphi_i
$$
その固有値、固有関数 $(\lambda_i, \varphi_i)_{i=0}^\infty$ の具体的表現は以下のとおりである [11].

$$
\begin{cases}
\lambda_0 = a_1, & \varphi_0(x) \equiv \nu_0 \\
\lambda_i = i^2 \pi^2 + \frac{\alpha^2}{4} + a_1, & \varphi_i(x) = \nu_i \left( e^{\frac{x}{2} \pi} \cos i\pi x - \frac{\alpha}{2i\pi} e^{\frac{x}{2} \pi} \sin i\pi x \right), \text{ for } i \geq 1
\end{cases}
$$

ここで

$$
\nu_0 := \sqrt{\frac{\alpha}{1 - e^{-\alpha}}}, \quad \nu_i := \sqrt{\frac{2}{1 + \frac{\alpha^2}{4i\pi^2}}}, \text{ for } i \geq 1
$$

以降、(6) 式における初期値 $z_{10}, z_{20}$ は $L^2_\alpha(0, 1)(= L^2(0, 1))$ の中から選ばれていると仮定する。いま、新しい変数

$$
x_1(t) = A_1^{-\frac{1}{4} - \epsilon} z_1(t, \cdot), \quad x_2(t) = A_1^{-\frac{1}{4} - \epsilon} z_2(t, \cdot)
$$

を導入する (e.g. [7]). ここで $0 < \epsilon < \frac{1}{4}$. 包含関係 $H^2(0, 1) \subset D(A_1^{-\frac{3}{4} - \epsilon}) \subset D(A_1^{\frac{1}{4} + \epsilon})$ に注意すると、(6) 式より次式を導くことができる。

$$
\begin{cases}
\frac{dx_1(t)}{dt} = -A_1 x_1(t) + A_1^{3/4 - \epsilon} \psi(t), \quad x_1(0) = A_1^{1/4 - \epsilon} z_{10} =: x_{10} \\
\frac{dx_2(t)}{dt} = (-A_1 + a_1) x_2(t) + a_2 x_1(t), \quad x_2(0) = A_1^{1/4 - \epsilon} z_{20} =: x_{20} \\
y(t) = [y_1(t), y_2(t)]^T = [\langle A_1^{3/4 - \epsilon} H, A_1^{1/4 + 2\epsilon} x_1(t) \rangle_\alpha, \langle A_1^{3/4 - \epsilon} H, A_1^{1/4 + 2\epsilon} x_2(t) \rangle_\alpha]^T
\end{cases}
$$

ここで $\psi \in H^2(0, 1)$ は境界値問題

$$
\mathcal{L}\psi = 0 \quad \text{in } (0, 1), \quad \frac{\partial\psi}{\partial n} = g \quad \text{on } \{0, 1\}
$$
の一意解であり、具体的に以下のように求まる。

$$
\psi(x) = -\frac{\alpha - \sqrt{D}}{2a_1(e^\sqrt{D} - 1)} e^{\frac{a_1}{2} \pi x} + \frac{(\alpha + \sqrt{D}) e^{\sqrt{D}}}{2a_1(e^\sqrt{D} - 1)} e^{\frac{a_1}{2} \pi x}, \quad D := \alpha^2 + 4a_1
$$

$H \in H^2(0, 1)$ は境界値問題

$$
\mathcal{L}H = 0 \quad \text{in } (0, 1), \quad \frac{\partial H}{\partial n} = \overline{g} \quad \text{on } \{0, 1\}
$$
の一意解である。ただし $\overline{g} : \{0, 1\} \to \mathbb{R}$ は

$$
\overline{g}(\xi) = \begin{cases}
0, & \text{if } \xi = 0 \\
e^\alpha, & \text{if } \xi = 1
\end{cases}
$$

によって定義される関数である。上記の解 $H$ は具体的に以下のように求まる。

$$
H(x) = -\frac{(\alpha - \sqrt{D}) e^{\frac{a_1}{2} \pi x}}{2a_1(e^\sqrt{D} - 1)} e^{\frac{a_1}{2} \pi x} + \frac{(\alpha + \sqrt{D}) e^{\sqrt{D}}}{2a_1(e^\sqrt{D} - 1)} e^{\frac{a_1}{2} \pi x}, \quad D := \alpha^2 + 4a_1
$$
3 (7) 式の導出

本節では (7) 式を導出する。その第 1 式、第 2 式は文献 [11] と同じであるが、本節では、さらにそれについて振り返ってみる。その後に、主要結果である第 3 式を導出する。

3.1 (7) の第 1 式、第 2 式の導出

\( \psi \in H^2(0, 1) \) は境界値問題 (8) の一意解であるので、(6) の第 1 式から

\[
\frac{\partial z_1}{\partial t}(t, x) = -\mathcal{L}z_1(t, x) = -\mathcal{L}(z_1(t, x) - \psi(x)u(t)) \tag{10}
\]

を得る。ここで

\[
\frac{\partial}{\partial n}(z_1(t, x) - \psi(x)u(t)) \bigg|_{x=\xi} = g(\xi)u(t) - g(\xi)u(t) = 0
\]

に注意すると、\( z_1(t, \cdot) - \psi u(t) \in D(A_1) \) が従うことがわかる。よって、(10) 式は

\[
\frac{dz_1}{dt}(t, \cdot) = -A_1(z_1(t, \cdot) - \psi u(t)) \tag{11}
\]

となる。ここで、(11) 式の両辺に \( A_1^{-\frac{1}{2}-\varepsilon} \) \((0 < \varepsilon < \frac{1}{2})\) を作用すると次式を得る。

\[
\frac{d}{dt}A_1^{-\frac{1}{2}-\varepsilon}z_1(t, \cdot) = -A_1^{\frac{3}{2}-\varepsilon}(z_1(t, \cdot) - \psi u(t)) = -A_1^{\frac{3}{2}-\varepsilon}z_1(t, \cdot) + A_1^{\frac{3}{2}-\varepsilon}\psi u(t) = -A_1A_1^{-\frac{1}{2}-\varepsilon}z_1(t, \cdot) + A_1^{\frac{3}{2}-\varepsilon}\psi u(t) \tag{12}
\]

ここで、包含関係 \( H^2(0, 1) \subset D(A_1^{-\frac{1}{2}-\varepsilon}) \) を用いた。したがって (12) 式において、新しい変数 \( x_1(t) = A_1^{-\frac{1}{2}-\varepsilon}z_1(t, \cdot) \) を定義することにより

\[
\frac{dx_1(t)}{dt} = -A_1x_1(t) + A_1^{\frac{3}{2}-\varepsilon}\psi u(t)
\]

を得る。このようにして (7) の第 1 式を得る。

つぎに、\( z_2(t, \cdot) \in D(A_1) \) であることに注意すると、(6) の第 2 式から

\[
\frac{dz_2}{dt}(t, \cdot) = -A_1z_2(t, \cdot) + a_1z_2(t, \cdot) + a_2z_1(t, \cdot) \tag{13}
\]

を得る。ここで、(13) 式の両辺に \( A_1^{-\frac{1}{2}-\varepsilon} \) を作用させ、別の変数 \( x_2(t) = A_1^{-\frac{1}{2}-\varepsilon}z_2(t, \cdot) \) を新たに定義すると

\[
\frac{dx_2(t)}{dt} = -A_1x_2(t) + a_1x_2(t) + a_2x_1(t)
\]

を得る。これは (7) の第 2 式そのものである。

3.2 (7) の第 3 式の導出 (主要結果)

(7) の最後の式は以下のようにして導出できる。はじめに、\( y_1(t) = z_1(t, 1) \) を以下のように表す。

\[
y_1(t) = z_1(t, 1) = \int_0^1 (h(x)z_1(t, x)e^{-\alpha x})_x \, dx \tag{14}
\]

ここで、\( h \) は

\[
h(0) = 0, \quad h(1) = e^\alpha
\]
を満たすある滑らかな関数である。一方、(14) 式はつぎのように計算できる。

\[ y_1(t) = z_1(t, 1) \]
\[ = \int_0^1 (h'(x)z_1(t, x)e^{-\alpha x} - \alpha h(x)z_1(t, x)e^{-\alpha x})dx + \int_0^1 h(x)z_1(x, x)e^{-\alpha x}dx \]
\[ = \int_0^1 (H''(x) - \alpha H'(x))z_1(x, x)e^{-\alpha x}dx + \int_0^1 H'(x)z_1(x, x)e^{-\alpha x}dx \]  

(15)

ここで

\[ H'(x) = h(x) \]

とおっている。ここで、部分積分と境界条件

\[ \frac{\partial z_1}{\partial n}(t, \xi) = g(\xi)u(t), \quad (t, \xi) \in (0, \infty) \times \{0, 1\} \]

を用いることにより、(15) 式最右辺の第 2 項目は以下のようになる。

\[ \int_0^1 H'(x)z_{1x}(t, x)e^{-\alpha x}dx = H(1)z_{1x}(t, 1)e^{-\alpha} - H(0)z_{1x}(t, 0) \]
\[ - \int_0^1 H(x)(z_{1xx}(t, x) - \alpha z_{1x}(t, x))e^{-\alpha x}dx \]
\[ = H(0)u(t) - \int_0^1 H(x)(z_{1xx}(t, x) - \alpha z_{1x}(t, x))e^{-\alpha x}dx \]

(16)

したがって、(15)，(16) 式から次式を得る。

\[ y_1(t) = \langle H'' - \alpha H', z_1(t, \cdot) \rangle_\alpha + H(0)u(t) - \langle H, z_{1xx}(t, \cdot) - \alpha z_{1x}(t, \cdot) \rangle_\alpha \]
\[ = \langle H'' - \alpha H', z_1(t, \cdot) \rangle_\alpha - \langle a_1H, z_1(t, \cdot) \rangle_\alpha \]
\[ + H(0)u(t) - \langle H, z_{1xx}(t, \cdot) - \alpha z_{1x}(t, \cdot) \rangle_\alpha + \langle a_1H, z_1(t, \cdot) \rangle_\alpha \]
\[ = \langle -\mathcal{L}H, z_1(t, \cdot) \rangle_\alpha + H(0)u(t) + \langle H, \mathcal{L}z_1(t, \cdot) \rangle_\alpha \]

(17)

以降、\( H \in H^2(0, 1) \) を境界値問題 (9) の一意解とする。そのとき、(17) 式は

\[ y_1(t) = H(0)u(t) + \langle H, \mathcal{L}z_1(t, \cdot) \rangle_\alpha \]
\[ = H(0)u(t) + \langle H, \mathcal{L}(z_1(t, \cdot) - \psi u(t)) \rangle_\alpha \]
\[ = H(0)u(t) + \langle H, A_1(z_1(t, \cdot) - \psi u(t)) \rangle_\alpha \]

(18)

となる。ここで、\( z_1(t, \cdot) - \psi u(t) \in D(A_1) \) となる事実を用いている。さらに、包含関係

\[ H^2(0, 1) \subset D(A_1^{3-\epsilon}) \subset D(A_1^{4+\epsilon}) \]

に注意すると、(18) 式は

\[ y_1(t) = H(0)u(t) + \langle A_1^{3-\epsilon}H, A_1^{4+\epsilon}z_1(t, \cdot) - \psi u(t) \rangle_\alpha \]
\[ = H(0)u(t) + \langle A_1^{3-\epsilon}H, A_1^{4+\epsilon}z_1(t, \cdot) - A_1^{4+\epsilon}\psi u(t) \rangle_\alpha \]
\[ = \langle A_1^{3-\epsilon}H, A_1^{4+\epsilon}z_1(t, \cdot) \rangle_\alpha + \langle H(0) - \langle A_1^{3-\epsilon}H, A_1^{4+\epsilon}\psi \rangle_\alpha, u(t) \rangle \]
\[ = \langle A_1^{3-\epsilon}H, A_1^{4+\epsilon}z_1(t, \cdot) \rangle_\alpha + D u(t) \]

(19)

となる。ただし

\[ D := H(0) - \langle A_1^{3-\epsilon}H, A_1^{4+\epsilon}\psi \rangle_\alpha \]
直接的な計算により

\[ H(0) = \langle A_1^{\frac{3}{2} - \varepsilon} H, A_1^{\frac{1}{2} + \varepsilon} \psi \rangle_\alpha = \frac{\nu_0^2}{a_1} + \sum_{i=1}^{\infty} \frac{e^{\frac{\pi i}{4}} v_0^2 (-1)^i}{i + \frac{1}{2} + i^2 \pi^2} \]

が従うが、これは \( \tilde{D} = 0 \) を意味する。よって、(19) 式より

\[ y_1(t) = z_1(t, 1) = \langle A_1^{\frac{\gamma}{2} - \varepsilon} H, A_1^{\frac{1}{2} + \varepsilon} z_1(t, \cdot) \rangle_\alpha = \langle A_1^{\frac{\gamma}{2} - \varepsilon} H, A_1^{\frac{1}{2} + 2\varepsilon} x_1(t) \rangle_\alpha \]

を得る。

つぎに、\( y_2(t) = z_2(t, 1) \) を作用素 \( A_1 \) の分数幂を用いて表そう。この場合、境界条件は

\[ \frac{\partial z_2}{\partial n}(t, \xi) = 0, \quad (t, \xi) \in (0, \infty) \times \{ 0, 1 \} \]

で与えられることに注意する。上と同様の議論により次式を導くできる。

\[ y_2(t) = z_2(t, 1) = \langle H, L z_2(t, \cdot) \rangle_\alpha = \langle H, A_1 z_2(t, \cdot) \rangle_\alpha = \langle A_1^{\frac{\gamma}{2} - \varepsilon} H, A_1^{\frac{1}{2} + \varepsilon} z_2(t, \cdot) \rangle_\alpha = \langle A_1^{\frac{3}{2} - \varepsilon} H, A_1^{\frac{1}{2} + 2\varepsilon} x_2(t) \rangle_\alpha \]

このようにして、(7) の最後の式が導出できる。

4 有限次元安定化コントローラの構成

まずはじめに、二つの有界作用素 \( B_1 : \mathbb{R} \rightarrow L^2_0(0, 1), \ C : L^2_0(0, 1) \rightarrow \mathbb{R} \) を

\[ B_1 v = A_1^{\frac{3}{2} - \varepsilon} \psi v, \quad v \in \mathbb{R} \]
\[ C \varphi = \langle A_1^{\frac{3}{2} - \varepsilon} H, \varphi \rangle_\alpha, \quad \varphi \in L^2_0(0, 1) \]

のように定義し、システム (7) をつぎのように表す。

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= -A_1 x_1(t) + B_1 u(t), \quad x_1(0) = x_{10} \\
\frac{dx_2(t)}{dt} &= (-A_1 + a_1) x_2(t) + a_2 x_1(t), \quad x_2(0) = x_{20} \quad \text{(20)} \\
y(t) &= \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} C A_1^\gamma x_1(t) \\ C A_1^\gamma x_2(t) \end{bmatrix}
\end{align*}
\]

ここで

\[ \gamma = \frac{1}{2} + 2\varepsilon \in (1/2, 1) \]

4.1 システムの分割

システム (20) に対する有限次元モデルを導出するために直交射影作用素 \( P_k \) を用いる。

\[ P_k f = \sum_{i=0}^{k} \langle f, \varphi_i \rangle_\alpha \varphi_i \]

まずははじめに、正数 \( \kappa \) が与えられているとする。

(1) そして、\( -\lambda_{l+1} + a_1 < -\kappa \) を満たすように整数 \( l (l \geq 0) \) を選ぶ。
(2) さらに、別の整数 $n$ を $n > l$ となるように選ぶ。

作用素 $P_l$ および $P_n$ ($n > l$) を用いて、状態変数 $x_1(t)$ と $x_2(t)$ を以下のように分割する。

$$
x_1(t) = x_{1,1}(t) + x_{1,2}(t) + x_{1,3}(t), \quad x_2(t) = x_{2,1}(t) + x_{2,2}(t) + x_{2,3}(t)
$$

ここで $x_{1,1}(t) := P_l x_1(t), x_{1,2}(t) := (P_n - P_l) x_1(t), x_{1,3}(t) := (I - P_n) x_1(t)$ ($i = 1, 2$)。また、これに伴って空間 $L_{A}(0, 1)$ は以下のように表される。

$$
L_{A}^2(0, 1) = P_l L_{A}^2(0, 1) \oplus (P_n - P_l) L_{A}^2(0, 1) \oplus (I - P_n) L_{A}^2(0, 1)
$$

ただし、各空間の次元は $\dim P_l L_{A}^2(0, 1) = l + 1, \dim (P_n - P_l) L_{A}^2(0, 1) = n - l, \dim (I - P_n) L_{A}^2(0, 1) = \infty$。
したがって、システム (20) は等価的に以下のように表される。

$$
\begin{align*}
\frac{dx_{1,1}(t)}{dt} &= -A_{1,1} x_{1,1}(t) + B_{1,1} u(t), \quad x_{1,1}(0) = x_{10} \\
\frac{dx_{1,2}(t)}{dt} &= -A_{1,2} x_{1,2}(t) + B_{1,2} u(t), \quad x_{1,2}(0) = x_{20} \\
\frac{dx_{1,3}(t)}{dt} &= -A_{1,3} x_{1,3}(t) + B_{1,3} u(t), \quad x_{1,3}(0) = x_{30} \\
\frac{dx_{2,1}(t)}{dt} &= (A_{1,1} + a_1) x_{2,1}(t) + a_2 x_{1,1}(t), \quad x_{2,1}(0) = x_{20} \\
\frac{dx_{2,2}(t)}{dt} &= (A_{1,2} + a_1) x_{2,2}(t) + a_2 x_{1,2}(t), \quad x_{2,2}(0) = x_{20} \\
\frac{dx_{2,3}(t)}{dt} &= (A_{1,3} + a_1) x_{2,3}(t) + a_2 x_{1,3}(t), \quad x_{2,3}(0) = x_{30} \\
y(t) &= \left( C_{1,1} A_{1,1} x_{1,1}(t) + C_{1,2} A_{1,2} x_{1,2}(t) + C_{1,3} A_{1,3} x_{1,3}(t) \right) \\
&\left( C_{2,1} A_{1,1} x_{2,1}(t) + C_{2,2} A_{1,2} x_{2,2}(t) + C_{2,3} A_{1,3} x_{2,3}(t) \right)
\end{align*}
$$

ここで

$$
A_{1,1} := P_l A_1 P_l, \quad A_{1,2} := (P_n - P_l) A_1 (P_n - P_l), \quad A_{1,3} := (I - P_n) A_1 (I - P_n)
$$

$$
B_{1,1} := P_l B_1, \quad B_{1,2} := (P_n - P_l) B_1, \quad B_{1,3} := (I - P_n) B_1
$$

$$
C_{1} := C P_l, \quad C_{2} := C (P_n - P_l), \quad C_{3} := C (I - P_n)
$$

$$
x_{10} := P_l x_{10}, \quad x_{20} := (P_n - P_l) x_{10}, \quad x_{10} := (I - P_n) x_{10}
$$

この中で、作用素 $A_{1,3}, A_{1,3}^*$ は非有界であるが、他の作用素はすべて有界である。

今後、有限次元ヒルベルト空間 $P_l L_{A}^2(0, 1)$ を基底 $(\varphi_0, \varphi_1, \ldots, \varphi_l)$ に関してユークリッド空間 $R^{l+1}$ と同一視する。このようにして、$P_l L_{A}^2(0, 1)$ の要素は $(l + 1)$ 次元ベクトルと同一視され、作用素 $A_{1,1}, B_{1,1}, C_1$ は適当なサイズの行列と同一視される。同様に、$(P_n - P_l) L_{A}^2(0, 1)$ の要素は $(n - l)$ 次元ベクトルと同一視され、作用素 $A_{1,2}, B_{1,2}, C_2$ は適当なサイズの行列と同一視される。

ここで、有限次元モデルおよびRMFを導出するために、システム (21) をつぎのように書き直す。まず、システム (21) の第1式と第3式を用いて

$$
\frac{d}{dt} \begin{bmatrix} x_{1,1}(t) \\ x_{2,1}(t) \end{bmatrix} = \begin{bmatrix} -A_{1,1} & 0 \\ a_2 I_{l+1} - A_{1,1} + a_1 I_{l+1} \end{bmatrix} \begin{bmatrix} x_{1,1}(t) \\ x_{2,1}(t) \end{bmatrix} + \begin{bmatrix} B_{1,1} \\ 0 \end{bmatrix} u(t), \quad \begin{bmatrix} x_{1,1}(0) \\ x_{2,1}(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}
$$

とし、つぎに第2式と第5式から

$$
\frac{d}{dt} \begin{bmatrix} x_{1,2}(t) \\ x_{2,2}(t) \end{bmatrix} = \begin{bmatrix} -A_{1,2} & 0 \\ a_2 I_{n-l} - A_{1,2} + a_1 I_{n-l} \end{bmatrix} \begin{bmatrix} x_{1,2}(t) \\ x_{2,2}(t) \end{bmatrix} + \begin{bmatrix} B_{1,2} \\ 0 \end{bmatrix} u(t), \quad \begin{bmatrix} x_{1,2}(0) \\ x_{2,2}(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}
$$
とする．さらに第3式と第6式を用いて
\[
\frac{d}{dt} \begin{bmatrix} x_{1,1}(t) \\ x_{2,1}(t) \end{bmatrix} = \begin{bmatrix} -A_{1,1} & 0 \\ a_2 I & -A_{1,1} + a_1 I \end{bmatrix} \begin{bmatrix} x_{1,1}(t) \\ x_{2,1}(t) \end{bmatrix} + \begin{bmatrix} B_{1,1} \\ 0 \end{bmatrix} u(t), \quad \begin{bmatrix} x_{1,3}(0) \\ x_{2,3}(0) \end{bmatrix} = \begin{bmatrix} x_{10}^3 \\ x_{20}^3 \end{bmatrix}
\]
とする．これらの表現に伴い、システム (21) の最後の式は
\[
y(t) = \begin{bmatrix} C_1 A_{1,1}^γ & 0 \\ 0 & C_1 A_{1,1}^γ \end{bmatrix} \begin{bmatrix} x_{1,1}(t) \\ x_{2,1}(t) \end{bmatrix} + \begin{bmatrix} C_2 A_{1,2}^γ & 0 \\ 0 & C_2 A_{1,2}^γ \end{bmatrix} \begin{bmatrix} x_{1,2}(t) \\ x_{2,2}(t) \end{bmatrix} + \begin{bmatrix} C_3 A_{1,3}^γ & 0 \\ 0 & C_3 A_{1,3}^γ \end{bmatrix} \begin{bmatrix} x_{1,3}(t) \\ x_{2,3}(t) \end{bmatrix}
\]
と表せる．以上をまとめると以下のようになる．
\[
\left\{ \begin{array}{l}
\frac{d\bar{x}_1(t)}{dt} = \overline{A}_1 \bar{x}_1(t) + \overline{B}_1 u(t), \quad \bar{x}_1(0) = \bar{x}_{10} \\
\frac{d\bar{x}_2(t)}{dt} = \overline{A}_2 \bar{x}_2(t) + \overline{B}_2 u(t), \quad \bar{x}_2(0) = \bar{x}_{20} \\
\frac{d\bar{x}_3(t)}{dt} = \overline{A}_3 \bar{x}_3(t) + \overline{B}_3 u(t), \quad \bar{x}_3(0) = \bar{x}_{30} \\
y(t) = \tilde{C}_1 \bar{x}_1(t) + \tilde{C}_2 \bar{x}_2(t) + \tilde{C}_3 \bar{x}_3(t)
\end{array} \right. \tag{22}
\]
ここで
\[
\bar{x}_1(t) := \begin{bmatrix} x_{1,1}(t) \\ x_{2,1}(t) \end{bmatrix}, \quad \bar{x}_{10} := \begin{bmatrix} x_{10}^1 \\ x_{20}^1 \end{bmatrix} \in \mathbb{R}^{(l+1)} \\
\bar{x}_2(t) := \begin{bmatrix} x_{1,2}(t) \\ x_{2,2}(t) \end{bmatrix}, \quad \bar{x}_{20} := \begin{bmatrix} x_{10}^2 \\ x_{20}^2 \end{bmatrix} \in \mathbb{R}^{(l-n)} \\
\bar{x}_3(t) := \begin{bmatrix} x_{1,3}(t) \\ x_{2,3}(t) \end{bmatrix}, \quad \bar{x}_{30} := \begin{bmatrix} x_{10}^3 \\ x_{20}^3 \end{bmatrix} \in [(I - P_n)L_n^2(0, 1)]^2
\]
であり，行列および作用素はつぎのとおりである．
\[
\overline{A}_1 := \begin{bmatrix} -A_{1,1} & 0 \\ a_2 I_{l+1} & -A_{1,1} + a_1 I_{l+1} \end{bmatrix}, \quad \overline{B}_1 := \begin{bmatrix} B_{1,1} \\ 0 \end{bmatrix}, \quad \tilde{C}_1 := \begin{bmatrix} C_1 A_{1,1}^γ & 0 \\ 0 & C_1 A_{1,1}^γ \end{bmatrix} \\
\overline{A}_2 := \begin{bmatrix} -A_{1,2} & 0 \\ a_2 I_{l-n} & -A_{1,2} + a_1 I_{l-n} \end{bmatrix}, \quad \overline{B}_2 := \begin{bmatrix} B_{1,2} \\ 0 \end{bmatrix}, \quad \tilde{C}_2 := \begin{bmatrix} C_2 A_{1,2}^γ & 0 \\ 0 & C_2 A_{1,2}^γ \end{bmatrix} \\
\overline{A}_3 := \begin{bmatrix} -A_{1,3} & 0 \\ a_2 I & -A_{1,3} + a_1 I \end{bmatrix}, \quad \overline{B}_3 := \begin{bmatrix} B_{1,3} \\ 0 \end{bmatrix}, \quad \tilde{C}_3 := \begin{bmatrix} C_3 A_{1,3}^γ & 0 \\ 0 & C_3 A_{1,3}^γ \end{bmatrix}
\]

4.2 RMF を用いた有限次元コントローラ
分割されたシステム (22) より，有限次元系
\[
\left\{ \begin{array}{l}
\frac{d\bar{x}_1(t)}{dt} = \overline{A}_1 \bar{x}_1(t) + \overline{B}_1 u(t) \\
y(t) = \tilde{C}_1 \bar{x}_1(t)
\end{array} \right. \tag{23}
\]
を考えるシステム (20) に対する有限次元モデルとする。
【定理 1】有限次元モデル (23) において、\( (\underline{A}_1, \underline{B}_1) \) は可制御、\( (\tilde{C}_1, \tilde{A}_1) \) は可観測である。
(証明) まず、行列 \( A_{1,1}, B_{1,1} \) がつぎのように表せることに注意する。
\[
A_{1,1} = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_l), \quad B_{1,1} = \begin{bmatrix} b_0 & b_1 & \cdots & b_l \end{bmatrix}^T
\]
ここで
\[
b_i := \lambda_i^{\frac{3}{4} - \varepsilon} \langle \psi, \varphi_i \rangle_a, \quad 0 \leq i \leq l
\]
であるが、直接的な計算により
\[
b_i = \begin{cases} \frac{\lambda_i^{\frac{3}{4} - \varepsilon} \nu_0}{a_1} (\neq 0), & \text{if } i = 0 \\ \frac{\lambda_i^{\frac{3}{4} - \varepsilon} \nu_i + \pi^2}{a_1^2 + \pi^2} (\neq 0), & \text{if } 1 \leq i \leq l \end{cases}
\]
が従う。よって、\( 0 < \lambda_0 < \lambda_1 < \cdots < \lambda_l \) に注意すると、行列
\[
\begin{bmatrix} \underline{A}_1 - \lambda I_{2(l+1)} & \underline{B}_1 \end{bmatrix} = \begin{bmatrix} -A_{1,1} - \lambda I_{l+1} & 0 \\ a_2 I_{l+1} & -A_{1,1} + a_1 I_{l+1} - \lambda I_{l+1} & B_{1,1} \end{bmatrix}
\]
はすべての \( \lambda \in \mathbb{C} \) に対して行フルランクをもつことがわかる。これは \( (\underline{A}_1, \underline{B}_1) \) が可制御であることを意味する [14]。
つきに、有限次元モデルの可観測性を示そう。行列 \( A_{1,1}^\gamma, C_1 \) が
\[
A_{1,1}^\gamma = \text{diag}(\lambda_0^\gamma, \lambda_1^\gamma, \ldots, \lambda_l^\gamma), \quad C_1 = \begin{bmatrix} c_0 & c_1 & \cdots & c_l \end{bmatrix}
\]
のように表せるので、行列 \( C_1 A_{1,1}^\gamma \) は
\[
C_1 A_{1,1}^\gamma = \begin{bmatrix} c_0 \lambda_0^\gamma & c_1 \lambda_1^\gamma & \cdots & c_l \lambda_l^\gamma \end{bmatrix}
\]
と表せることに注意する。ここで
\[
c_i := \lambda_i^{\frac{3}{4} - \varepsilon} \langle H, \varphi_i \rangle_a, \quad 0 \leq i \leq l
\]
直接的な計算により
\[
c_i = \begin{cases} \frac{\lambda_i^{\frac{3}{4} - \varepsilon} \nu_0}{a_1} (\neq 0), & \text{if } i = 0 \\ \frac{\lambda_i^{\frac{3}{4} - \varepsilon} \nu_i + \pi^2}{a_1^2 + \pi^2} (\neq 0), & \text{if } 1 \leq i \leq l \end{cases}
\]
を得る。よって、\( 0 < \lambda_0 < \lambda_1 < \cdots < \lambda_l \) であるので、行列
\[
\begin{bmatrix} \underline{A}_1 - \lambda I_{2(l+1)} & \tilde{C}_1 \end{bmatrix} = \begin{bmatrix} -A_{1,1} - \lambda I_{l+1} & 0 \\ a_2 I_{l+1} & -A_{1,1} + a_1 I_{l+1} - \lambda I_{l+1} \\ C_1 A_{1,1}^\gamma & 0 \\ C_1 A_{1,1}^\gamma \end{bmatrix}
\]
はすべての \( \lambda \in \mathbb{C} \) に対して列フルランクをもつことが従う。これは \( (\tilde{C}_1, \underline{A}_1) \) は可観測であることを意味する [14].
定理 1 より，行列 \( \overline{A}_1 - \overline{B}_1 F_1 \) が安定になるように行列 \( F_1 \) を選ぶことができ，行列 \( \overline{A}_1 - G_1 \hat{C}_1 \) が安定になるように行列 \( G_1 \) を選ぶことができる [14]。つぎの制御則を考えよう。

\[
\begin{align*}
\frac{dw_1(t)}{dt} &= (\overline{A}_1 - G_1 \hat{C}_1)w_1(t) + G_1 y(t) + \overline{B}_1 u(t), \quad w_1(0) = w_{10} \\
\hat{y}_1(t) &= \overline{C}_1 w_1(t)
\end{align*}
\]

制御則 (24) は有限次元モデル (23) に対して安定化コントローラとして機能するが，元のシステム (20) に対しては閉ループ系の安定性が保証されない。そこで，RMF

\[
\begin{align*}
\frac{dw_2(t)}{dt} &= \overline{A}_2 w_2(t) + \overline{B}_2 u(t), \quad w_2(0) = w_{20} \\
\hat{y}_2(t) &= \hat{C}_2 w_2(t)
\end{align*}
\]

を制御則 (24) に付け加える。そのとき，コントローラ全体は以下のようになる。

\[
\begin{align*}
\frac{dw_2(t)}{dt} &= \overline{A}_2 w_2(t) + \overline{B}_2 u(t), \quad w_2(0) = w_{20} \\
\hat{y}_2(t) &= \hat{C}_2 w_2(t) \\
\frac{dw_1(t)}{dt} &= (\overline{A}_1 - G_1 \hat{C}_1)w_1(t) + G_1 (y(t) - \hat{y}_2(t)) + \overline{B}_1 u(t), \quad w_1(0) = w_{10} \\
\hat{y}_1(t) &= \overline{C}_1 w_1(t)
\end{align*}
\]

【定理 2】 ある与えられた正数 \( \kappa \) に対して，整数 \( l (l \geq 0) \) が \( -\lambda_{l+1} + a_1 < -\kappa \) となるように選ばれているとする。さらに，別の整数 \( n \) が \( n > l \) となるように選ばれているとする。そのとき，(25) と (26) から構成される制御則は，整数 \( n \) が十分大きいときに，システム (20) に対する有限次元安定化コントローラになる。加えて，その閉ループ系を記述する \( C_0 \) 半群の減衰率は，\( n \to \infty \) のとき -\( \kappa \) に近づく。

(証明) 境界制御・分布観測を伴うシステムを扱った文献 [11] では，出力作用素の非有界性の度合いは \( 1/4 < \gamma < 1/2 \) であった。今回のように，出力作用素の非有界性の度合いが \( 1/2 < \gamma < 1 \) に増しても定理の証明は全く同様である。□

5 まとめ

本研究では，二つの線形移流拡散方程式によって記述される境界制御・境界観測を伴う移流拡散系の，有限次元コントローラによる安定化問題を取り上げた。はじめに，観測方程式が \( A_1 \) の分数幕を用いて表現できるものを示した。特に境界入力系を分布入力系に変換した後の，最終的な観測方程式の非有界性の度合いが \( 1/2 < \gamma < 1 \) となったことに注意されたい。つぎに，有限次元モデルが可制御かつ可観測になることを示した。したがって文献 [11] と同様に，有限次元モデルに対して構成された有限次元コントローラと RMF からなるコントローラが，RMF の次数が十分大きいときに，境界制御・境界観測を伴う移流拡散系に対する有限次元安定化コントローラになり得る。

最後に，本研究を行う際に対議論いただいた神戸大学大学院教授中村信一先生ならびに神戸大学大学院教授南部隆夫先生に厚く御礼申し上げます。

参考文献


ROBUST CONTROL DESIGN OF PSS IN WIDE
AREA POWER SYSTEM CONSIDERING
INFORMATION RELIABILITY

Hiroyuki Ukai * Goh Toyosaki *
Yoshiki Nakachi * Surech Chand Verma **

* Nagoya Institute of Technology,
Gokiso, Showa, Nagoya, 466-8555, JAPAN
** Electric Power R&D Center, Chubu Electric Power Co.
Inc.,
20-1 Kitasekiyama Odaka, Midori, Nagoya 459-8522,
JAPAN

Abstract: In recent years, electric power system becomes larger and more complex
due to the enlargement of power flow in wide area, the deregulation of electric utilities
and the full-scale entry of Power Product Companies. Under the background
both inter-area and local oscillations are significant. In order to cope with these
problems, the advancement of the PSS (Power Stabilizing System) are expected.
In this paper, we propose the control system on basis of the robust control theory,
which can improve stability of power system in wide area. The proposed control
system is based on the "hierarchical power and information network"; that is, the
feedback control system is hierarchically constructed by using information of wide
area system in correspondence with the level of reliability. In particular, we identify
the dynamic impedance model of large power system by measuring voltage and
current at the interconnection node.

Keywords: power system stabilizer, wide area power system, system
identification, PMU, reliability

1. INTRODUCTION

Power system stabilizer (PSS) has been a cost-
efficient measure to improve stability in power
systems. The conventional PSS is designed on
the basis of classical control theory for the
linearized model around a certain operating con-
dition. Therefore, it is adequate only for a nar-
row range around a design point. Since a wide
area power system is highly nonlinear and the
network operating condition frequently changes,
it brings discrepancies between the mathematical
linearized model and the real nonlinear system.
Moreover it cannot compensate for the multi-
modes of power oscillation in wide area power sys-
tem. Under these backgrounds many robust con-
trol techniques have been applied with increasing
demands for high quality electricity supply(Qihua
However, most of these robust controllers are de-
signed on basis of a single machine and a infinite
bus model. Therefore, even if it is robustly de-
signed for uncertainties of oscillation modes, it is
difficult to compensate for both the local and the
inter-area modes of power oscillation. Moreover, when the operating condition undergoes large variations, it is often impossible to achieve high performance over an entire operating range.

On the other hand, in recent years, phasor measurement unit (PMU) in power system by using GPS (Global Positioning System) was introduced and applied to a static state estimation, various protections, and so on (R.O. Burnett et al., 1994) (M. Hojo et al., 2003). One of the authors is carrying on a research project among universities in Japan for developing an online global monitoring system for power system dynamics by using the PMU. In the near future, it is expected applying the PMU based technologies to various fields in power system. In stabilizing control of power system, power information which are measured or estimated at multi-points by using the PMU will be aggressively utilized.

In that case, however, it is important to consider the reliability of various power data measured in a wide area power system. In other words, it is necessary to properly use several information data in a wide area according the level of reliability. Reliability of information used in this paper implies two meanings. One is the measure of degradation of power data owing to the lack of data in acquisition systems, time delay of transmitted data, transmission noise, and so on. The other is parameter uncertainties of mathematical model used for designing controllers. The power information of the generator to be controlled is highly reliable, highly frequent, and has large data numbers in which generator speed, active power, terminal voltage, and so on are included. Moreover almost parameters of the generator are able to be known in advance. The reliability, the frequency, and data number of power information in the local area belong to the medium level, and the parameters of generators, network impedance, and load are somewhat uncertain. Moreover, in case that there are many generators and load in a local area, they should be aggregated. Therefore the local area power system is forced to include parameter uncertainty. On the other hand, the power information of a large power system or another power system area connected to the interconnection node is poor, that is, at most only information at the interconnection node can be useful. Moreover, its reliability and frequency belong to the low level. As the result, the feedback data to be directly used for the PSS inputs are divided to three categories according to the information reliability and acquisition frequency. Moreover, the model parameters to be used to design the controller are categorized to three levels.

2. HIERARCHICAL FEEDBACK CONTROL SYSTEM

In this section, the concept of the proposed control design method based on the hierarchical feedback control system is explained. The configuration of the system is shown in Fig. 1. The basic idea is to construct feedback control loops according to reliability of information fed back from multi-points in a wide area power system. As mentioned before, the reliability of information used in this paper implies two meanings. One is the measure of degradation of power data owing to the lack of data in acquisition systems, time delay of transmitted data, transmission noise, and so on. The other is parameter uncertainties of mathematical model used for designing controllers. The power information of the generator to be controlled is highly reliable, highly frequent, and has large data numbers in which generator speed, active power, terminal voltage, and so on are included.

Moreover almost parameters of the generator are able to be known in advance. The reliability, the frequency, and data number of power information in the local area belong to the medium level, and the parameters of generators, network impedance, and load are somewhat uncertain. Moreover, in case that there are many generators and load in a local area, they should be aggregated. Therefore the local area power system is forced to include parameter uncertainty. On the other hand, the power information of a large power system or another power system area connected to the interconnection node is poor, that is, at most only information at the interconnection node can be useful. Moreover, its reliability and frequency belong to the low level. As the result, the feedback data to be directly used for the PSS inputs are divided to three categories according to the information reliability and acquisition frequency. Moreover, the model parameters to be used to design the controller are categorized to three levels.

3. CONTROL DESIGN METHOD

In this section, the control model is constructed based on the idea of the hierarchical feedback control system. Moreover, the \( H^\infty \) controller to stabilize the total power system is designed based on the aggregated control model.
3.1 System model for control design

First of all, let $G_1$ be a generator to be controlled, and let $G_2 \sim G_k$ be generators belonging a local area power system. It may be assumed that the generator group is aggregated in a local area power system. Then state space model for $k$ generators is defined by

$$\dot{x}_L = A_L x_L + B_L \Delta i_L + B_L u_{ex}, \quad (1)$$

where

$$x_L = [x_{L1}, x_{L2}, \ldots, x_{Lk}],$$

$$A_L = diag(A_{L1}, A_{L2}, \ldots, A_{Lk}),$$

$$B_L1 = diag(B_{L11}, B_{L12}, \ldots, B_{L1k}),$$

$$B_L2 = diag(B_{L21}, 0, \ldots, 0),$$

$$\Delta i_L = [\Delta i_{d1}, \Delta i_{q1}, \ldots, \Delta i_{dk}, \Delta i_{qk}],$$

where, $x_{Li}$ is a state variable for each generator, $u_{ex}$ is a control input from PSS, and $\Delta i_{di}, \Delta i_{qi}$ are $d-q$ axis terminal currents of generator $i$. The percise model of the generator to be controled is derived by linearizing the following nonlinear model.

$$\dot{\delta} = \omega_B (\omega - 1) \quad (2)$$

$$\dot{\omega} = \frac{1}{2H} [T_m - T_e - D(\omega - 1)] \quad (3)$$

$$T_e = e'_q i_d + (x_d - x'_d) i_d i_q \quad (4)$$

$$e'_q = \frac{1}{T_d} [E_{FD} - e'_q - (x_d - x'_d) i_d] \quad (5)$$

$$v_d = x_q i_q \quad (6)$$

$$v_q = e'_q - x'_d i_d \quad (7)$$

with the governor and AVR as shown in Fig.2 and Fig.3. On the other hand, for other generators in local power system the simplified models are used, where the same model of generator is used and the governor and the AVR are reduced to first order models.

These generator models are not connected to power network. In order to embedding the network impedance and load in the local power system to the above system models, it is necessary to give the impedance characteristics of both the local power system and the large power system connected to the local power system. The local power system is constructed by appropriately aggregated generaors and loads. The local oscillation modes are represented in this way. On the other hand, the large power system includes many generators and loads. As mentioned above, however, the impedance characteristics of the large power system is uncertain. Thus it is necessary to model the inter-area oscillation mode between the generators in the local power system and those in the other large power system. In this paper, it is proposed that the network impedance of the large power system is presented as the dynamical model by applying the system identification method. Therefore, the effects from other power systems connected to the considered local power system can be embedded in the control design model.

Let the small deviations of node voltage and current at the interconnection point be $\Delta v_R, \Delta i_R$, and the deviation of the frequency at this node be $\omega_R$. If there are several nodes connected to the considered local power system, the identification is done at each connection node. Then the dy-
namical model for each connected power system is identified as follows:

$$\dot{x}_R = A_R x_R + B_R \Delta v_R,$$  \hspace{1cm} (8)

and

$$\begin{bmatrix} \Delta i_R \\ \Delta \omega_R \end{bmatrix} = \begin{bmatrix} C_R & 0 \\ \omega & D_R \end{bmatrix} x_R + \begin{bmatrix} D_R \\ 0 \end{bmatrix} \Delta v_R.$$  \hspace{1cm} (9)

where, the input signal for the identification is \( \Delta v_R \) and the output signals the \( \Delta i_R \) and \( \Delta \omega_R \) at the interconnected node, respectively. In this manner, the deviation of the terminal currents and voltages at generator nodes are expressed by network impedances as follows;

$$I = L \dot{i} + R i,$$  \hspace{1cm} (10)

where, \( I \), \( v \), and \( i \) represent the network impedance, the voltages, and the currents at generator nodes, respectively.

Now we embed the network equation to the system model including the inter-area oscillation problem. In this section, the numerical simulation results are shown by applying the proposed method to the IEEE West10-machines system in Fig.4, which consists of 10 machines and 30 nodes. It is generally known that the longitudinal structure produces the typical long term oscillation and local power system, respectively. In the similar manner, the deviation of the terminal voltage \( V_L \) are expressed by

\[ \Delta V_L = F_3 x_L + F_4 \Delta I_L \]  \hspace{1cm} (12)

Substituting Eqs.(9) and (10) to Eq.(12), we have

\[ \Delta V_L = K_L x_L + K_R x_R \]  \hspace{1cm} (13)

Then the following relation is finally obtained.

$$\begin{bmatrix} \Delta i_L \\ \Delta v_L \end{bmatrix} = \begin{bmatrix} F_1 + F_2 K_L & F_2 K_R \\ K_L & K_R \end{bmatrix} \begin{bmatrix} x_L \\ x_R \end{bmatrix}.$$  \hspace{1cm} (14)

As the result, by substituting the above relation to (1) and (8), the control design is formulated by the generator power outputs in local power system.

On the other hand, the measured output is defined by

\[ y = [\Delta \omega_1, \Delta \omega_2, \cdots, \Delta \omega_k, \Delta \omega_R] \]  \hspace{1cm} (16)

Each output is fed back to PSS according to their information levels. In this paper, the output to be controlled is defined by the generator power outputs in local power system.

\[ z = [\Delta P_{e1}, \Delta P_{e2}, \cdots, \Delta P_{ek}] \]  \hspace{1cm} (17)

3.2 Control design

The controller is designed based on this mathematical model of the total system by applying the \( H^\infty \) control theory. In this paper, the control problem is formulated by the robust stabilization problem of \( H^\infty \) control. The weight functions are designed by considering the model errors due to the aggregation of generators in local power system, the identification error, and the data error of frequency at each measuring point. The solution to this problem is obtained by using MATLAB “Robust control toolbox”.

4. NUMERICAL SIMULATION RESULTS

In this section, the numerical simulation results are shown by applying the proposed method to the IEEE West10-machines system in Fig.4, which consists of 10 machines and 30 nodes. It is generally known that the longitudinal structure produces the typical long term oscillation and local
generator oscillations. It is assumed that the generator to be controlled is G1, and generators in the local area power system is aggregated to the generator G2. The interconnection node is the tie line between G2 and G3.

First of all, the identification result is shown in Fig.5. The input and output data are used in case that the three phase-to-ground fault occurs at the transmission line near the generator 9. It is found that the estimated frequency well coincides with the measured one. The order of the identified system model is five. In order to confirm the effectiveness of the proposed method, the nonlinear simulation is done when the three phase-to-ground fault occurs at the transmission line near the generator 1 as shown in Fig.5. The simulation is achieved under the conditions;

1. superimpose the random noise of maximum ±10% to the frequencies of the generator G2 and the interconnection node,
2. assume the time delay 80[msec] for the transmitted data,
3. and thin the data acquisition of the frequencies of G2 and the interconnection node compared with the one of generator G1.

The bode diagrams from the control input to each generator active power are shown in Fig. 6. It is found that each transfer function has resonance modes at 0.4Hz and 1Hz. The former corresponds to the inter-area oscillation mode, and the latter local mode. The controller is designed to suppress these resonance modes against modeling errors. The transient responses of the active power and the frequency of each generator are shown in Fig.7 and Fig.8. In both figures, the solid line represents the case of proposed controller and the broken line the case that only the frequency of G1 is fed back. In the latter case, the $H^\infty$ control method is applied to a single machine and infinite bus model with respect to the generator 1. It is found that the proposed method improves the transient stability compared with the case of the conventional $H^\infty$ control case against parameter uncertainty. Moreover, the frequency at the interconnection node is shown in Fig.9. It is also the oscillation of the frequency at the interconnection node is well suppressed.
5. CONCLUSIONS

In this paper, the design method of the advanced PSS is proposed to compensate for multiple oscillation modes in a wide area power system. The main ideas and results of the proposed method are:

(1) the feedback controller is designed by taking account of information reliability in power system.

(2) To do this, the control sign model is derived in correspondence with three levels of information reliability. The first is exact model of the controlled generator, the second is reduced model of the local generators, and third is the estimation model of the large power system connected to the considered local power system.

(3) The dynamical impedance model of a large power system connected to the considered local power systems is obtained by using the system identification method. The 4SID method is applied to obtain the dynamical impedance model. This method can be realized based on the PMU system.

(4) The simulation results show the good performance of the proposed method compared with the conventional $H_\infty$ applied to control design model based on a single machine and infinite bus system.

REFERENCES


1. Introduction: Feedback stabilization for linear boundary control systems of parabolic type has a long history of more than three decades. Although some difficult problems are left unresolved, it seems that the study has reached a degree of maturity in a sense. The so called dynamic compensators are introduced in the feedback loop to cope with the most difficult case such as the scheme of boundary observation - boundary input (see the literature, e.g., [2, 6 – 11]). The approach based on the fractional powers of the coefficient elliptic operators is effective to systems equipped with boundary conditions of the 2nd or the 3rd kind. However, this approach has two serious defects: One is that it is no more applied to systems with Dirichlet boundaries. The other is that it is generally implausible to obtain complete informations on the fractional powers of the elliptic operators with more complicated boundary conditions. An alternative algebraic approach is much superior to the former one in the sense that no Riesz basis is assumed, corresponding to the coefficient elliptic operators equipped with complicated boundary operators (see [8 – 10]).

It remains one of the main interests of control theory to determine the minimum numbers of the parameters such as the sensors, the actuators, and the dimension of dynamic compensators in the feedback control scheme. We raise in this paper the problem of seeking the minimum number, say \( N \), of the sensors necessary for stabilization. The problem is closely related to the finite-dimensional substructure of the elliptic operator. We also discuss the same problem on the actuators in the end of the paper.

To clarify the issue, let us begin with the formulation of the boundary control system. Let \( \Omega \) denote a bounded domain in \( \mathbb{R}^m \) with the boundary \( \Gamma \) which consists of a finite number of smooth components of \((m - 1)\)-dimension. Our control plant \( \Sigma_\rho \) is characterized by the pair of linear differential operators \((\mathcal{L}, \tau)\) described by

\[
\mathcal{L}u = - \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + \sum_{i=1}^{m} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u, \\
\tau u = \alpha(\xi)u + (1 - \alpha(\xi)) \frac{\partial u}{\partial \nu}.
\]

In \((\mathcal{L}, \tau)\), the coefficients in \( \mathcal{L} \) and on \( \Gamma \) satisfy the conditions:

(i) \( a_{ij}(x) = a_{ji}(x), \) for \( x \in \mathcal{O}, \ 1 \leq i, j \leq m; \) \( \{a_{ij}(x)\}, \ x \in \mathcal{O} \) is positive-definite;
(ii) \( 0 \leq \alpha(\xi) \leq 1; \) and
(iii) \( \frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{m} a_{ij}(\xi) \nu_i(\xi) \frac{\partial u}{\partial x_j}|_{\Gamma} \) with \( \nu(\xi) = (\nu_1(\xi), \ldots, \nu_m(\xi)) \) being the outward normal at \( \xi \in \Gamma \).

Necessary regularity on these coefficients is assumed tacitly. Our control system is described by the following system of differential equations:

\[
\begin{aligned}
\frac{\partial u}{\partial t} + \mathcal{L}u &= 0 \quad \text{in } \mathbb{R}_+^1 \times \Omega, \\
\tau u &= \sum_{k=1}^{M} (\nu, \rho_k) \delta_t h_k \quad \text{on } \mathbb{R}_+^1 \times \Gamma, \\
\frac{dv}{dt} + B_1v &= \sum_{k=1}^{N} p_k(u)\xi_k \quad \text{in } \mathbb{R}_+^1, \\
u(0, \cdot) &= \varphi(\cdot) \quad \text{in } \Omega, \quad v(0) = v_0.
\end{aligned}
\]

In (1), the controlled plant \( \Sigma_\rho \) has state \( u = u(t, \cdot) \). The compensator \( \Sigma_\psi \) with state \( v = v(t) \) is described by the differential equation in the Euclidean space \( \mathbb{R}^L \). The compensator will be finally reduced from the differential equation with the coefficient operator \( B \) in a separable Hilbert space \( H \) (see eqn. (10) below).

---

\(^1\)“Workshop on Mathematical Control Theory in Kobe,” Kobe University, January 8 – 10, '10
The fundamental structure of the $\Sigma_p$ consists of a set of the outputs $p_k(u)$, $1 \leq k \leq N$, and a set of the inputs $f_k(t) = \langle v(t), p_k \rangle_{\mathbb{R}^l}$ in our case which enter the differential equation through the actuators $h_k(\xi)$ on the boundary $\Gamma'$. Let $w_k$ be in $L^2(\Gamma)$, $1 \leq k \leq N$. The outputs $p_k(u)$ are characterized by

$$p_k(u) = \begin{cases} \langle u, w_k \rangle, & \text{if } \alpha(\xi) \neq 1, \\ \langle \frac{\partial u}{\partial \nu}, w_k \rangle, & \text{if } \alpha(\xi) = 1. \end{cases}$$

(2)

We assume that the actuators $h_k(\xi)$, $1 \leq k \leq M$ belong to $C^{2+\omega}(\Gamma')$, $0 < \omega < 1$. The outputs of the $\Sigma_e$ are $\langle v, p_k \rangle_{\mathbb{R}^l}$, $1 \leq k \leq M$.

The stabilization problem for the system (1) is stated as follows: Given a set of the actuators $h_k(\xi)$ and the sensors $w_k(\xi)$, determine the feedback parameters, i.e., the matrix $B_1$, the vectors $\xi_k$, and the $p_k$, so that the state $(u(t), v(t))$ of the system (1) decays exponentially as $t \to \infty$ for every initial state $(u_0, v_0)$. The stabilization for (1) is established in [8–10], where the number $N$ of the sensors $w_k$ must be equal to or greater than the maximum of the algebraic multiplicities of a finite number of the unstable eigenvalues. This gives the best possible $N$, as long as the pair $(L, \tau)$ admits the Riesz basis consisting of the eigenfunctions, or in other words, the finite-dimensional structure of the elliptic operator is similar to the diagonal matrix. We show in this paper that the smallest $N = 1$ is possible even in the case where the algebraic multiplicities of the unstable eigenvalues are greater than 1.

2. Preliminaries: Let us begin with characterizing the operators $L$ and $B$. Let $\hat{L}$ denote the operator defined as

$$\hat{L}u = Lu,$$

$$\mathcal{D}(\hat{L}) = \{u \in C^2(\Omega) \cap C^4(\partial \Omega); \mathcal{L}u \in L^2(\Omega), \tau u = 0\}.$$

The closure of the $\hat{L}$ in $L^2(\Omega)$ is denoted as $L$. It is well known [3] that the $\hat{L}$ has a compact resolvent $(\lambda - L)^{-1}$; that the spectrum $\sigma(L)$ lies in the complement $(\Sigma - b)^c$ of some sector $\Sigma - b$, where $\Sigma = \{\lambda \in \mathbb{C}; \theta_0 \leq |\arg \lambda| \leq \pi\}$, $0 < \theta_0 < \pi/2$, $b \in \mathbb{R}^1$; and that the decay estimate

$$\|\lambda - L\|^{-1} \leq \text{const} \frac{1}{1 + |\lambda|}, \quad \lambda \in \Sigma - b$$

holds, where the norm $\||\|$ denote the $L^2(\Omega)$- or the $L(L^2(\Omega); L^2(\Omega))$-norm. Thus, $-L$ is the infinitesimal generator of an analytic semigroup $e^{-tL}$, $t > 0$.

There is a set of generalized eigenpairs $\{(\lambda_i, \varphi_{ij}); i \geq 1, 1 \leq j \leq m_i(< \infty)\}$ such that

(i) $\sigma(L) = \{\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots\}$;

(ii) $\lambda_i \neq \lambda_j$, $i \neq j$;

(iii) $L\varphi_{ij} = \lambda_i \varphi_{ij} + \sum_{k<j} \alpha_{ij}^k \varphi_{ik}$, $i \geq 1, 1 \leq j \leq m_i(< \infty)$.

The set $\{\varphi_{ij}\}$ spans $L^2(\Omega)$ [1], but does not necessarily form a Riesz basis. The eigenvalue $\lambda_i$ is the pole of $(\lambda - L)^{-1}$ of order $i$ ($m_i$).

Let $(\mathcal{L}^*, \tau^*)$ be the formal adjoint of $(\mathcal{L}, \tau)$:

$$\mathcal{L}^* \varphi = - \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial \varphi}{\partial x_j}) - \text{div}(b(x) \varphi) + c(x) \varphi,$$

$$\tau^* \varphi = \alpha(\xi) \varphi + (1 - \alpha(\xi)) \left( \frac{\partial \varphi}{\partial \nu} + (b(\xi) \cdot \nu(\xi)) \varphi \right),$$

(4)

where $b(x) = (b_1(x), \ldots, b_m(x))$. The pair $(\mathcal{L}^*, \tau^*)$ defines the closable operator $\hat{L}^*$ like the $\hat{L}$. The adjoint of $L$, denoted by $L^*$, is given as the closure of $\hat{L}^*$ in $L^2(\Omega)$. There is a set of generalized eigenpairs $\{(\lambda_i, \psi_{ij}); i \geq 1, 1 \leq j \leq m_i\}$ such that

(i) $\sigma(L^*) = \{\lambda_i\}_{i \geq 1}$;

(ii) $L^* \psi_{ij} = \sum \psi_{ij} + \sum_{k<j} \beta_{ij}^{jk} \psi_{ik}$.

Let $P_\lambda$ be the projector for the eigenvalue $\lambda_i$ of $L$. Then the adjoint $P_\lambda^*$ is the projector for the eigenvalue $\lambda_i$ of $L^*$. Setting $P_\lambda u = \sum_{i=1}^{m_i} u_{ij} \varphi_{ij}$, we have the relationship:

$$u_{ij}; \quad (j \downarrow 1, \ldots, m_i)$$

$$= \Pi_\lambda^{-1} \langle u, \psi_{ij}; j \downarrow 1, \ldots, m_i \rangle,$$

(5)

where $\Pi_\lambda$ means the $m_i \times m_i$ matrix consisting of the elements, $\langle \varphi_{ij}, \psi_{il} \rangle$, $j \downarrow 1, \ldots, m_i$ and $l \downarrow 1, \ldots, m_i$. The characterization of the domain $\mathcal{D}(L)$ is not so clear in terms of the Sobolev spaces as in the cases of the standard 1st, the 2nd, and the 3rd boundaries. However, it is well known [3] that, for a
Thus we see that the boundary value problem:

\[(c + \mathcal{L})u = f \quad \text{in } \Omega, \quad \tau u = 0 \quad \text{on } \Gamma \]  

admits a unique solution \( u \in \mathcal{D}(\mathcal{L}) \). The same result holds for the \( L^* \). When \( c > 0 \) is chosen large enough, the fractional powers \( L_c^\alpha \) of the \( L_c = L + c \) is well defined. In our problem, the boundary is more complicated so that the boundary of the 1st kind is continuously connected with the boundary of the 3rd kind. The property of the fractional powers thus looks less clearer. However, we have at least the relation: \( \mathcal{D}(L^{\alpha/2}) \subset H^{-\alpha}(\Omega), 0 \leq \alpha \leq 1 \).

Let us turn to the operator \( B \). Let \( H \) be a separable Hilbert space equipped with the inner product: \( \langle \cdot, \cdot \rangle_H \), and choose an orthonormal basis for \( H \). We relabel the basis as \( \{ \eta^\pm_i; i \geq 1, 1 \leq j \leq n_i (\langle \infty) \}. \) Every vector \( v \in H \) is expressed in terms of the basis \( \{ \eta^\pm_i \} \) as \( v = \sum_{i,j} v^\pm_{ij} \eta^\pm_i \). Let \( \{ \mu_i \} \) be a sequence of increasing positive numbers: \( 0 < \mu_1 < \mu_2 < \cdots < \infty \), and define \( B \) as

\[ Bv = \sum_{i,j} \mu_i |v^\pm_{ij}|^2 \eta^\pm_i, \quad \mathcal{D}(B) = \{ v \in H; \sum_{i,j} |v^\pm_{ij}|^2 < \infty \}, \]

\[ \omega^\pm = a \pm i \sqrt{1 - a^2}, \quad 0 < a < 1. \]

It is easily seen that \( B \) is a closed operator with dense domain \( \mathcal{D}(B) = \{ v \in H; \sum_{i,j} |v^\pm_{ij}|^2 < \infty \}. \) In addition,

(i) \( \sigma(B) = \{ \mu_i \omega^\pm; i \geq 1 \}; \) and

(ii) \( \mu_i \omega^\pm - B \eta^\pm_i = 0, \quad i \geq 1, \quad 1 \leq j \leq n_i. \)

Thus we see that \( -B \) is the generator of an analytic semigroup \( e^{-tB}, \ t > 0 \) with the decay estimate:

\[ \| e^{-tB} \|_H \leq e^{-\alpha \omega t}, \ t \geq 0. \]

We have characterized the operators \( L \) and \( B \). Since the actuator \( h_k \) belongs to \( C^{2+\omega}(\Gamma) \), the boundary value problem:

\[(c + \mathcal{L})u = f \quad \text{in } \Omega, \quad \tau u = h_k \quad \text{on } \Gamma \]  

admits a unique solution \( u = N_{-\alpha} h_k \in C^2(\Omega) \cap C^1(\overline{\Omega}) \).

Let us choose the vectors \( \rho_k, 1 \leq k \leq M \) in \( \mathcal{D}(B^*) \). Our boundary control system (1) will be derived from the following system which is well posed in \( L^2(\Omega) \times H \):

\[
\begin{cases}
\frac{\partial u}{\partial t} + \mathcal{L}u = 0 & \text{in } \mathbb{R}_+^1 \times \Omega, \\
\tau u = \sum_{k=1}^M \langle v, \rho_k \rangle_H h_k & \text{on } \mathbb{R}_+^1 \times \Gamma, \\
\frac{dv}{dt} + Bv = \sum_{k=1}^N \langle v, \mu_k \rangle_H \zeta_k & \text{in } \mathbb{R}_+^1 \times H, \\
u(0, \cdot) = \mu_0(\cdot) \in L^2(\Omega), \quad v(0) = v_0 \in H.
\end{cases}
\]

The semigroup generated by eqn. (10) is analytic in \( t > 0 \), and every \( u(t, \cdot) \) belongs to \( C^2(\Omega) \cap C^1(\overline{\Omega}) \), \( t > 0 \) [9].

3. Main result: Let \( K \geq 1 \) be the integer such that \( \Re \lambda_k < 0 < \Re \lambda_{K+1} \), and set \( P_k = P_{\lambda_1} \cdots \cdots P_{\lambda_K} \). The restriction of the \( L \) onto the invariant subspace \( P_k L^2(\Omega) \) is, according to the basis \( \{ \varphi_{i,j} \}_{1 \leq j \leq n_i} \), equivalent to the \( m_i \times m_i \) upper triangular matrix \( A_i = \lambda_i + N_i \), where

\[ A_i|_{(j,k)} = \begin{cases} \lambda_i & \text{for } k = j; \\
\alpha_{i,j}^{k+1} & \text{for } k > j, \end{cases} \]

and \( N_i \) is nilpotent. The minimum integer \( n \geq 1 \) such that \( \ker N_i^n = \ker N_i^{n+1} \) is called the ascent of the \( \lambda_i - L \), and denoted as \( l_i (\leq m_i) \) [12]. It is well known that the ascent \( l_i \) coincides with the order of the pole \( \lambda_i \) of the resolvent \( \lambda - L \)^{-1} [12, Theorem 5.8-A]. The restriction of the \( L \) onto the \( P_k L^2(\Omega) \) is thus equivalent to the upper triangular matrix \( A = \text{diag} (A_1 A_2 \cdots A_K) \).

*Let \( R \) be a non-unique operator of prolongation from \( \Gamma \) to \( \overline{\Omega} \) such that \( Rh_k \in C^{2+\omega}(\overline{\Omega}) \), \( Rh_k|_\Gamma = \partial (R h_k)/\partial n|_\Gamma = h_k \). Then the solution is expressed as \( u = Rh_k - L_c^{-1}(c + \mathcal{L})Rh_k \).
Setting $P_{\lambda_i}N_{-c}h_k = \sum_{j=1}^{m_i} \zeta_{ij}^k\varphi_{ij}$, we calculate via Green’s formula - that
\[
\begin{align*}
\langle \zeta_{ij}^k; j \downarrow 1, \ldots , m_i \rangle \\
= R^{-1}_{\lambda_i} \left((N_{-c}h_k, \psi_j); j \downarrow 1, \ldots , m_i \right) \\
= R_i \left((h_k, \sigma \psi_{ij})_R; j \downarrow 1, \ldots , m_i \right).
\end{align*}
\]
Here $R_i$ denotes the $m_i \times m_i$ non-singular matrix, and $\sigma$ the boundary operator defined as
\[
\sigma \psi_{ij} = \left(1 - b(\xi) \cdot u(\xi)\right) \psi_{ij} - \frac{\partial \psi_{ij}}{\partial \nu}.
\]
Let $H$ be the controllability matrix defined as
\[
H = \left((h_k, \sigma \psi_{ij})_R; k \rightarrow 1, \ldots , M, (i, j) \downarrow (1, 1), \ldots , (K, M, K)\right).
\]
The set of the functions $P_K N_{-c}h_1 P_K N_{-c}h_2 \ldots P_K N_{-c}h_M$ is equivalent to the matrix
\[
Z = (Z_1 Z_2 \ldots Z_M) = RH,
\]
where $R = \text{diag}(R_1 R_2 \ldots R_K)$. The controllability condition for the pair $(A, Z)$:
\[
\text{rank}(ZAZ \ldots A^{S-1}Z) = S, \quad S = m_1 + \cdots + m_K
\]
is equivalent to the controllability condition for the pair $(R^{-1}AR, H)$.$^1$

As for the sensors $w_k$, we define the $N \times m_i$ matrices $W_i$ by
\[
W_i = \left(p_k(\varphi_{ij}); j \rightarrow 1, \ldots , N, i \uparrow 1, \ldots , m_i \right), \quad i \geq 1.
\]
Let us begin with the stabilization for eqn. (10). Suppose that the pair $(R^{-1}AR, H)$ is controllable, and choose an arbitrary $r$, $0 < r < \text{Re} \lambda_{K+1}$. Suppose further that
\[
\text{rank} \left(W_i A_i^k; k \uparrow 0, \ldots , m_i - 1\right) = m_i, \quad 1 \leq i \leq K,
\]
where $\Xi_i$ means the complex conjugate of $\xi_{ij}^k$. The reason for this setting is to design the dynamic compensator $\Sigma_i$ in eqn. (1) (or (34) below) as the system of differential equations with real coefficients. Furthermore, we suppose that, for $0 < \epsilon < 1/4$,
\[
\sum_{i,j} |\xi_{ij}^k \mu_i|^{3/4+\epsilon} < \infty, \quad \text{if} \alpha(\xi) \neq 1,
\]
\[
\sum_{i,j} |\xi_{ij}^k \mu_i|^{3/4+\epsilon} < \infty, \quad \text{if} \alpha(\xi) = 1.
\]

With these preparations, our main result is stated as follows:

**Theorem 1.** (i) Suppose that the pair $(R^{-1}AR, H)$ is controllable, and choose an arbitrary $r$, $0 < r < \text{Re} \lambda_{K+1}$. Suppose further that
\[
\text{rank} \left(W_i A_i^k; k \uparrow 0, \ldots , m_i - 1\right) = m_i, \quad 1 \leq i \leq K,
\]
\[
\text{rank} \Xi_i = N, \quad i \geq 1. \quad (15)
\]
Then, for $r < \forall r_1 < \text{Re} \lambda_{K+1}$, we find the vectors $\zeta_k$ and $p_k$ such that the decay estimate:
\[
\|u(t, \cdot)\| + \|v(t)\|_H \leq \text{const} e^{-r t} \left\{\|u_0\| + \|v_0\|_H\right\}, \quad t \geq 0. \quad (16)
\]

(ii) The control system (1) is derived from eqn. (10) by suitable choice of the integer $l$, and is well posed in $L^2(\Omega) \times \mathbb{R}^l$. Every solution $(u(t, \cdot), v(t))$ to eqn. (1) satisfies the decay estimate:
\[
\|u(t, \cdot)\| + \|v(t)\| \leq \text{const} e^{-r t} \left\{\|u_0\| + \|v_0\|\right\}, \quad t \geq 0. \quad (17)
\]

**Remark.** The novelty of Theorem 1 consists in the condition (15) on the $w_k$. Since $N_i = m_i$, $j < k$, the best choice: $N = 1$ is possible when $\alpha_{ij}^{(j+1)} \neq 0$, $1 \leq j < m_i$, $1 \leq i \leq K$. In fact, if $p_1(\varphi_{ii}) \neq 0$, $1 \leq i \leq K$, the above rank condition is fulfilled. In the case where $N_i = 0$, $1 \leq i \leq K$, however, the condition means that rank $W_i = m_i$, $1 \leq i \leq K$, which requires that $N$ be equal to or greater than $\max(m_1, \ldots , m_K)$: In our previous papers [6, 8 – 10], the condition posed on the $w_k$ is that rank $W_i = m_i$, $1 \leq i \leq K$ in general situations. Thus our present result guarantees the possibility of reducing the number $N$ in the case where the operator $L$ really admits the generalized eigenfunctions.

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$^1$The more concrete expression of the condition is obtained. See the Remark in the end of the paper.
Sketch of the proof of Theorem 1. Consider the operator equation (the so called Silvester equation) on $\mathcal{D}(L)$:

$$XL - BX = C, \quad C = -\sum_{k=1}^{N} p_k(\cdot)\xi_k. \quad (18)$$

**Proposition 2** [9]. (i) The operator equation (18) admits a unique solution $X \in \mathcal{L}(L^2(\Omega); H)$. The solution $X$ is expressed as

$$Xu = \sum_{i,j} \sum_{k=1}^{N} \left( f_k(\mu_i\omega^+; u)\xi_j^k\eta_{ij}^k + f_k(\mu_i\omega^-; u)\xi_j^k\eta_{ij}^k \right), \quad (19)$$

where $f_k(\lambda; u) = p_k((\lambda - L)^{-1}u)$.

(ii) In addition, we have the relations: $XL^2(\Omega) \subset \mathcal{D}(B)$ and $X^*H \subset \mathcal{D}(L_{\epsilon}^\alpha)$, $0 \leq \alpha < \frac{3}{4}$.

The following Theorem 3 forms the key result in Theorem 1:

**Theorem 3.** With the assumptions (13) and (15), we have the inclusion relation:

$$P_K^eL^2(\Omega) \subset \overline{X^*H}. \quad (20)$$

The left-hand side is the subspace spanned by the $\psi_{ij}$, $1 \leq i \leq K$, $1 \leq j \leq m_i$.

Proof. We have to show the inclusion relation: $\ker X \subset \ker P_K$. The set $\{\eta_{ij}^k\}$ forms the orthonormal basis for $H$. In view of (19), we see that

$$\sum_{k=1}^{N} f_k(\mu_i\omega^+; u)\xi_j^k = \sum_{k=1}^{N} f_k(\mu_i\omega^-; u)\xi_j^k = 0, \quad \forall \lambda \in \rho(L). \quad (21)$$

Here note the obvious relation:

$$(\lambda - L)^{-1} = -L_{\epsilon}^{-1} + (\lambda + c)(\lambda - L)^{-1}L_{\epsilon}^{-1}. \quad (\lambda - L)^{-1} = -L_{\epsilon}^{-1} + (\lambda + c)(\lambda - L)^{-1}L_{\epsilon}^{-1}.$$

For each $k$, $1 \leq k \leq N$, let us define the sequence of meromorphic functions $f_k^l(\lambda; u)$, $l = 0, 1, \ldots$ as

$$f_k^l(\lambda; u) = f_k(\lambda; u), \quad f_k^{l+1}(\lambda; u) = \frac{f_k^l(\lambda; u)}{\lambda + c}, \quad l = 0, 1, \ldots. \quad (22)$$

It is easily seen that the $f_k^l(\lambda; u)$ are expressed as

$$f_k^l(\lambda; u) = \begin{cases} 
(\lambda - L)^{-1}L_{\epsilon}^{-1}u, w_k)_{\Gamma} - \sum_{i=1}^{l} \left( \frac{1}{\lambda(i + 1)} \right) L_{\epsilon}^{-1}(\lambda - L)^{-1}(\lambda + c)^{-1}(\lambda(i + 1) - \lambda)u, w_k)_{\Gamma}, 
\text{if } \alpha(\xi) \neq 1, 
\left( \frac{\partial}{\partial \nu} L_{\epsilon}^{-1}u, w_k \right)_{\Gamma}, 
\text{if } \alpha(\xi) \equiv 1. 
\end{cases} \quad (23)$$

In addition,

$$f_k^l(\mu_i\omega^+; u) = 0, \quad (24)$$

Let us begin with the $f_k^0(\lambda; u) (= f_k(\lambda; u))$. The function $f_k(\lambda; u)$ has the following properties:

(i) The function $f_k(\lambda; u)$ is meromorphic;

(ii) it has at least the zeros $\lambda = \mu_i\omega^\pm$, $i \geq 1$; and

(iii) the growth rate $\gamma$ of these zeros, $\mu_i\omega^\pm$ is smaller than 2 by our assumption (13).

Based on these properties combined with Carleman’s theorem [5, 13], we can conclude that

$$f_k(\lambda; u) = 0, \quad \lambda \in \rho(L). \quad (25)$$

See, e.g., [10] for the detailed proof. We next consider the $f_k^l(\lambda; u)$, $l \geq 1$. The difference in this case is that $-c$ is added as a possible singularity of the $f_k^l(\lambda; u)$. Thus we similarly find - via Carleman’s theorem - that

$$f_k^l(\lambda; u) = 0, \quad l \geq 1, \quad \lambda \in \rho(L). \quad (26)$$

The Laurent expansion of the resolvent $(\lambda - L)^{-1}$ in a neighborhood of the pole $\lambda_i$ is

$$(\lambda - L)^{-1} = \sum_{j=1}^{l_1} \frac{A_{-j}}{(\lambda - \lambda_i)^j} + \sum_{j=0}^{\infty} (\lambda - \lambda_i)^j A_j, \quad (\lambda - L)^{-1} = \sum_{j=1}^{l_1} \frac{A_{-j}}{(\lambda - \lambda_i)^j} + \sum_{j=0}^{\infty} (\lambda - \lambda_i)^j A_j,$$

where $A_{-1}u = P\lambda_i u = \sum_{j=1}^{m_1} u_{ij}\varphi_{ij}$. 

Thus we see that
\[ f_k(\lambda; u) = \begin{cases} \sum_{j=1}^l (A_j u, w_k)_\Gamma + \sum_{j=0}^\infty (\lambda - \lambda_j)^j (A_j u, w_k)_\Gamma, \\
\quad \text{if } \alpha(\xi) \neq 1, \\
\sum_{j=1}^l \frac{1}{(\lambda - \lambda_j)^j} \left( \frac{\partial}{\partial \nu} A_j u, w_k \right)_\Gamma \\
\quad + \sum_{j=0}^\infty (\lambda - \lambda_j)^j \left( \frac{\partial}{\partial \nu} A_j u, w_k \right)_\Gamma, \\
\quad \text{if } \alpha(\xi) \equiv 1. 
\end{cases} \]

Calculating the residue of the \( f_k(\lambda; u) \) at each pole \( \lambda_i \), we obtain the relation:
\[ \sum_{j=1}^{m_i} (\bar{\varphi}_{ij}, w_k)_\Gamma \bar{u}_{ij} = 0, \quad \text{if } \alpha(\xi) \neq 1, \]
\[ \sum_{j=1}^{m_i} \left( \frac{\partial \bar{\varphi}_{ij}}{\partial \nu}, w_k \right)_\Gamma \bar{u}_{ij} = 0, \quad \text{if } \alpha(\xi) \equiv 1. \]

In other words, setting \( u_i = \{u_{i1}, u_{i2}, \ldots, u_{imi}\} \), we obtain the relation:
\[ W_i u_i = 0, \quad i \geq 1. \quad (25) \]

The function \( f_k^l(\lambda; u) \), \( l \geq 1 \) has the expression in a neighborhood of the pole \( \lambda_i \):
\[ \sum_{j=1}^l \frac{1}{(\lambda - \lambda_j)^j} \left( \frac{\partial}{\partial \nu} A_j u, w_k \right)_\Gamma + h_k^l(\lambda), \quad \text{if } \alpha(\xi) \neq 1, \]
\[ \sum_{j=1}^l (\bar{\varphi}_{ij}, w_k)_\Gamma \bar{u}_{ij} = 0, \quad \text{if } \alpha(\xi) \equiv 1. \]

Here both \( g_k^l(\lambda) \) and \( h_k^l(\lambda) \) are analytic at \( \lambda_i \). The restriction of the operator \( L_c^{-1} \) onto the subspace \( P_cL^2(\Omega) \) is equivalent to the matrix \( (A_i + c)^{-l} \). Thus calculating the residue of the \( f_k^l(\lambda; u) \) at each pole \( \lambda_i \), we similarly obtain the relation:
\[ (p_k(\varphi_{i1}) p_k(\varphi_{i2}) \ldots p_k(\varphi_{imi}))(A_i + c)^{-l} u_i = 0, \]
or
\[ W_i(A_i + c)^{-l} u_i = 0, \quad i \geq 1, \quad l \geq 1. \]

Altogether we have the relation:
\[ \left( W_i(A_i + c)^{-k}; k \downarrow 0 \ldots, m_i - 1 \right) u_i = 0, \quad i \geq 1. \quad (26) \]

The rank of the matrix in (26) is apparently equal to
\[ \text{rank} \left( W_i A_i^k; k \downarrow 0 \ldots, m_i - 1 \right) \]
which is equal to \( m_i \) by (15). Thus we see that \( u_i = 0, 1 \leq i \leq K \), or \( P_K \neq 0 \). The proof of Theorem 3 is now complete.

Q.E.D.

Our boundary control system (10) has a unique solution \( (u(t, \cdot), v(t)) \in L^2(\Omega) \times H \). The state \( u(t, \cdot) \in C^2(\Omega) \cap C^1(\partial \Omega) \) of the controlled plant \( \Sigma_p \) satisfies the equation:
\[ \frac{d}{dt} u + L_c (u - \sum_{k=1}^M (v, \rho_k)_H N_{-c} h_k) = cu. \]

Applying the operator \( X \) on both sides, we have
\[ \frac{d}{dt} X u + (B_c X + C) u = \sum_{k=1}^M (v, \rho_k)_H (B_c X + C) N_{-c} h_k + c X u. \]

Here, \( B_c = B + c \), and we have used the fact that \( X u \in D(B) \) (see Proposition 2, (iii)). By designing the actuators \( c_k \) of the compensator \( \Sigma_c \) as
\[ c_k = (B_c X + C) N_{-c} h_k, \quad 1 \leq k \leq M, \]
\( X u - v \) satisfies the equation:
\[ \frac{d}{dt} (X u - v) + B(X u - v) = 0, \quad t \geq 0, \]
or \( X u(t, \cdot) - v(t) = e^{-tB}(X u_0 - v_0) \). Thus, by the estimate (8),
\[ \|X u(t, \cdot) - v(t)\|_H \leq e^{-\alpha t}\|X u_0 - v_0\|_H, \quad t \geq 0. \quad (27) \]

**Operator \( F \).** Let \( y_k \in L^2(\Omega), 1 \leq k \leq M, \) and let \( \tau_f \) be the boundary operator containing the feedback term defined as
\[ \tau_f u = \tau u - \sum_{k=1}^M (u, y_k) h_k. \quad (28) \]

In exactly the same manner as in \( \hat{L} \), we define the operator \( \hat{F} \) as
\[ \hat{F} u = \hat{L} u, \quad D(\hat{F}) = \{ u \in C^2(\Omega) \cap C^1(\partial \Omega); \}
\[ \hat{L} u \in L^2(\Omega), \quad \tau_f u = 0 \} \quad (29) \]

The operator \( \hat{F} \) has the closure \( F \) in \( L^2(\Omega) \).
Proposition 4 [9]. (i) The domain \( \mathcal{D}(F) \) is dense. The semigroup \( e^{-tF} \) generated by \(-F\) is analytic in \( t > 0 \). If \( y_k, 1 \leq k \leq M \) belong to \( \mathcal{D}(L^2(\beta)) \), \( \beta > 0 \), then the function \( e^{-tF}u_0 \) belongs to \( C^2(\Omega) \cap C^1(\overline{\Omega}) \), \( t > 0 \) for each \( u_0 \in L^2(\Omega) \).

(ii) If the pair \((R^{-1}AR, H)\) or \((A, Z)\) is a controllable pair, we find suitable \( y_k \in P^*_KL^2(\Omega), 1 \leq k \leq M \) such that
\[
\|e^{-tF}\| \leq \text{const} \ e^{-\tau_it}, \quad t \geq 0,
\]

\[ r_1 < r_2 < \text{Re} \lambda_{K+1}. \tag{30} \]

Add a small perturbation to the above \( y_k \) in (30) to yield the perturbed \( \tilde{y}_k \). The corresponding perturbed \( F \) is then denoted as \( \tilde{F} \). The following proposition is just a classical result as long as the boundary condition is the standard one of the 1st, the 2nd, or the 3rd kind:

Proposition 5 [9]. If \( \sum_{k=1}^M \|\tilde{y}_k - y_k\| \) is small enough, we have the decay estimate
\[
\|e^{-t\tilde{F}}\| \leq \text{const} \ e^{-r_1t}, \quad t \geq 0. \tag{31} \]

Stabilization. We rewrite the equation of \( u(t, \cdot) \) in eqn. (10) as
\[
d\frac{du}{dt} + Lu = 0, \quad u(0, \cdot) = u_0, \tag{32} \]
\[
\tau u - \frac{M}{k=1} \langle u, X^{*} \rho_k \rangle_h \bar{h}_k = \sum_{k=1}^M f_k(t)h_k,
\]
where \( f_k(t) = (e^{-tB}(v_0 - Xu_0), \rho_k)_H \). Choose the \( y_k \in P^*_KL^2(\Omega), 1 \leq k \leq M \) stated in Proposition 4, (ii). In view of the inclusion relation (20) in Theorem 3, we can find suitable \( X^* \rho_k \) which arbitrarily approximate the \( y_k \) in the topology of \( L^2(\Omega) \). Since the set \( \{\eta_{ij}^t\} \) forms an orthonormal basis for \( H \), we may assume with no loss of generality that the \( \rho_k \) are expressed by a finite number of \( \eta_{ij}^t \), say, \( 1 \leq i \leq n \).

The operator \( \hat{F} \) in (28) and (29) and its closure \( \hat{F} \) with the \( y_k \) being replaced by the above \( X^* \rho_k \) are denoted, respectively as \( \hat{F} \) and \( \hat{F} \) with the same symbols. Then we have the decay estimate (31). Let \(-c \in \rho(L), c > 0\). Given a \( g \in C^2+B(\Gamma) \), the boundary value problem:
\[
(c + \mathcal{L})u = 0 \quad \text{in} \quad \Omega,
\]
\[
\tau_fu = g \quad \text{on} \quad \Gamma, \tag{33} \]
admits a unique solution \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \), which will be denoted as \( u = N_f^c g \). Set then
\[
p(t, \cdot) = u(t, \cdot) - \sum_{k=1}^M f_k(t)N_f^c_h_k.
\]

The function \( p(t), t > 0 \) belongs to \( \mathcal{D}(\hat{F}) \), and satisfies the equation:
\[
\frac{dp}{dt} = p(t) = \sum_{k=1}^M (c \tilde{f}_k(t) - f_k(t)) \bar{N}_eh_k,
\]
\[
\bar{p}(0) = u_0 - \sum_{k=1}^M f_k(0)N_f^c_h_k.
\]

Since \( \rho_k \) belong to \( \mathcal{D}(B^*) \), we see that
\[
|f_k(t)| + |\tilde{f}_k(t)| \quad \leq \text{const} \ e^{-a_iu(t)}(\|u_0\| + \|v_0\|_H), \quad t \geq 0,
\]
which yields the decay estimate:
\[
\|p(t)\| \leq \text{const} \ e^{-t_0l}(\|u_0\| + \|v_0\|_H), \quad t \geq 0.
\]

Thus we immediately obtain the decay estimate (16). In other words, the stabilization of (10) is thereby achieved.

Finite-dimensional compensator. To derive the finite-dimensional compensator \( \Sigma_c \) in eqn. (1), we go back to eqn. (10). Let \( P_n \) be the projector in \( H \) corresponding to the eigenvalues \( \{\mu_i, \omega^{\pm}_i\} \subseteq \mathbb{C} \) of the \( B \). In view of the fact that \( \rho_k \in \mathcal{P}_nH \), set \( \nu_1(t) = \mathcal{P}_n \nu(t) \). Applying the \( \mathcal{P}_n \) to the both sides of the equation in \( \nu \) in (10), we obtain
\[
\frac{d\nu_1}{dt} + B_1\nu_1 = \sum_{k=1}^M \langle \nu_1, \rho_k \rangle_h \mathcal{P}_n \xi_k
\]
\[
+ \sum_{k=1}^M \langle \nu_1, \rho_k \rangle_h \mathcal{P}_n \xi_k \tag{34} \]
\[
\nu_1(0) \in L^2(\Omega), \quad \nu_1(0) = \mathcal{P}_n \nu_0 \in \mathcal{P}_n H.
\]

Here we have set \( B_1 = B|_{\mathcal{P}_nH} \). The control system (34) is clearly well posed in \( L^2(\Omega) \times P_nH \), and generates an analytic semigroup in \( t > 0 \). The state
$u(t, \cdot)$ belongs to $C^2(\Omega) \cap C^1(\overline{\Omega})$, $t > 0$. Thus every solution $(u(t, \cdot), v(t))$ to eqn. (34) is derived from eqn. (10), and satisfies the decay estimate (17). The differential equation in $v_1$ in (34) means the finite-dimensional compensator $Z_c$ of dimension $l = \dim \mathcal{P}_n H$. The second term of the right-hand side is contained in the matrix $B_1$ in eqn. (1).

**Remark.** We have obtained a general result in Theorem 1 of ensuring the smallest necessary number $N$ of the sensors $v_k$ for the stabilization. Thus, the best choice: $N = 1$ is possible even in the case where the algebraic multiplicities $m_i$ of the operator $L$ are greater than 1. The same possible choice of the minimum number $M$ of the actuators $h_k$ is ensured by the controllability condition on the pair $(A, Z)$ of the subpairs $(A_i, Y_i)$ is equivalent to the set of the controllability conditions of the subpairs $(A_i, Y_i)$, or

\[
\text{rank}(Y_i A_i Y_i A_i^2 Y_i \ldots A_i^{m_i - 1} Y_i) = \text{rank}(Y_i N Y_i N_i^2 Y_i \ldots N_i^{m_i - 1} Y_i) = m_i, \quad 1 \leq i \leq K.
\]

Since $N_i|_{(j, k)} = \alpha_{ij}^k$, $j < k$, the best choice: $M = 1$ is possible when $\alpha_{ij}^{(j+1)k} \neq 0$, $1 \leq j < m_i$, $1 \leq i \leq K$. The situation is similar to the case of the $w_k$: In each $m_i \times m_i$ matrix $(Y_i N_i Y_i N_i^2 Y_i \ldots N_i^{m_i - 1} Y_i)$, the condition

\[
\det(Y_i N_i Y_i N_i^2 Y_i \ldots N_i^{m_i - 1} Y_i) \neq 0, \quad 1 \leq i \leq K
\]

is fulfilled, as long as the single actuator $h_1$ is chosen such that

\[
(Y_i)_{(1, m_i)} = \zeta_{m_i}^1 = \sum_{j=1}^{m_i} R_i \langle (m_i, j), (h_1, \sigma \psi_{ij}) \rangle \neq 0,
\]

where

\[
\langle h_1, \sigma \psi_{ij} \rangle = \left( h_1, (1 - b(\xi) \nu(\xi)) \psi_{ij} - \frac{\partial \psi_{ij}}{\partial \nu} \right)_{\Gamma}.
\]

**References**


Nonlocal Controllability for the Semilinear Fuzzy Integrodifferential Equations in \( n \)-Dimensional Fuzzy Vector Space

Young Chel Kwun\(^1\) and Jin Han Park\(^2\).

\(^1\) Department of Mathematics, Dong-A University, Pusan 604-714, South Korea
\texttt{yckwun@dau.ac.kr}

\(^2\) Division of Math. Sci., Pukyong National University, Pusan 608-737, South Korea
\texttt{jihpark@pknu.ac.kr}

Abstract. In this paper, we study the existence and uniqueness of solutions and controllability for the semilinear fuzzy integrodifferential equations in \( n \)-dimensional fuzzy vector space \( (E_N)^n \) by using Banach fixed point theorem. That is an extension of the result of Park et al. [J.H. Park, J.S. Park and Y.C. Kwun, Controllability for the semilinear fuzzy integrodifferential equations with nonlocal conditions, Lecture Notes in Artificial Intelligence 4223 (2006), 221–230] to \( n \)-dimensional fuzzy vector space.

1 Introduction

Many authors have studied several concepts of fuzzy systems. Diamond and Kloeden [3] proved the fuzzy optimal control for the following system:

\[
\dot{x}(t) = a(t)x(t) + u(t), \quad x(0) = x_0,
\]

where \( x(\cdot) \) and \( u(\cdot) \) are nonempty compact interval-valued functions on \( E^1 \). Kwun and Park [6] proved the existence of fuzzy optimal control for the nonlinear fuzzy differential system with nonlocal initial condition in \( E^1_N \) by using Kuhn-Tucker theorems. Fuzzy integrodifferential equations are a field of interest, due to their applicability to the analysis of phenomena with memory where imprecision is inherent. Balasubramaniam and Muralisankar [1] proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equation with nonlocal initial condition. They considered the semilinear one-dimensional heat equation on a connected domain \( (0,1) \) for material with memory. In one-dimensional fuzzy vector space \( E^1_N \), Park et al. [10] proved the existence and uniqueness of fuzzy solutions and presented the sufficient condition of nonlocal controllability for the following semilinear fuzzy integrodifferential equation with nonlocal initial condition:

\[
\frac{dx(t)}{dt} = A \left[ x(t) + \int_0^t G(t-s)x(s)ds \right] + f(t, x) + u(t), \quad t \in J = [0, T],
\]

\[
x(0) + g(t_1, t_2, \ldots, t_p, x(t_m)) = x_0 \in E_N, \quad m = 1, 2, \ldots, p,
\]
where \( T > 0, A : J \to E_N \) is a fuzzy coefficient, \( E_N \) is the set of all upper semicontinuous convex normal fuzzy numbers with bounded \( \alpha \)-level intervals, \( f : J \times E_N \to E_N \) is a nonlinear continuous function, \( g : J^p \times E_N \to E_N \) is a nonlinear continuous function, \( G(t) \) is an \( n \times n \) continuous matrix such that \( \frac{dG(t)x}{dt} \) is continuous for \( x \in E_N \) and \( t \in J \) with \( \|G(t)\| \leq K, K > 0 \), with all nonnegative elements, \( u : J \to E_N \) is control function.


In this paper, we study the the existence and uniqueness of solutions and controllability for the following semilinear fuzzy integrodifferential equations:

\[
\frac{dx_i(t)}{dt} = A_i \left[ x_i(t) + \int_0^t G(t-s)x_i(s)ds \right] + f_i(t, x_i(t)) + u_i(t) \text{ on } E_N^i, \quad (1)
\]

\[
x_i(0) + g_i(x_i) = x_{i0} \in E_N^i, \quad (i = 1, 2, \cdots, n),
\]

where \( A_i : [0, T] \to E_N^i \) is fuzzy coefficient, \( E_N^i \) is the set of all upper semicontinuously convex fuzzy numbers on \( R \) with \( E_N^i \neq E_N^j \) \((i \neq j)\), \( f_i : [0, T] \times E_N^i \to E_N^i \) is a nonlinear regular fuzzy function, \( g_i : E_N^i \to E_N^i \) is a nonlinear continuous function, \( G(t) \) is an \( n \times n \) continuous matrix such that \( \frac{dG(t)x}{dt} \) is continuous for \( x_i \in E_N^i \) and \( t \in [0, T] \) with \( \|G(t)\| \leq k, k > 0, u_i : [0, T] \to E_N^i \) is control function and \( x_{i0} \in E_N^i \) is initial value.

\section{Preliminaries}

A fuzzy set of \( R^n \) is a function \( u : R^n \to [0, 1] \). For each fuzzy set \( u \), we denote by \( [u]_\alpha = \{x \in R^n : u(x) \geq \alpha\} \) for any \( \alpha \in [0, 1] \), its \( \alpha \)-level set.

Let \( u, v \) be fuzzy sets of \( R^n \). It is well known that \([u]_\alpha = [v]_\alpha \) for each \( \alpha \in [0, 1] \) implies \( u = v \).

Let \( E^n \) denote the collection of all fuzzy sets of \( R^n \) that satisfies the following conditions:

1. \( u \) is normal, i.e., there exists an \( x_0 \in R^n \) such that \( u(x_0) = 1 \);
2. \( u \) is fuzzy convex, i.e., \( u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\} \) for any \( x, y \in R^n \), \( 0 \leq \lambda \leq 1 \);
3. \( u(x) \) is upper semi-continuous, i.e., \( u(x_0) \geq \lim_{k \to \infty} u(x_k) \) for any \( x_k \in R^n \) \((k = 0, 1, 2, \cdots)\), \( x_k \to x_0 \);
4. \( [u]_0 \) is compact.

We call \( u \in E^n \) a \( n \)-dimension fuzzy number.

Wang, Li and Wen[12] defined \( n \)-dimensional fuzzy vector space and investigated its properties.
For any \( u_i \in E, \ i = 1, 2, \ldots, n \), we call the ordered one-dimension fuzzy number class \( u_1, u_2, \ldots, u_n \) (i.e., the Cartesian product of one-dimension fuzzy number \( u_1, u_2, \ldots, u_n \)) a \( n \)-dimension fuzzy vector, denote it as \((u_1, u_2, \ldots, u_n)\), and call the collection of all \( n \)-dimension fuzzy vectors (i.e., the Cartesian product \( E \times E \times \cdots \times E \)) \( n \)-dimensional fuzzy vector space, and denote it as \((E)^n\).

**Definition 2.1** [12] If \( u \in E^n \), and \([u]^\alpha\) is a hyperrectangle, i.e., \([u]^\alpha\) can be represented by \( \prod_{i=1}^{n}[u_{il}^\alpha, u_{ir}^\alpha] \), i.e., \([u_{il}^\alpha, u_{ir}^\alpha] \times [u_{2l}^\alpha, u_{2r}^\alpha] \times \cdots \times [u_{nl}^\alpha, u_{nr}^\alpha] \) for every \( \alpha \in [0, 1] \), where \( u_{il}^\alpha, u_{ir}^\alpha \in R \) with \( u_{il}^\alpha \leq u_{ir}^\alpha \) when \( \alpha \in (0, 1) \), \( i = 1, 2, \ldots, n \), then we call \( u \) a fuzzy \( n \)-cell number. We denote the collection of all fuzzy \( n \)-cell numbers by \( L(E^n) \).

**Theorem 2.1** [12] For any \( u \in L(E^n) \) with \([u]^\alpha = \prod_{i=1}^{n}[u_{il}^\alpha, u_{ir}^\alpha] \) (\( \alpha \in [0, 1] \)), there exists a unique \((u_1, u_2, \ldots, u_n) \in (E)^n\) such that \([u_i]^\alpha = [u_{il}^\alpha, u_{ir}^\alpha] \) (\( i = 1, 2, \ldots, n \) and \( \alpha \in [0, 1] \)).

Conversely, for any \((u_1, u_2, \ldots, u_n) \in (E)^n\) with \([u_i]^\alpha = [u_{il}^\alpha, u_{ir}^\alpha] \) (\( i = 1, 2, \ldots, n \) and \( \alpha \in [0, 1] \)), there exists a unique \( u \in L(E^n) \) such that \([u]^\alpha = \prod_{i=1}^{n}[u_{il}^\alpha, u_{ir}^\alpha]\) (\( \alpha \in [0, 1] \)).

**Note 2.1** [12] Theorem 2.1 indicates that fuzzy \( n \)-cell numbers and \( n \)-dimension fuzzy vectors can represent each other, so \( L(E^n) \) and \((E)^n\) may be regarded as identity. If \((u_1, u_2, \ldots, u_n) \in (E)^n\) is the unique \( n \)-dimension fuzzy vector determined by \( u \in L(E^n) \), then we denote \( u = (u_1, u_2, \ldots, u_n) \).

Let \((E_N^N)^n = E_N^1 \times E_N^2 \times \cdots \times E_N^n\), \( E_N^i(i = 1, 2, \ldots, n) \) is fuzzy subset of \( R \). Then \((E_N^N)^n \subseteq (E)^n\).

**Definition 2.2** [12] The complete metric \( D_L \) on \((E_N^N)^n\) is defined by

\[
D_L(u, v) = \sup_{0 < \alpha < 1} d_L([u]^\alpha, [v]^\alpha)
\]

\[
= \sup_{0 < \alpha < 1} \max \{|u_{il}^\alpha - v_{il}^\alpha|, |u_{ir}^\alpha - v_{ir}^\alpha|\}
\]

for any \( u, v \in (E_N^N)^n \), which satisfies \( d_L(u + w, v + w) = d_L(u, v) \).

**Definition 2.3** Let \( u, v \in C([0, T] : (E_N^N)^n) \)

\[H_1(u, v) = \sup_{0 \leq t \leq T} D_L(u(t), v(t)).\]

**Definition 2.4** [12] The derivative \( x'(t) \) of a fuzzy process \( x \in (E_N^N)^n \) is defined by

\[x'(t)]^\alpha = \prod_{i=1}^{n} \langle (x_{il}^\alpha)'(t), (x_{ir}^\alpha)'(t) \rangle\]

provided that equation defines a fuzzy \( x'(t) \) in \((E_N^N)^n\).

**Definition 2.5** [12] The fuzzy integral \( \int_b^a x(t)dt \), \( a, b \in [0, T] \) is defined by

\[
\left[ \int_b^a x(t)dt \right]^\alpha = \prod_{i=1}^{n} \left[ \int_b^a x_{il}^\alpha(t)dt, \int_b^a x_{ir}^\alpha(t)dt \right]
\]

provided that the Lebesgue integrals on the right hand side exist.
3 Existence and Uniqueness

In this section we consider the existence and uniqueness of the fuzzy solution for the equations (1)-(2) \((u \equiv 0)\).

We define
\[
A = (A_1, A_2, \cdots, A_n),
\]
\[
x = (x_1, x_2, \cdots, x_n),
\]
\[
f = (f_1, f_2, \cdots, f_n),
\]
\[
u = (u_1, u_2, \cdots, u_n),
\]
\[
g = (g_1, g_2, \cdots, g_n),
\]
and
\[
x_0 = (x_{01}, x_{02}, \cdots, x_{0n}).
\]

Then
\[
A, \ x, \ f, \ x_0, \ u, \ g \in (E_N^n)^n.
\]

Instead of the equations (1)-(2), we consider the following fuzzy integrodifferential equations in \((E_N^n)^n\).

\[
\frac{dx(t)}{dt} = A \left[ x(t) + \int_0^t G(t - s)x(s)ds \right] + f(t, x(t)) + u(t) \quad \text{on} \quad (E_N^n)^n \tag{3}
\]
\[
x(0) + g(x) = x_0 \in (E_N^n)^n \tag{4}
\]

with fuzzy coefficient \(A : [0, T] \to (E_N^n)^n\), initial value \(x_0 \in (E_N^n)^n\), and \(u : [0, T] \to (E_N^n)^n\) is a control function. Given nonlinear regular fuzzy function \(f : [0, T] \times (E_N^n)^n \to (E_N^n)^n\) satisfies a global Lipschitz condition, i.e. there exists a finite \(k > 0\) such that

\[
d_L([f(x(s))]^n, [f(y(s))]^n) \leq kd_L([x(s)]^n, [y(s)]^n) \tag{5}
\]

for all \(x(s), y(s) \in (E_N^n)^n\). The nonlinear function \(g : (E_N^n)^n \to (E_N^n)^n\) is a continuous function and satisfies the Lipschitz condition

\[
d_L([g(x(\cdot))]^n, [g(y(\cdot))]^n) \leq hd_L([x(\cdot)]^n, [y(\cdot)]^n) \tag{6}
\]

for all \(x(\cdot), y(\cdot) \in (E_N^n)^n\), \(h\) is a finite positive constant.

**Definition 3.1** The fuzzy process \(x : I = [0, T] \to (E_N^n)^n\) with \(\alpha\)-level set \([x(t)]^n = \Pi_{i=1}^n [x_i]^n = \Pi_{i=1}^n [x_i^\alpha, x_i^\alpha]\) is a fuzzy solution of the equations (3)-(4) without nonhomogeneous term if and only if

\[
(x_i^\alpha)'(t) = \min \{A_{ij}^\alpha(t)[x_j^\alpha(t)] + \int_0^t G(t - s)x_{ik}^\alpha(s)ds : j, k = l, r\},
\]
\[
(x_i^\alpha)'(t) = \max \{A_{ij}^\alpha(t)[x_j^\alpha(t)] + \int_0^t G(t - s)x_{ik}^\alpha(s)ds : j, k = l, r\},
\]
\[
x_i^\alpha(0) + g_{il}(x_i^\alpha) = x_{0i}, \quad x_{ir}^\alpha(0) + g_{ir}(x_r^\alpha) = x_{0r}, \quad i = 1, 2, \cdots, n.
\]
For the sequel, we need the following assumptions:

(H1) $S(t)$ is a fuzzy number satisfying, for $y \in (E_N^n)$, \( \frac{d}{dt}S(t)y \in C^1(I : (E_N^n)) \cap C(I : (E_N^n)) \), the equation

\[
\frac{d}{dt}S(t)y = A \left[ S(t)y + \int_0^t G(t-s)S(s)yds \right]
\]

\[= S(t)Ay + \int_0^t S(t-s)AG(s)yds, \quad t \in I,
\]

where

\[
[S(t)]^\alpha = \Pi_{i=1}^n [S_i(t)]^\alpha = \Pi_{i=1}^n [S_{ij}^0(t), S_{ir}^0(t)],
\]

and $S_{ij}^0(t)$ is continuous with $|S_{ij}^0(t)| \leq c$, $c > 0$, for all $t \in I = [0,T]$.

(H2) $c\{h(1 + T + cT) + kT(1 + cT)\} < 1$.

In view of Definition 3.1 and (H1), equations (3)-(4) can be expressed as

\[
x(t) = S(t)(x_0 - g(x)) + \int_0^t S(t-s)(f(s,x(s)) + u(s))ds,
\]

\[
x(0) + g(x) = x_0.
\]

**Theorem 3.1** Let $T > 0$. If hypotheses (H1)-(H2) are hold. Then, for every $x_0 \in (E_N^n)$, equations (7)-(8)(u ≡ 0) have a unique fuzzy solution $x \in C([0,T])$:

\[(E_N^n).
\]

**Proof.** For each $x(t) \in (E_N^n)$ and $t \in [0,T]$, define $(G_0x)(t) \in (E_N^n)$ by

\[
(G_0x)(t) = S(t)(x_0 - g(x)) + \int_0^t S(t-s)f(s,x(s))ds.
\]

Thus, $G_0x : [0,T] \to (E_N^n)$ is continuous, so $G_0$ is a mapping from $C([0,T])$ to $(E_N^n)$ into itself. By Definition 2.2, Definition 2.3, some properties of $d_L$ and inequalities (5) and (6), we have following inequalities. For $x, y \in C([0,T])$:

\[
d_L([G_0x(t)]^\alpha, [G_0y(t)]^\alpha)
\]

\[= d_L(\left[ S(t)(x_0 - g(x)) + \int_0^t S(t-s)f(s,x(s))ds \right]^\alpha,
\]

\[
\left[ S(t)(x_0 - g(y)) + \int_0^t S(t-s)f(s,y(s))ds \right]^\alpha
\]

\[= d_L(\left[ - S(t)g(x) + \int_0^t S(t-s)f(s,x(s))ds \right]^\alpha,
\]

\[
\left[ - S(t)g(y) + \int_0^t S(t-s)f(s,y(s))ds \right]^\alpha
\]

\leq d_L([S(t)g(x)]^\alpha, [S(t)g(y)]^\alpha)
Therefore

\[ y(t) = \max_{1 \leq i \leq n} \left\{ \left| S_i^\alpha(t) \{ g_i^\alpha(x) - g_i^\alpha(y) \} \right|, \left| S_i^\alpha(t) \{ g_i^\alpha(x) - g_i^\alpha(y) \} \right| \right\} + \int_0^t \max_{1 \leq i \leq n} \left\{ \left| S_i^\alpha(s, x(s)) - f_i^\alpha(s, y(s)) \right|, \left| S_i^\alpha(s, x(s)) - f_i^\alpha(s, y(s)) \right| \right\} ds \]

\[ \leq c \max_{1 \leq i \leq n} \left\{ \left| (g_i^\alpha(x) - g_i^\alpha(y)) \right|, \left| (g_i^\alpha(x) - g_i^\alpha(y)) \right| \right\} + c \int_0^t \max_{1 \leq i \leq n} \left\{ \left| f_i^\alpha(s, x(s)) - f_i^\alpha(s, y(s)) \right|, \left| f_i^\alpha(s, x(s)) - f_i^\alpha(s, y(s)) \right| \right\} ds \]

\[ = cdL \left( (y(x))^\alpha, (y(y))^\alpha \right) + c \int_0^t dL \left( (f(s, x(s))^\alpha, (f(s, y(s))^\alpha \right) ds \]

\[ \leq chdL \left( (x(\cdot))^\alpha, (y(\cdot))^\alpha \right) + ck \int_0^t dL \left( (x(s))^\alpha, (y(s))^\alpha \right) ds. \]

Therefore

\[ D_L(\{G_0x(t), (G_0y(t)\}) = \sup_{0<\alpha \leq 1} dL \left( \{ (G_0x(t))^\alpha, (G_0y(t))^\alpha \} \right) \]

\[ \leq ch \sup_{0<\alpha \leq 1} dL \left( \{ (x(\cdot))^\alpha, (y(\cdot))^\alpha \} \right) + ck \sup_{0<\alpha \leq 1} \int_0^t dL \left( \{ (x(s))^\alpha, (y(s))^\alpha \} \right) ds \]

\[ \leq chD_L(\{x(\cdot), y(\cdot)\}) + ck \int_0^t D_L \left( (x(s), y(s)) \right) ds. \]

Hence

\[ H_1(G_0x, G_0y) = \sup_{0 \leq t \leq T} D_L(\{ (G_0x(t), (G_0y(t)) \}) \]

\[ \leq ch \sup_{0 \leq t \leq T} D_L(\{x(\cdot), y(\cdot)\}) + ck \sup_{0 \leq t \leq T} \int_0^t D_L \left( (x(s), y(s)) \right) ds \]

\[ \leq ch H_1(x, y) + ckT H_1(x, y) = c(h + kT) H_1(x, y). \]

By hypotheses (H2), \( G_0 \) is a contraction mapping.

Using the Banach fixed point theorem, equations (7)-(8) have a unique fixed point \( x \in C([0, T]: (E_N)^\alpha) \).
4 Controllability

In this section, we show the nonlocal controllability for the control system (1)-(2).

**Definition 4.1** The equations (1)-(2) are nonlocal controllable. Then there exists \( u(t) \) such that the fuzzy solution \( x(t) \) for the equations (7)-(8) as \( x(T) = x^1 - g(x) \) (i.e., \([x(T)]^\alpha = [x^1 - g(x)]^\alpha\)) where \( x^1 \in (E_N^n) \) is target set.

Define the fuzzy mapping \( \tilde{\beta} : \tilde{P}(R^n) \to (E_N^n) \) by

\[
\tilde{\beta}^\alpha(v) = \begin{cases}
\int_0^T S^\alpha(T - s)v(s)ds, & v \subset \mathcal{T}_u,
0, & \text{otherwise},
\end{cases}
\]

where \( \mathcal{T}_u \) is closed support of \( u \). Then there exists

\[
\tilde{\beta}_i : \tilde{P}(R) \to E_N^1 \quad (i = 1, 2, \cdots, n)
\]

such that

\[
\tilde{\beta}_i^\alpha(v_i) = \begin{cases}
\int_0^T S^\alpha_i(T - s)v_i(s)ds, & v_i(s) \subset \mathcal{T}_{u_i},
0, & \text{otherwise},
\end{cases}
\]

Then exists \( \tilde{\beta}_{ij}^\alpha \) \((j = l, r)\) such that

\[
\tilde{\beta}_{il}^\alpha(v_{il}) = \int_0^T S^\alpha_{il}(T - s)v_{il}(s)ds, \quad v_{il}(s) \in [u_{il}^0(s), u_{il}^1],
\]

\[
\tilde{\beta}_{ir}^\alpha(v_{ir}) = \int_0^T S^\alpha_{ir}(T - s)v_{ir}(s)ds, \quad v_{ir}(s) \in [u_{ir}^1(s), u_{ir}^0(s)].
\]

We assume that \( \tilde{\beta}_{il}^\alpha, \tilde{\beta}_{ir}^\alpha \) are bijective mappings.

We can introduce \( \alpha \)-level set of \( u(s) \) of the equations (5)-(6)

\[
[u(s)]^\alpha = \prod_{i=1}^n [u_i(s)]^\alpha = \prod_{i=1}^n [u_{il}^0(s), u_{il}^1(s)]
\]

\[
= \prod_{i=1}^n \left[ (\tilde{\beta}_{il}^\alpha)^{-1} \left( ((x^1)_i^\alpha - g_{il}(x_{il}^\alpha)) - S_{il}(T)(x_{0il}^\alpha - g_{il}(x_{il}^\alpha)) \right) - \int_0^T S_{il}^\alpha(T - s)f_{il}^\alpha(s, x_{0il}^\alpha)ds \right],
\]

\[
(\tilde{\beta}_{ir}^\alpha)^{-1} \left( ((x^1)_i^\alpha - g_{ir}(x_{ir}^\alpha)) - S_{ir}(T)(x_{0ir}^\alpha - g_{ir}(x_{ir}^\alpha)) \right) - \int_0^T S_{ir}^\alpha(T - s)f_{ir}^\alpha(s, x_{0ir}^\alpha)ds \right].
\]
Therefore substituting this expression into the equations (7)-(8) yields \( \alpha \)-level of \( x(T) \).

For each \( i = 1, 2, \ldots, n \),

\[
[x_i(T)]^\alpha = \left[ S_{ii}^\alpha(T)(x_{0_i}(x_{0_i}^\alpha)) + \int_0^T S_{ii}^\alpha(T - s)f_{ii}^\alpha(s, x_{ii}^\alpha(s))ds \right.
\]

\[
+ \int_0^T S_{ii}^\alpha(T - s)(\bar{\tilde{\alpha}}_{ii})^{-1}\left((x_{1i}^\alpha - g_{ii}^\alpha(x_{ii}^\alpha)) - S_{ii}^\alpha(T)(x_{0_i}^\alpha - g_{ii}^\alpha(x_{ii}^\alpha))\right)
\]

\[
- \int_0^T S_{ii}^\alpha(T - s)f_{ii}^\alpha(s, x_{ii}^\alpha(s))ds\left.\right]
\]

\[
= \left[ (x^1 - g(x))^{\alpha}_{ii}, \ (x^1 - g(x))^{\alpha}_{ii} \right] = \left[ (x^1 - g(x))^{\alpha} \right].
\]

Therefore

\[
[x(T)]^\alpha = \prod_{i=1}^n [x_i(T)]^\alpha = \prod_{i=1}^n [(x^1 - g(x))^{\alpha}_i] = [(x^1 - g(x))]^\alpha.
\]

We now set

\[
\phi x(t) = S(t)(x_0 - g(x)) + \int_0^t S(t - s)f(s, x(s))ds \tag{9}
\]

\[
+ \int_0^t S(t - s)\bar{\tilde{\alpha}}^{-1}(x^1 - g(x)) - S(T)(x_0 - g(x))
\]

\[
- \int_0^T S(T - s)f(s, x(s))ds\),
\]

where the fuzzy mapping \( \bar{\tilde{\alpha}}^{-1} \) satisfies above statements.

Notice that \( \phi x(T) = x^1 - g(x) \), which means that the control \( u(t) \) steers the equations (7)-(8) from the origin to \( x^1 - g(x) \) in time \( T \) provided we can obtain a fixed point of the operator \( \phi \).

(H3) Assume that the linear system of equations (7)-(8) \( (f \equiv 0) \) is controllable.

**Theorem 4.1** Suppose that hypotheses (H1)-(H3) are satisfied. Then the equations (7)-(8) are nonlocal controllable.
\textbf{Proof.} We can easily check that \( \Phi \) is continuous function from \( C([0, T] : (E_N)^n) \) to itself. By Definition 2.2, Definition 2.3, some properties of \( d_L \) and inequalities (5) and (6), we have following inequalities. For any \( x, y \in C([0, T] : (E_N)^n) \),

\[
d_L(\[\Phi x(t)\]^\alpha, [\Phi y(t)]^\alpha) = d_L(\left[S(t)(x_0 - g(x)) + \int_0^t S(t - s)f(s, x(s))ds + \int_0^t S(t - s)\tilde{\beta}^{-1} \times \left(x^1 - g(x) - S(T)(x_0 - g(x)) - \int_0^T S(T - s)f(s, x(s))ds \right)ds \right]^\alpha, \right]

\[
\left[S(t)(x_0 - g(y)) + \int_0^t S(t - s)f(s, y(s))ds + \int_0^t S(t - s)\tilde{\beta}^{-1} \times \left(x^1 - g(y) - S(T)(x_0 - g(y)) - \int_0^T S(T - s)f(s, y(s))ds \right)ds \right]^\alpha) \right]
\]

\[
\leq d_L(\left[S(t)(g(x))^\alpha, [S(t)g(y)]^\alpha \right)
+ \int_0^t d_L(\left[S(t - s)f(s, x(s))]^\alpha, [S(t - s)f(s, y(s))]^\alpha \right)ds
+ \int_0^t d_L(\left[S(t - s)\tilde{\beta}^{-1}g(x)]^\alpha, [S(t - s)\tilde{\beta}^{-1}g(y)]^\alpha \right)ds
+ \int_0^t d_L(\left[S(t - s)\tilde{\beta}^{-1}S(T)g(x)]^\alpha, [S(t - s)\tilde{\beta}^{-1}S(T)g(y)]^\alpha \right)ds
+ \int_0^t d_L(\left[S(t - s)\tilde{\beta}^{-1}\int_0^T S(T - s)f(s, x(s))ds \right]^\alpha,
\]

\[
\left[S(t - s)\tilde{\beta}^{-1}\int_0^T S(T - s)f(s, y(s))ds \right]^\alpha \right)ds
\]

\[
= \max_{1 \leq i \leq n} \left\{ \left| S_{\alpha_{ii}}^n(t)(g_{ii}^n(x) - g_{ii}^n(y)) \right|, \left| S_{\alpha_{ii}}^n(t)(g_{ii}^n(x) - g_{ii}^n(y)) \right| \right\}
\]

\[
+ \int_0^t \max_{1 \leq i \leq n} \left\{ \left| S_{\alpha_{ii}}^n(t - s)(f_{ii}^n(s, x(s)) - f_{ii}^n(s, y(s)) \right|, \neq \left| S_{\alpha_{ii}}^n(t - s)(f_{ii}^n(s, x(s)) - f_{ii}^n(s, y(s)) \right| \right\}ds
\]

\[
+ \int_0^t \max_{1 \leq i \leq n} \left\{ \left| S_{\alpha_{ii}}^n(t - s)(\tilde{\beta}_{ii}^{-1})^{-1}(g_{ii}^n(x) - g_{ii}^n(y)) \right|, \neq \left| S_{\alpha_{ii}}^n(t - s)(\tilde{\beta}_{ii}^{-1})^{-1}(g_{ii}^n(x) - g_{ii}^n(y)) \right| \right\}ds
\]

\[
+ \int_0^t \max_{1 \leq i \leq n} \left\{ \left| S_{\alpha_{ii}}^n(t - s)(\tilde{\beta}_{ii}^{-1})^{-1}S_{\alpha_{ii}}^n(T)(g_{ii}^n(x) - g_{ii}^n(y)) \right|, \neq \left| S_{\alpha_{ii}}^n(t - s)(\tilde{\beta}_{ii}^{-1})^{-1}S_{\alpha_{ii}}^n(T)(g_{ii}^n(x) - g_{ii}^n(y)) \right| \right\}ds
\]

\[
+ \int_0^t \max_{1 \leq i \leq n} \left\{ \left| S_{\alpha_{ii}}^n(t - s)(\tilde{\beta}_{ii}^{-1})^{-1}\int_0^T S_{\alpha_{ii}}^n(T - s)f_{ii}^n(s, x(s))ds \right|, \neq \left| S_{\alpha_{ii}}^n(t - s)(\tilde{\beta}_{ii}^{-1})^{-1}\int_0^T S_{\alpha_{ii}}^n(T - s)f_{ii}^n(s, x(s))ds \right| \right\}ds
\]
\[- \int_{0}^{T} S_{\alpha}^{\infty}(T - s) f_{\alpha}^{n}(s, y(s))ds \leq \sup_{\alpha \leq 1} \max_{0 \leq t \leq 1} \left\{ \left| g_{\alpha}^{n}(x) - g_{\alpha}^{n}(y) \right|, \left| g_{\alpha}^{n}(x) - g_{\alpha}^{n}(y) \right| \right\} ds \]

\[ + c \int_{0}^{t} \max_{\alpha \leq 1} \left\{ \left| f_{\alpha}^{n}(s, x(s)) - f_{\alpha}^{n}(s, y(s)) \right|, \left| f_{\alpha}^{n}(s, x(s)) - f_{\alpha}^{n}(s, y(s)) \right| \right\} ds \]

\[ + c \int_{0}^{t} \max_{\alpha \leq 1} \left\{ \left| g_{\alpha}^{n}(x) - g_{\alpha}^{n}(y) \right|, \left| g_{\alpha}^{n}(x) - g_{\alpha}^{n}(y) \right| \right\} ds \]

\[ + c^{2} \int_{0}^{t} \int_{0}^{T} \max_{\alpha \leq 1} \left\{ \left| f_{\alpha}^{n}(s, x(s)) - f_{\alpha}^{n}(s, y(s)) \right|, \left| f_{\alpha}^{n}(s, x(s)) - f_{\alpha}^{n}(s, y(s)) \right| \right\} ds \]

\[ = c d_{L} \left( \left[ g(x) \right]^{\alpha}, \left[ g(y) \right]^{\alpha} \right) + c \int_{0}^{t} d_{L} \left( \left[ f(s, x(s)) \right]^{\alpha}, \left[ f(s, y(s)) \right]^{\alpha} \right) ds \]

\[ + c \int_{0}^{t} d_{L} \left( \left[ g(x) \right]^{\alpha}, \left[ g(y) \right]^{\alpha} \right) ds + c^{2} \int_{0}^{t} d_{L} \left( \left[ g(x) \right]^{\alpha}, \left[ g(y) \right]^{\alpha} \right) ds \]

\[ + c \int_{0}^{t} \int_{0}^{T} d_{L} \left( \left[ f(s, x(s)) \right]^{\alpha}, \left[ f(s, y(s)) \right]^{\alpha} \right) ds \]

\[ \leq c h \left\{ \sup_{\alpha \leq 1} d_{L} \left( \left[ x(t) \right]^{\alpha}, \left[ y(t) \right]^{\alpha} \right) + (1 + c) \int_{0}^{t} \sup_{\alpha \leq 1} d_{L} \left( \left[ x(t) \right]^{\alpha}, \left[ y(t) \right]^{\alpha} \right) ds \right\} \]

\[ + c \int_{0}^{t} \sup_{\alpha \leq 1} d_{L} \left( \left[ x(s) \right]^{\alpha}, \left[ y(s) \right]^{\alpha} \right) ds \]

\[ + c \int_{0}^{t} \int_{0}^{T} \sup_{\alpha \leq 1} d_{L} \left( \left[ x(s) \right]^{\alpha}, \left[ y(s) \right]^{\alpha} \right) ds \]

Therefore

\[ D_{L} \left( \Phi x(t), \Phi y(t) \right) = \sup_{0 < \alpha \leq 1} d_{L} \left( \left[ \Phi x(t) \right]^{\alpha}, \left[ \Phi y(t) \right]^{\alpha} \right) \]

\[ \leq c h \left\{ \sup_{\alpha \leq 1} d_{L} \left( \left[ x(t) \right]^{\alpha}, \left[ y(t) \right]^{\alpha} \right) + (1 + c) \int_{0}^{t} \sup_{0 < \alpha \leq 1} d_{L} \left( \left[ x(t) \right]^{\alpha}, \left[ y(t) \right]^{\alpha} \right) ds \right\} \]

\[ + c \int_{0}^{t} \sup_{0 < \alpha \leq 1} d_{L} \left( \left[ x(s) \right]^{\alpha}, \left[ y(s) \right]^{\alpha} \right) ds \]

\[ + c \int_{0}^{t} \int_{0}^{T} \sup_{0 < \alpha \leq 1} d_{L} \left( \left[ x(s) \right]^{\alpha}, \left[ y(s) \right]^{\alpha} \right) ds \]
\[= ch \left\{ D_L(x(\cdot), y(\cdot)) + (1 + c) \int_0^t D_L(x(\cdot), y(\cdot)) ds \right\} + ck \left\{ \int_0^t D_L(x(s), y(s)) ds + c \int_0^t \int_0^T D_L(x(s), y(s)) ds ds \right\}.\]

Hence
\[H_1(\Phi x, \Phi y) = \sup_{0 \leq t \leq T} D_L(\Phi x(t), \Phi y(t)) \leq ch \left\{ \sup_{0 \leq t \leq T} D_L(x(\cdot), y(\cdot)) + (1 + c) \sup_{0 \leq t \leq T} \int_0^t D_L(x(\cdot), y(\cdot)) ds \right\}
+ ck \left\{ \sup_{0 \leq t \leq T} \int_0^t D_L(x(s), y(s)) ds + c \sup_{0 \leq t \leq T} \int_0^t \int_0^T D_L(x(s), y(s)) ds ds \right\}
\leq ch \{ H_1(x, y) + (1 + c)T H_1(x, y) \} + ck \{ T H_1(x, y) + cT^2 H_1(x, y) \}
= c(h(1 + T + cT) + kT(1 + cT)) H_1(x, y).\]

By hypotheses (H2), \(\Phi\) is a contraction mapping. Using the Banach fixed point theorem, the equation (9) has a unique fixed point \(x \in C([0, T] : (E_N)^n)\).

5 Example

Consider the two semilinear one dimensional heat equations on a connected domain \((0, 1)\) for material with memory on \(E_N, i = 1, 2\), boundary condition \(x_i(t, 0) = x_i(t, 1) = 0, i = 1, 2\) and with initial conditions \(x_i(0, z_i) + \sum_{k=1}^p (c_k)x_i(t_k, z_i) = x_{0_i}(z_i)\), where \(x_{0_i}(z_i) \in E_N, \sum_{k=1}^p (c_k)x_i(t_k, z_i) = g_i(x_i), i = 1, 2\). Let \(x_i(t, z_i), i = 1, 2\) be the internal energy and \(f_i(t, x_i(t, z_i)) = 2tx_i(t, z_i)^2, i = 1, 2\) be the external heat.

Let
\[A = (A_1, A_2) = (2 \frac{\partial^2}{\partial z_1^2}, 2 \frac{\partial^2}{\partial z_2^2}),\]
\[f(t, x(t)) = (f_1(t, x_1(t)), f_2(t, x_2(t))) = (2tx_1(t, z_1)^2, 2tx_2(t, z_2)^2),\]
\[g(x) = (g_1(x_1), g_2(x_2)) = (\sum_{k=1}^p (c_k)x_1(t_k, z_1), \sum_{k=1}^p (c_k)x_2(t_k, z_2)),\]
\[x(0) + g(x) = (x_1(0) + g_1(x), x_2(0) + g_2(x)), x_0 = (x_{0_1}, x_{0_2}) = (0, 0),\]
and
\[G(t - s) = (e^{-(t-s)}, e^{-(t-s)}).\]
then the balance equations become
\[
\frac{dx(t)}{dt} = A \left[ x(t) + \int_0^t G(t-s)x(s)ds \right] + f(t, x(t)) \text{ on } (E^*_N)^2, \quad (10)
\]
\[
x(0) + g(x) = x_0 \in (E^*_N)^2. \quad (11)
\]

The α-level set of fuzzy numbers are the following:
\[ [0]^\alpha = [\alpha - 1, 1 - \alpha], \quad [2]^\alpha = [\alpha + 1, 3 - \alpha] \text{ for all } \alpha \in [0, 1]. \] Then α-level sets of \( f(t, x(t)) \) is
\[
[f(t, x(t))]^\alpha = [2tx_1(t)^2]^\alpha \times [2tx_2(t)^2]^\alpha
\]
\[
= [\alpha + 1, 3 - \alpha] \cdot t[x_1(t)^2, x_2(t)^2] \times [\alpha + 1, 3 - \alpha] \cdot t[(x_1^\alpha(t))^2, (x_2^\alpha(t))^2]
\]
\[
= [(\alpha + 1)t(x_1^\alpha(t))^2, (3 - \alpha)t(x_1^\alpha(t))^2] \times [(\alpha + 1)t(x_2^\alpha(t))^2, (3 - \alpha)t(x_2^\alpha(t))^2]
\]
Further, we have
\[
d_L([f(t, x(t))]^\alpha, f(t, y(t))^\alpha)
\]
\[
d_L(\alpha^\alpha \cdot t[x_1(t)^2, x_2(t)^2], [(\alpha + 1)t(y_1(t))^2, (3 - \alpha)t(y_2(t))^2])
\]
\[
= t \max_{1 \leq r \leq 2} \left\{ (\alpha + 1)[(x_1^\alpha(t))^2 - (y_1^\alpha(t))^2], (3 - \alpha)[(x_2^\alpha(t))^2 - (y_2^\alpha(t))^2] \right\}
\]
\[
\leq T(3 - \alpha) \max_{1 \leq r \leq 2} \left\{ |x_1^\alpha(t) - y_1^\alpha(t)|, |x_2^\alpha(t) + y_2^\alpha(t)| \right\}
\]
\[
\leq 3T|x_1^\alpha(t) + y_1^\alpha(t)| \times \max_{1 \leq r \leq 2} \left\{ |x_1^\alpha(t) - y_1^\alpha(t)|, |x_2^\alpha(t) - y_2^\alpha(t)| \right\}
\]
\[
= kd_L([x(t)]^\alpha, [y(t)]^\alpha),
\]
\[
d_L([g(x(\cdot))]^\alpha, [g(y(\cdot))]^\alpha)
\]
\[
d_L\left( \left[ \sum_{k=1}^p c_k(x(t_k)) \right]^\alpha, \left[ \sum_{k=1}^p c_k(y(t_k)) \right]^\alpha \right)
\]
\[
= \max_{1 \leq r \leq 2} \left\{ \left| \sum_{k=1}^p (c_k x_1^\alpha(t_k)) - \sum_{k=1}^p (c_k y_1^\alpha(t_k)) \right|, \left| \sum_{k=1}^p (c_k x_2^\alpha(t_k)) - \sum_{k=1}^p (c_k y_2^\alpha(t_k)) \right| \right\}
\]
\[
\leq \sum_{k=1}^p c_k \max_{1 \leq r \leq 2} \left\{ |x_1^\alpha(t_k) - y_1^\alpha(t_k)|, |x_2^\alpha(t_k) - y_2^\alpha(t_k)| \right\}
\]
\[
= \sum_{k=1}^p c_k d_L([x_1(t)]^\alpha, [y_1(t)]^\alpha)
where $k$ and $h$ satisfy the inequality (5) and (6) respectively. Choose $T$ such that $T < (1 - ch)/ek$. Then all conditions stated in Theorem 3.1 are satisfied, so the problem (10)-(11) has a unique fuzzy solution.

Let target set is $x^1 = (x^1_1, x^1_2) = (2, 3)$. The $\alpha$-level set of fuzzy numbers $\tilde{3}^{\alpha} = [\alpha + 2, 4 - \alpha]$.

From the definition of fuzzy solution,

\[
x^\alpha_{\alpha}(t) = S^\alpha_{\alpha}(t)(x_0^\alpha) - \sum_{k=1}^{p}(c_k)_1(x_\alpha^\alpha(t_k))
\]

\[
+ \int_0^T S^\alpha_{\alpha}(t - s)(\alpha + 1)s(x_\alpha^\alpha(s))^2 ds + \int_0^T S^\alpha_{\alpha}(t - s)u_\alpha^\alpha(s)ds
\]

and

\[
x^\alpha_{1r}(t) = S^\alpha_{1r}(t)(x_0^\alpha) - \sum_{k=1}^{p}(c_k)_1(x_\alpha^\alpha(t_k))
\]

\[
+ \int_0^T S^\alpha_{1r}(t - s)(3 - \alpha)s(x_\alpha^\alpha(s))^2 ds + \int_0^T S^\alpha_{1r}(t - s)u_\alpha^\alpha(s)ds,
\]

where $i = 1, 2$.

Thus the $\alpha$-level of $u(s)$ are

\[
u_\alpha^\alpha(s) = (\tilde{3}^{\alpha})^{-1}\left((\alpha + 1) - \sum_{k=1}^{p}(c_k)_1(x_\alpha^\alpha(t_k)) - \left[S^\alpha_{\alpha}(T)(x_0^\alpha) - \sum_{k=1}^{p}(c_k)_1(x_\alpha^\alpha(t_k)) + \int_0^T (\alpha + 1)S^\alpha_{\alpha}(T - s)s(x_\alpha^\alpha(s))^2 ds\right]\right),
\]

\[
u_\alpha_{1r}(s) = (\tilde{3}^{\alpha})^{-1}\left((3 - \alpha) - \sum_{k=1}^{p}(c_k)_1(x_\alpha^\alpha(t_k)) - \left[S^\alpha_{1r}(T)(x_0^\alpha) - \sum_{k=1}^{p}(c_k)_1(x_\alpha^\alpha(t_k)) + \int_0^T (3 - \alpha)S^\alpha_{1r}(T - s)s(x_\alpha^\alpha(s))^2 ds\right]\right),
\]

\[
u_{2l}(s) = (\tilde{3}^{\alpha})^{-1}\left((\alpha + 2) - \sum_{k=1}^{p}(c_k)_2(x_\alpha^\alpha(t_k)) - \left[S^\alpha_{2l}(T)(x_0^\alpha) - \sum_{k=1}^{p}(c_k)_2(x_\alpha^\alpha(t_k)) + \int_0^T (\alpha + 1)S^\alpha_{2l}(T - s)s(x_\alpha^\alpha(s))^2 ds\right]\right),
\]
Hence (10)-(11) is nonlocal controllable on $[0, T]$. Then all the conditions stated in Theorem 4.1 are satisfied, so the system (10)-(11) is nonlocal controllable on $[0, T]$. 

\[ u_{2r}(s) = (\tilde{\beta}_{2r})^{-1} \left( (4 - \alpha) - \sum_{k=1}^{p} (c_k)_2 (x_{2r}^k(t_k)) \right) - \left[ S_{2r}(T) \left( x_0^2 \right)_{2r} - \sum_{k=1}^{p} (c_k)(x_{2r}^k(t_k)) \right] + \int_{0}^{T} (3 - \alpha) S_{2r}(T - s) s (x_{2r}^2(s))^2 ds \]
References

Inverse problem for a heat equation in the bar with piecewise constant thermal conductivity

Junhong Ha
School of Liberal Arts, Korea University of Technology and Education
Cheonan, 330-708, KOREA
hjh@kut.ac.kr

Abstract
This paper is to summarize the identification results in [3, 4]. That is, the conductivity in a one dimensional heat conduction process can be uniquely identified from observations of the process at finitely many points. The main tool for the identification is the Marching Algorithm. Numerical experiments show that the algorithm identifies the discontinuity points and the conductivity values.

1 Introduction
Let $a(x), 0 < \nu \leq a(x) \leq \mu, x \in [0,1]$ be a variable conductivity in a rod of the unit length. Then the heat conduction in it can be described by

$$
\begin{align*}
  u_t - (a(x)u_x)_x &= f(x,t), & Q &= (0,1) \times (0,T), \\
  u(0,t) &= q_1(t), & u(1,t) &= q_2(t), & t &\in (0,T), \\
  u(x,0) &= g(x), & x &\in (0,1).
\end{align*}
$$

The parameter identification problem for (1.1) consists of finding variable parameters $a, f, q_1, q_2$ and $g$ such that the solution $u(x, t)$ fits given observations $z$ in a prescribed (e.g the best fit to data) sense.

The identifiability problem for (1.1) consists in establishing the uniqueness of the above identification.

Our goal is to study the conductivity identifiability problem, that is the functions $g, f, q_1, q_2$ are assumed to be known, and the conductivity $a$ is to be found from some observations of the system (1.1). Some identifiability results for smooth or constant conductivities were obtained previously, see [7,9,8].

The main goal of this paper is the identifiability results for piecewise constant coefficients $a(x)$ in (1.1), given observations of the process at finitely many points $p_k \in (0,1)$. We start by recalling some basic results in Section 2. In Section 3 Marching Algorithm
is given. Identifiability of piecewise constant conductivities $a$ is discussed in Section 4, where we present an algorithm for such an identification. In Section 5 the theoretical constructions are tested in several numerical experiments.

2 Auxiliary results

Let us give some descriptions of the classes of $a(x)$. See [3] for the detailed. The notation $\mathcal{PS}_N$ is the class of piecewise smooth function on $[0,1]$ with $N$ discontinuities. $\mathcal{PS} = \bigcup_{N=1}^{\infty} \mathcal{PS}_N$ and $\mathcal{PC} \subset \mathcal{PS}$ is the class of piecewise constant conductivities. $\mathcal{PC}_N = \mathcal{PC} \cap \mathcal{PS}_N$ and $\mathcal{PC}(\sigma) = \{a \in \mathcal{PC} : x_n - x_{n-1} \geq \sigma, \ n = 1, 2, \ldots, N\}$.

Everywhere in the sequel the conductivities $a$ are assumed to be in $\mathcal{PS}$. If $a \in \mathcal{PS}_N$ then the regularity conditions on $a$ and the uniqueness of the weak solutions imply that for any $t > 0$ the weak solution $u(x,t;a)$ of (1.1) satisfies the equation in the classical sense on any subinterval $(x_i, x_{i+1})$, $i = 0, \ldots, N - 1$. Also $u$ satisfies the matching conditions for the continuity of the solution and its conormal derivative at $x_i \in (0,1)$, $i = 1, 2, \ldots, N - 1$:

$$
\begin{align*}
&u_t - (a(x)u_x)_x = 0, \ x \neq x_i, \ t \in (0,T) \\
&u(0,t) = u(1,t) = 0, \ t \in (0,T), \\
&u(x_i+,t) = u(x_{i-},t), \\
&a(x_i+)u_x(x_i+,t) = a(x_{i-})u_x(x_{i-},t), \\
&u(x,0) = g(x), \ x \in (0,1),
\end{align*}
$$

(2.2)

where $g \in L^2(0,1), q_1, q_2 \in C^1[0,T]$ and $f \in L^2(Q)$.

We collect some results for the solution $u(x,t;a)$ of (1.1), as well as for its associated Sturm-Liouville problem, see [2,4,5].

**Theorem 2.1.** Let $a \in \mathcal{PS}_N$. Then

i. The associated Sturm-Liouville problem

$$
\begin{align*}
&(a(x)v(x))' = -\lambda v(x), \ x \neq x_i, \\
v(0) = v(1) = 0, \\
v(x_i+) = v(x_{i-}), \\
a(x_i+)v_x(x_i+) = a(x_{i-})v_x(x_{i-})
\end{align*}
$$

(2.3)

has infinitely many eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots \rightarrow \infty$$

and corresponding orthonormal eigenfunctions $\{\psi_k\}_{k=1}^{\infty}$ satisfying

$$
\lambda_k = \int_0^1 a(x)[\psi'_k(x)]^2 dx.
$$

(2.4)
The normalized eigenfunctions \( \{ \psi_k \}_{k=1}^{\infty} \) form a basis in \( L^2(0,1) \).

ii. Each eigenvalue is simple. For each eigenvalue \( \lambda_k \) there exists a unique continuous, piecewise smooth normalized eigenfunction \( \psi_k(x) \) such that \( \psi'_k(0+) > 0 \), and the function \( a(x)\psi'_k(x) \) is continuous on \([0,1]\).

iii. Eigenvalues \( \{ \lambda_k \}_{k=1}^{\infty} \) satisfy the inequality
\[
\nu \pi^2 k^2 \leq \lambda_k \leq \mu \pi^2 k^2.
\]

iv. First eigenfunction \( \psi_1 \) satisfies \( \psi_1(x) > 0 \) for any \( x \in (0,1) \).

v. First eigenfunction \( \psi_1 \) has a unique point of maximum \( q \in (0,1) \) : \( \psi_1(x) < \psi_1(q) \) for any \( x \neq q \).

In general, the solution of (1.1) is understood in the weak sense. See [6] for the existence and continuity properties of the weak solutions of (1.1). When the conductivity \( a \) is piecewise smooth one can obtain the following representation of its solution.

Denote by \( \| \cdot \|, < \cdot, \cdot > \) the norm and the inner product in \( L^2(0,1) \).

**Theorem 2.2.** Let \( a \in A_{ad} \), \( f \in L^2(Q) \), \( g \in L^2(0,1) \), \( q_1 \), \( q_2 \in C^1[0,T] \). Then the solution \( u \) of (2.2) is given by
\[
\begin{align*}
    u(x,t; a) &= \sum_{k=1}^{\infty} < g, \psi_k > e^{-\lambda_k t} \psi_k(x) \\
    &\quad + \int_0^t \left[ \sum_{k=1}^{\infty} < f(s,\tau), \psi_k(s) > e^{-\lambda_k (t-\tau)} \psi_k(x) \right] d\tau \\
    &\quad + a(0) \int_0^t q_1(\tau) \left[ \sum_{k=1}^{\infty} \psi'_k(0) e^{-\lambda_k (t-\tau)} \psi_k(x) \right] d\tau \\
    &\quad - a(1) \int_0^t q_2(\tau) \left[ \sum_{k=1}^{\infty} \psi'_k(1) e^{-\lambda_k (t-\tau)} \psi_k(x) \right] d\tau.
\end{align*}
\]

**3 Marching Algorithm**

In this section, the Marching Algorithm which is the main tool for the identification is established. The algorithm is related to Sturm-Liouville problem.

Since \( a \in PC_N \) has the form \( a(x) = a_i \) for \( x \in [x_{i-1}, x_i) \), \( i = 1, 2, ..., N \), the governing system (2.2) is
\[
\begin{align*}
    u_t - a_i u_{xx} &= 0, \quad x \in (x_{i-1}, x_i), \quad t \in (0,T) \\
    u(0,t) &= u(1,t) = 0, \quad t \in (0,T), \\
    u(x_i+, t) &= u(x_i-, t), \\
    a_{i+1} u_x(x_i+, t) &= a_i u_x(x_i-, t), \\
    u(x,0) &= g(x), \quad x \in (0,1).
\end{align*}
\]
where $g \in L^2(0,1)$ and $i = 1, 2, \ldots, N - 1$. The associated Sturm-Liouville problem is
\begin{equation}
\begin{aligned}
a_i v''(x) &= -\lambda v(x), \quad x \in (x_{i-1}, x_i) \\
v(0) &= v(1) = 0, \\
v(x_{i+}) &= v(x_i-), \\
a_{i+1} v'(x_{i+}) &= a_i v'(x_i-)
\end{aligned}
\end{equation}
for $i = 1, 2, \ldots, N - 1$.

We are interested only in the first eigenfunction $v_1$ of (3.7). Let $\lambda_1$ be the first eigenvalue. Suppose that $p^* \in (x_{i-1}, x_i)$. Then
\[ v_1(x) = A \cos \left( \frac{\lambda_1}{a_i} (x - p^*) + \gamma \right), \quad A > 0, \quad -\frac{\pi}{2} < \gamma < \frac{\pi}{2} \]
where $x \in (x_{i-1}, x_i)$. The range for $\gamma$ in the above representation follows from the fact that $v_1(p^*) = A \cos \gamma > 0$.

The identifiability of piecewise constant conductivities is based on the following three Lemmas.

Fix $\delta > 0$. Let
\[ P = \left\{ (A, \omega, \gamma) : A > 0, \quad 0 < \omega < \frac{\pi}{2\delta}, \quad -\frac{\pi}{2} < \gamma < \frac{\pi}{2} \right\} . \]

**Lemma 3.1.** Assume $Q_1, Q_3 \geq 0$, $Q_2 > 0$ and $0 < Q_1 + Q_3 < 2Q_2$. Then the system of equations
\[ A \cos(\omega \delta - \gamma) = Q_1, \quad A \cos \gamma = Q_2, \quad A \cos(\omega \delta + \gamma) = Q_3 \]
has a unique solution $(A, \omega, \gamma) \in P$ given by
\[ \omega = \frac{1}{\delta} \arccos \frac{Q_1 + Q_3}{2Q_2}, \quad \gamma = \arctan \left( \frac{Q_1 - Q_3}{2Q_2 \sin \omega \delta} \right), \quad A = \frac{Q_2}{\cos \gamma} . \]

Note that function $A \cos(\omega(x - p^*) + \gamma)$ interpolates the points $(p^* - \delta, Q_1), (p^*, Q_2)$ and $(p^* + \delta, Q_3)$.

**Lemma 3.2.** Let $(B_1, \Omega_1, \Gamma_1), (B_2, \Omega_2, \Gamma_2) \in P, \Omega_1 \neq \Omega_2$. Also let $0 < \eta \leq 2\delta$, $0 \leq p < p + \eta \leq 1$ and
\[ w(x) = B_1 \cos[\Omega_1(x - p) + \Gamma_1], \quad v(x) = B_2 \cos[\Omega_2(x - p - \eta) + \Gamma_2] . \]

Then system
\begin{equation}
\begin{aligned}
w(q) &= v(q), \\
\Omega_2^2 w(q) &= \Omega_1^2 v(q), \\
w(q) &= w(q) > 0, \quad v(q) > 0
\end{aligned}
\end{equation}
admits at most one solution $q$ on $[p, p + \eta]$. This unique solution $q$ can be found as follows:
If $\Gamma_1 \geq 0$ then
\[
q = p + \frac{1}{\Omega_1} \left[ \arctan \left( \Omega_1 \sqrt{\frac{B_2^2 - B_1^2}{B_1^2\Omega_2^2 - B_2^2\Omega_1^2}} \right) - \Gamma_1 \right].
\] (3.9)

If $\Gamma_2 \leq 0$ then
\[
q = p + \eta + \frac{1}{\Omega_2} \left[ - \arctan \left( \frac{B_2^2 - B_1^2}{B_1^2\Omega_2^2 - B_2^2\Omega_1^2} \right) - \Gamma_2 \right].
\] (3.10)

Otherwise compute $q_1$ and $q_2$ according to formulas (3.9) and (3.10) and discard the one that does not satisfy the conditions (3.8) of the Lemma.

**Marching Algorithm**

The algorithm uniquely identifies piecewise constant conductivity $a \in \mathcal{PC}(\sigma)$ from the first eigenvalue $\lambda_1$ and a nonzero multiple of the first eigenfunction $\beta\psi_1(p_m)$ of (2.3), where $p_m = m/M$, $m = 1, 2, \ldots, M - 1$. The eigenvalue $\lambda_1$ is used only in the last step of the algorithm. The Marching algorithm identifies the number $N - 1$ of the discontinuities of $a$ on the interval $(0, 1)$, their locations $\{x_i\}_{i=1}^{N-1}$ as well as the values $\{a_i\}_{i=1}^N$ of the conductivity $a$ between the discontinuity points, see (3.6).

**Step 1.** Let $G_m = \beta\psi_1(p_m)$ for $m = 1, 2, \ldots, M - 1$ and $G_0 = G_M = 0$.

**Step 2.** Find $l$, $0 < l < M$ such that $G_l = \max\{G_m : m = 1, 2, \ldots, M - 1\}$ and $G_m < G_l$ for any $0 \leq m < l$.

**Step 3.** Let $i = 1$, $m = 0$.

**Step 4.** Let $Q_1 = G_0$, $Q_2 = G_1$, $Q_3 = G_2$ and find $(A_i, \omega_i, \gamma_i)$ using formulas (3.8) in Lemma 3.1. Let $p_i^* = p_{m+1}$ and $H_i(x) = A_i \cos(\omega_i(x - p_i^*) + \gamma_i)$.

**Step 5.** If $m + 3 \geq l$ then go to Step 8.

If $H_i(p_{m+3}) = G_{m+3}$ and $H_i'(p_{m+3}) > 0$ then let $m := m + 1$ and repeat Step 5. Otherwise proceed to Step 6 ($a$ has a discontinuity $x_i$ on interval $[p_{m+2}, p_{m+3})$).

**Step 6.** Let $Q_1 = G_{m+3}$, $Q_2 = G_{m+4}$, $Q_3 = G_{m+5}$ and find $(A_{i+1}, \omega_{i+1}, \gamma_{i+1})$ using formulas (3.8) in Lemma 3.1. Let $p_{i+1}^* = p_{m+4}$ and $H_{i+1}(x) = A_{i+1} \cos(\omega_{i+1}(x - p_{i+1}^*) + \gamma_{i+1})$.

**Step 7.** Let $Q_1 = H_i(p_{m+1})$, $Q_2 = H_i(p_{m+2})$, $Q_3 = H_i(p_{m+3})$ and find $(B_1, \Omega_1, \Gamma_1) \in P$ using formulas (3.8) in Lemma 3.1. Let
\[
w(x) = B_1 \cos(\Omega_1(x - p_{m+2}) + \Gamma_1), \quad |\Gamma_1| < \pi/2.
\]
Let $Q_1 = H_{i+1}(p_{m+2})$, $Q_2 = H_{i+1}(p_{m+3})$, $Q_3 = H_{i+1}(p_{m+4})$ and find $(B_2, \Omega_2, \Gamma_2) \in P$ using formulas (3.8) in Lemma 3.1. Let
\[ v(x) = B_2 \cos(\Omega_2(x - p_{m+3}) + \Gamma_2), \quad |\Gamma_2| < \pi/2. \]
Let $\eta = \delta$ and find the discontinuity $q \in [p_{m+2}, p_{m+3})$ using Lemma 3.2. Let $x_i = q$.

If $m < l$ then return to Step 5.

If $m \geq l$ then go to Step 8.

Step 8. Do Steps 3-7 in the reverse direction of $x$, advancing from $x = 1$ to $x = p_{l+1}$.

Identify the values and the discontinuity points of $a$ on $[p_{l+1}, 1]$, as well as determine the corresponding functions $H_i(x)$.

Step 9. By now the discontinuities $x_i$ of $a$ on $[0, p_{l-1})$ as well as the other related quantities $\omega_i$, $A_i$, $p_i^*$, $\gamma_i$ have been determined in Steps 3-7. Similarly, the discontinuities $x_j$ of $a$ on $(p_{l+1}, 1]$ have been determined in Step 8. The only issue left is whether there is an additional discontinuity $q$ of $a$ on interval $[p_{l-1}, p_{l+1}]$. Reorder all the previously determined discontinuities on $[0, 1]$ in the increasing order. Let $x_i$ be the largest discontinuity on $[0, p_{l-1})$, and $x_j$ be the smallest one on $(p_{l+1}, 1]$.

If $\omega_{i+1} = \omega_j$ then there is no additional discontinuity. Proceed to Step 10.

If $\omega_{i+1} \neq \omega_j$ then there is an additional discontinuity $q$ of $a$ on $[p_{l-1}, p_{l+1}]$. To find it let $H_{i+1}(x) = A_{i+1} \cos(\omega_{i+1}(x - p_{i+1}^*) + \gamma_{i+1})$.

Let $Q_1 = H_{i+1}(p_{l-2})$, $Q_2 = H_{i+1}(p_{l-1})$, $Q_3 = H_{i+1}(p_l)$. Find $(B_1, \Omega_1, \Gamma_1) \in P$ using formulas (3.8) in Lemma 3.1.

Let $H_j(x) = A_j \cos(\omega_j(x - p_j^*) + \gamma_j)$ and $Q_1 = H_j(p_l)$, $Q_2 = H_j(p_{l+1})$, $Q_3 = H_j(p_{l+2})$. Find $(B_2, \Omega_2, \Gamma_2) \in P$ using formulas (3.8) in Lemma 3.1.

Let $\eta = 2\delta$ and find the discontinuity $q \in [p_{l-1}, p_{l+1}]$ using Lemma 3.2.

Step 10. Let $\{x_i\}_{i=1}^{N-1} \subset (0, 1)$ be the set of all found discontinuity points. Let $x_0 = 0$, $x_N = 1$.

Find the conductivity $a(x) = a_i$, $x \in [x_{i-1}, x_i)$, $i = 1, 2, \cdots, N$ by
\[ a_i = \frac{\lambda_1}{\omega_i^2}. \]

Stop.

4 Identifiability of piecewise constant conductivities

In this section we present the identifiability of piecewise constant conductivities. Besides the Marching Algorithm we need the following result on the uniqueness of the representation of a function by a Dirichlet series, see [3, 10].
Lemma 4.1. Let $\lambda_k > 0$, $k = 1, 2, \cdots$ be a strictly increasing sequence. Suppose that $T_1 \geq 0$ and $\sum_{k=1}^{\infty} |C_k| < \infty$. If

$$g(t) = \sum_{k=1}^{\infty} C_k e^{-\lambda_k t} \quad \text{for all } t \in (T_1, T_2)$$

then the exponents $\lambda_k$ and the corresponding coefficients $C_k \neq 0$, $k = 1, 2, \cdots$ are uniquely identifiable from this representation.

A numerical determination of the exponents and the coefficients from the Dirichlet series representation of $g$ is difficult since it is an ill-posed problem, see [1, What not to compute, p. 252]. A numerical method for the extraction of data for the Marching Algorithm from the Dirichlet series representation is discussed in the next section.

The following theorem is main in this paper and see [4] for the proof.

Theorem 4.1. Given $\sigma > 0$ let an integer $M$ be such that

$$M \geq \frac{3}{\sigma} \quad \text{and} \quad M > 2 \sqrt{\frac{\mu}{\nu}}.$$ 

Suppose that the observations $z_m(t) = u(p_m, t; a)$ for $p_m = m/M$, $m = 1, 2, \cdots, M - 1$ and $0 \leq T_1 < t < T_2$ of the heat conduction process (3.6) are given. Then the conductivity $a \in A_{ad}$ is identifiable in the class of piecewise constant functions $\mathcal{PC}(\sigma)$ in either one of the following five cases.

i. $f = 0$, $q_1 = 0$, $q_2 = 0$, $g > 0$, $g \in L^2(0, 1)$.

ii. $g = 0$, $q_1 = 0$, $q_2 = 0$, $f(x, t) = h(x)r(t) \neq 0$, $h > 0$, $h \in L^2(0, 1)$, $r \in L^2(0, T)$.

iii. $g = 0$, $f = 0$, $q_2 = 0$, $q_1 \neq 0$, $q_1 \in C^1[T_1, T_2]$.

iv. $g = 0$, $f = 0$, $q_1 = 0$, $q_2 \neq 0$, $q_2 \in C^1[T_1, T_2]$.

v. $g > 0$, $f = 0$, $q_2 = 0$ and there exist $L > 0$, $\zeta > 0$ such that $q_1(t + 2L) = q_1(t)$ for $0 < t < \zeta$. In addition we assume that there exists $0 \leq \xi < 2L$ such that $q_1(t)$ is nonnegative on $(0, \xi)$, nonpositive on $(\xi, 2L)$ and

$$\int_0^{2L} q_1(\tau) \, d\tau \leq 0. \quad (4.11)$$

Here $g \in L^2(0, 1)$, $q_1 \in C^1[T_1, T_2]$.

Remark. Condition (4.11) is satisfied for functions $q_1(t) = \sin \frac{\pi t}{L}$, a $2L$-periodic function, $\int_0^{2L} q_1(t)dt \leq 0$ as well as for many others.
5 Numerical results

The Marching Algorithm requires the first eigenvalue $\lambda_1$ of (3.6) and the data sequence $\{G_m = \beta \psi_1(p_m)\}_{m=1}^{M-1}$ as its input. They have to be obtained from the observations $z_m(t), \ m = 1, \ldots, M - 1, \ T_1 < t < T_2$. We have shown in Theorem 4.1 that this information can, in principle, be recovered from the observations because in every case considered in Theorem 4.1 the observations $z_m(t)$ or their combinations are represented by a Dirichlet series. However, it is known that the recovery of the coefficients and the exponents from a Dirichlet series is an ill-posed problem, see section 4.

For definiteness let us consider case (i) of Theorem 4.1. Other cases are treated similarly. In this case

$$z_m(t) = u(p_m, t) = \sum_{k=1}^{\infty} \langle g, \psi_k \rangle \psi_k(p_m) e^{-\lambda_k t},$$

$m = 1, 2, \ldots, M - 1, \ T_1 < t < T_2$

with $\lambda_1 < \lambda_2 < \ldots$. Our task is somewhat simplified since we only need to find the first eigenvalue $\lambda_1$ and the values $G_m = \langle g, \psi_1 \rangle \psi_1(p_m), \ m = 1, 2, \ldots, M - 1$.

One can show (see [3]) that the Dirichlet series in the above formula is a real analytic function for $t > 0$. We therefore assume that the data $z_m(t)$ is given for all $t > 0$. Arguing as in the proof of Theorem on the uniqueness for the Dirichlet series representation in [4] we have

$$\lim_{t \to \infty} e^{\lambda_1 t} z_m(t) = \lim_{t \to \infty} \sum_{k=1}^{\infty} \langle g, \psi_k \rangle \psi_k(p_m) e^{(\lambda_1 - \lambda_k) t} = \langle g, \psi_1 \rangle \psi_1(p_m).$$

Fix an $h > 0$. Then

$$\lim_{t \to \infty} \frac{z_m(t + h)}{z_m(t)} = e^{-\lambda_1 h} \lim_{t \to \infty} \frac{e^{\lambda_1 t} z_m(t + h)}{e^{\lambda_1 t} z_m(t)} = e^{-\lambda_1 h}.$$

This observation is the basis for the following algorithm for the recovery of the first eigenvalue $\lambda_1$ and the sequence $G_m = \langle g, \psi_1 \rangle \psi_1(p_m), \ m = 1, 2, \ldots, M - 1$.

**Dirichlet series algorithm**

Fix an $h > 0$, $\epsilon > 0$ and an integer $n$ between 1 and $M - 1$.

i. Let $j = 0$, $\xi_{old} = -1$.

ii. Let $j := j + 1$. Let $t = jh$. Compute

$$\xi_{new} = \frac{1}{h} \ln \frac{z_n(t + h)}{z_n(t)}.$$
iii. If \(|\xi_{new} - \xi_{old}| > \epsilon\) then let \(\xi_{new} = \xi_{old}\) and repeat Step 2. Otherwise proceed to the next Step.

iv. Let \(\lambda_1 = \xi_{new}\) and \(G_m = z_m(t + h)e^{\lambda_1(t + h)}\), \(m = 1, 2, \cdots, M - 1\). Stop.

We describe one set of numerical experiments for the conductivity identification. All the variables are consistent with the notations introduced in the earlier sections. Let the test conductivity be

\[
a(x) = \begin{cases} 
0.5, & 0 < x < 0.25, \\
0.8, & 0.25 < x < 0.37, \\
0.5, & 0.37 < x < 0.53, \\
0.2, & 0.53 < x < 0.7, \\
0.7, & 0.7 < x < 0.831, \\
0.4, & 0.831 < x < 1. 
\end{cases}
\]

Thus \(N = 6\) and the discontinuity points \(x_i, \ i = 1, 2, \cdots, 5\) are \(x_1 = 0.25, \ x_2 = 0.37\) etc. The values of \(\nu = 0.1\) and \(\mu = 1.0\) are fixed. The solution of (3.6) is given by (2.5). To compute it one has to find the eigenvalues and the eigenfunctions of (2.3). The first eigenvalue for this example is \(\lambda_1 = 4.67549602\), which is obtained via the shooting method. To compute the solution according to (2.5) we used the initial condition

\[
g(x) = \sin(\pi x)
\]

and the boundary input

\[
q_1(t) = \sin \frac{\pi t}{L}
\]

with \(L = 0.2\).

The observation points \(p_m\) were generated with \(M = 50\), that is \(\delta = 0.02\). The observation data \(z_m(t)\) was computed at the time instants \(hj, \ j = 1, 2, \cdots, 40\), where \(h = 0.1\). This value of \(h\) was used in the Dirichlet series algorithm with \(n = 6\) (arbitrarily chosen). The tolerance \(\epsilon\) was set at \(\epsilon = 10^{-8}\). The algorithm has stopped after 19 iterations. The results of this iterative identification of the first eigenvalue \(\lambda_1\) are shown in Table 2.

The identified value of \(\lambda_1 = 4.67549603\) and the data \(G_m\) were supplied to the Marching Algorithm according to Theorem 4.1. The relative \(L^1\) error \(E\) between the original \(a(x)\) and the identified \(a_{id}(x)\) conductivities for various noise levels \(\eta\) in the data are shown in Tables 3 and 4. The error \(E\) is defined as

\[
E = \frac{\int_0^1 |a_{id}(x) - a(x)| \, dx}{\int_0^1 a(x) \, dx}.
\]

Let \(r(\zeta)\) be a random variable uniformly distributed on interval \([0, 1]\). Data contaminated by noise of level \(\eta\) is defined as \(\hat{G}_m = G_m(1 + \eta(r(\zeta) - 0.5))\). Similar definition is used for the perturbation \(\hat{\lambda}_1\) of the first eigenvalue.
Table 1: Identification of $\lambda_1$ by the Dirichlet series algorithm.

<table>
<thead>
<tr>
<th>Iterations</th>
<th>$\lambda_1$</th>
<th>Iterations</th>
<th>$\lambda_1$</th>
</tr>
</thead>
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<td>1</td>
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<td>4.67551060</td>
</tr>
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<td>4.78755032</td>
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<td>4.71670051</td>
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<td>4</td>
<td>4.69077319</td>
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<td>6</td>
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</tr>
<tr>
<td>7</td>
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<td>17</td>
<td>4.67549606</td>
</tr>
<tr>
<td>8</td>
<td>4.67578338</td>
<td>18</td>
<td>4.67549603</td>
</tr>
<tr>
<td>9</td>
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<td>19</td>
<td>4.67549603</td>
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<tr>
<td>10</td>
<td>4.67553540</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Tables 2 and 3 show typical results of the conductivity identification corresponding to cases (i) and (iii) of Theorem 4.1. The algorithm also produces similar results in case (v) of Theorem 4.1. A typical program run time was about 1 second on a 2.8 MHz PC in either case.

References


Table 2: Conductivity identification in case (i) of Theorem 4.1.

<table>
<thead>
<tr>
<th>conductivities</th>
<th>$\eta = 0.0$</th>
<th>$\eta = 0.000001$</th>
<th>$\eta = 0.000003$</th>
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<tr>
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<td>0.5001</td>
<td>0.4999</td>
</tr>
<tr>
<td>$a_2$</td>
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<td>0.8004</td>
<td>0.8006</td>
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<td>0.4998</td>
</tr>
<tr>
<td>$a_4$</td>
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<td>0.2000</td>
<td>0.2000</td>
</tr>
<tr>
<td>$a_5$</td>
<td>0.7000</td>
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<td>0.6996</td>
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<tr>
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<td>0.4000</td>
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Table 3: Conductivity identification in case (iii) of Theorem 4.1.

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<td>0.2000</td>
<td>0.2000</td>
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Positive solutions of quasilinear elliptic
equations involving indefinite lower term

Kimiaki Narukawa

Department of Mathematics, Naruto University of Education
Takashima, Naruto 772-8502, Japan
e-mail knaru@naruto-u.ac.jp

1 Introduction

Let us consider a quasilinear elliptic boundary value problem in the form

\[
\begin{aligned}
-\text{div}(\phi(|\nabla u|)\nabla u) &= \lambda a(x)f(u) + b(x)g(u) \quad \text{in } \Omega, \\
\quad u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]

(1.1)

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\), \(N \geq 2\), \(a(x), b(x)\) are continuous functions in \(\Omega\), which may change their signs, \(f(u)\) and \(g(u)\) are nonnegative continuous functions and \(\lambda\) a positive parameter.

The existence and multiplicity of positive solutions for the problem (1.1) have been studied by many authors since the original work of Brezis and Nirenberg [4]. In [4], they give the existence and nonexistence results of positive solutions for the type of problem (1.1) in the case when \(\phi(t) = 1\), \(a(x) = b(x) = 1\), \(f(u) = u\) and \(g(u) = u^{\frac{N+q}{N-2q}}\), i.e., semilinear equations with critical nonlinearity. Following this, Ambrosetti, Brezis and Cerami [2] also study the semilinear equation with concave convex nonlinearities, that is, \(f(u) = u^{q-1}\), \(g(u) = u^{p-1}\) with \(1 < q < 2 < p \leq \frac{2N}{N-2}\), with \(a(x) = b(x) = 1\). Further, the problem for the \(p\)-Laplace equation, i.e., \(\phi(t) = t^{p-1}\), with constant coefficients has also been studied e.g. in [3], [9]. Following these works, de Figueiredo, Gossez and Ubilla investigated the semilinear equations and \(p\)-Laplace equations with variable coefficients \(a(x), b(x)\) in [6] and [7] respectively. In the case for the general \(\phi(t)\), Fukagai and Narukawa [8] give the multiple existence of positive solutions under the subcritical condition with constant coefficients. However, the general quasilinear equation (1.1) with indefinite nonlinearity does not seem to be treated well.

Here we consider the problem (1.1) with indefinite nonlinearity, including the critical case, and give the multiple existences of positive solutions.
2 Assumptions on $\phi(t)$ and Orlicz-Sobolev spaces

In order to state the results, we give assumptions on $\phi(t)$ and the definition of Orlicz-Sobolev spaces. Put
\[
\Phi(t) = \int_0^t \phi(s)ds \quad \text{for} \quad t \geq 0.
\]
Throughout the followings, we assume that $\phi(t)$ satisfies the hypotheses:
1. $\phi(t) \in C^1(0, \infty)$, increasing in $t > 0$,
2. there exist constants $\ell, m > 1$ such that
   \[
   \ell \leq \frac{t\Phi'(t)}{\Phi(t)} \leq m,
   \]
3. there exist constants $c_0, c_1 > 0$ such that
   \[
   c_0 \leq \frac{t\Phi''(t)}{\Phi'(t)} \leq c_1.
   \]

For the function $\Phi(t)$ satisfying the conditions above, we define the Orlicz space $L_\Phi(\Omega)$ as
\[
L_\Phi(\Omega) = \left\{ u : \text{measurable on } \Omega \left| \int_\Omega \Phi(|u(x)|)dx < \infty \right. \right\}
\]
with the norm
\[
\|u\|_\Phi = \inf \left\{ k > 0 \left| \int_\Omega \Phi(|u(x)|)dx \leq 1 \right. \right\},
\]
and the Orlicz-Sobolev space $W^{1}_0L_\Phi(\Omega)$ as the completion of $C^\infty_0(\Omega)$ with the norm
\[
\|u\|_{W^{1}_0L_\Phi(\Omega)} = \|\nabla u\|_\Phi + \|u\|_\Phi.
\]
From the Poincaré inequality
\[
\|u\|_\Phi \leq c\|\nabla u\|_\Phi, \quad u \in W^{1}_0L_\Phi(\Omega),
\]
we adopt $\|\nabla u\|_\Phi$ as the norm of $W^{1}_0L_\Phi(\Omega)$.

There are several examples of $\Phi(t)$ satisfying the assumptions above, which appear in the field of physics, e.g.,
- i) nonlinear elasticity: $\Phi(t) = (1 + t^2)^\gamma - 1, \gamma > \frac{1}{2}$,
- ii) plasticity: $\Phi(t) = t^\alpha (\log(1 + t))^\beta, \alpha \geq 1, \beta > 0$,
- iii) generalized Newtonian fluids: $\Phi(t) = \int_0^t s^{1-\alpha}(\sinh^{-1} s)^\beta ds, 0 \leq \alpha \leq 1, \beta > 0$.

3 Existence of a first solution

In the followings, let $a(x)$ and $b(x)$ be assumed to be continuous on $\overline{\Omega}$ and $f(t), g(t) \in C^0[0, \infty), f(0) = g(0) = 0$ and $f(t), g(t) > 0$ for $t > 0$. By putting 0 on the interval $(-\infty, 0)$, the functions $f(t), g(t)$ are assumed to be defined on $(-\infty, \infty)$.

Then we have
Theorem 3.1 Let $f$, $g$ satisfy
\[
\begin{align*}
(f_0) & \quad \frac{f(t)}{\phi(t)t} \to \infty \text{ as } t \to 0, \\
(g_0) & \quad \frac{g(t)}{\phi(t)t} \to 0 \text{ as } t \to \infty.
\end{align*}
\]
If there exists $x_0 \in \Omega$ with $a(x_0) > 0$, then there exists $\lambda_0 > 0$ such that (1.1) has a positive solution $u_\lambda$ in $W_0^1 L_\Phi(\Omega) \cap L^\infty(\Omega)$ for any $\lambda \in (0, \lambda_0)$.

Outline of the proof. The energy functional attached to the problem (1.1) is naturally given by
\[
I_\lambda(u) = \int_\Omega \Phi(|\nabla u|)dx - \lambda \int_\Omega a(x)F(u)dx - \int_\Omega b(x)G(u)dx,
\]
where
\[
F(t) = \int_0^t f(s)ds, \quad G(t) = \int_0^t g(s)ds.
\]
Without growth condition on $g(t)$ as $t \to \infty$, the functional $I_\lambda$ does not converge in general. Thus we consider a cut-off function
\[
g_\mu(t) = \begin{cases} 
g(\mu) & (t \geq \mu) \\
g(t) & (t \leq \mu) \end{cases}
\]
for $\mu > 0$, and the functional
\[
I_\mu^\mu(u) = \int_\Omega \Phi(|\nabla u|)dx - \lambda \int_\Omega a(x)F(u^+)dx - \int_\Omega b(x)G_\mu(u^+)dx,
\]
where
\[
G_\mu(t) = \int_0^t g_\mu(s)ds,
\]
and $u^+(x) = \max\{u(x), 0\}$. Then $I_\mu^\mu$ is well-defined on the space $W_0^1 L_\Phi(\Omega)$ and continuously Frechét differentiable in this space. Taking $\mu > 0$ sufficiently small, from the growth conditions $(f_0)$, $(g_0)$, we see that there exists $\lambda_1 > 0$ independent of $\mu$ such that $I_\mu^\mu$ has a local minimizer $u_\lambda^\mu$ in the ball $\{u \in W_0^1 L_\Phi(\Omega) \mid \|\nabla u\|_\Phi < \rho_\lambda\}$ with a positive constant $\rho_\lambda$ independent of $\mu$ and $\rho_\lambda \to 0$ as $\mu \to 0$. Since
\[
\inf \{I_\mu^\mu(u) \mid u \in W_0^1 L_\Phi(\Omega), \|\nabla u\|_\Phi < \rho_\lambda\} < 0,
\]
u_\lambda^\mu is nonzero. Further the Euler equation
\[
-\text{div}(\phi(|\nabla u_\lambda^\mu|)|\nabla u_\lambda^\mu|) = \lambda a(x)f((u_\lambda^\mu)^+) + b(x)g_\mu((u_\lambda^\mu)^+)
\]
holds in $\Omega$. Since $\|\nabla u_\lambda^\mu\|_\Phi \to 0$ as $\lambda \to 0$, by apriori estimate, we have
\[
\|\nabla u_\lambda^\mu\|_{L^\infty(\Omega)} \leq c_\lambda < \mu \quad \text{for small } \lambda > 0.
\]
Further, $u_\lambda^\mu$ is positive by maximum principle. These show that $u_\lambda^\mu$ is a positive solution of the original problem (1.1).

Now let us put
\[
\Lambda_0 = \sup \{\lambda > 0 \mid \text{there exists a positive solution for (1.1)}\}.
\]

Then, applying super-, subsolution arguments, we have
**Theorem 3.2** In addition to the assumptions in Theorem 3.1, assume
\[ a(x) \geq 0, \quad a(x) \neq 0 \quad \text{in } \Omega. \]

Then, there exists a positive solution \( u_\lambda \) of (1.1) in \( W^1_0 L_\Phi(\Omega) \cap L^\infty(\Omega) \) for any \( \lambda \in (0, \Lambda_0) \).

**Remark 3.1.** There exists the case \( \Lambda_0 = \infty \). In fact, if \( b(x) < 0 \) in \( \Omega \), then \( \Lambda_0 = \infty \).

Contrary to Remark 3.1, we have

**Theorem 3.3** Let us assume \( (f_0) \),
\[ (g_\infty) \quad \frac{g(t)}{\phi(t)} \to \infty \quad \text{as } t \to \infty, \]
and
\[ \phi(t) = \begin{cases} c_0 t^{p-1} + o(t^{p-1}) & \text{as } t \to 0, \\ c_1 t^{p-1} + o(t^{p-1}) & \text{as } t \to \infty \end{cases} \]
with some constants \( c_0, c_1 > 0 \).

If
\[ \{ x \in \Omega \mid a(x) > 0 \} \cap \{ x \in \Omega \mid b(x) > 0 \} \neq \emptyset, \]
then \( \Lambda_0 < \infty \).

**Proof.** Let us consider an eigenvalue problem
\[ \begin{cases} -\text{div}(\alpha_\lambda(x)|\nabla u|^p \nabla u) = \lambda |u|^{p-2}u & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{3.1} \]
where \( B \) is a ball contained in \( \{ x \in \Omega \mid a(x) > 0 \} \cap \{ x \in \Omega \mid b(x) > 0 \} \) and
\[ \alpha_\lambda(x) = \frac{\phi(|\nabla u_\lambda|)}{|\nabla u_\lambda|^{p-2}} \]
for a positive solution \( u_\lambda \) of (1.1). Since \( c_2 \leq \alpha(x) \leq c_3 \) with constants \( c_2, c_3 > 0 \) independent on \( \lambda \), the first eigenvalue \( \lambda_0^\alpha \) of (3.1) is estimated as
\[ \lambda_0^\alpha = \inf_{u \in W^{1,p}_0(B) \setminus \{0\}} \frac{\int_B \alpha(x)|\nabla u|^p}{\int_B |u|^p dx} \leq c_3 \inf_{u \in W^{1,p}_0(B) \setminus \{0\}} \frac{\int_B |\nabla u|^p}{\int_B |u|^p dx} = c_3 \lambda_0, \]
where \( \lambda_0 \) is the first eigenvalue of \( p \)-Laplacian in the ball \( B \). Take \( \overline{\lambda} > \lambda_0^\alpha \). If \( \lambda \) is sufficiently large,
\[ \lambda a(x)f(t) + b(x)g(t) \geq \overline{\lambda}^p \quad \text{for } t > 0, \quad x \in B. \tag{3.2} \]

Let assume that there exists a positive solution \( u_\lambda \) for this \( \lambda \). Then the inequality (3.2) shows that \( u_\lambda \) is a supersolution of (3.1) with \( \lambda = \overline{\lambda} \). The eigenfunction of (3.1) for the first eigenvalue \( \lambda = \lambda_0^\alpha \) is a subsolution of (3.1) with \( \lambda = \overline{\lambda} \). Then, by the super-, subsolution arguments, there exists a positive solution of (3.1) with \( \lambda = \overline{\lambda} \). This contradicts that \( \lambda_0^\alpha \) is the first eigenvalue of (3.1). Thus there is no positive solution for large \( \lambda \).

For \( \lambda = \Lambda_0 \), we have
**Theorem 3.4** In addition to the assumptions in Theorem 3.3,
\[ G(t) = o(t^{p^*-1}) \quad \text{as} \quad t \to \infty, \quad p^* = \frac{Np}{N-p} \]

Then, there exists a positive solution of (1.1) for \( \lambda = \Lambda_0 \).

### 4 Existence of a second solution - subcritical case

Let \( \Phi^*(t) \) be the Sobolev conjugate function of \( \Phi(t) \), which is defined by
\[
\Phi^{-1}(t) = \int_0^t \Phi^{-1}(s) \frac{ds}{s^{\frac{N}{p^*+1}}}.
\]

As for the properties of a Sobolev conjugate function, see e.g. [1]. In order to apply the strong maximum principle, we put the additional assumption:

\((g_1)\) for any \( s_0 > 0 \), there exists a constant \( \alpha = a(s_0) > 0 \) such that \( \alpha \phi(t)t + b(x)g(t) \) is increasing in \( t \) on \((0, s_0)\) for each \( x \in \Omega \).

Then, we have the multiplicity result of positive solutions in the subcritical case.

**Theorem 4.1** Let us assume the followings:

1. \( (f_0), (g_0), (g_1) \) and
   \[
   \lim_{t \to \infty} \frac{f(t)}{\phi(t)t} = 0 \quad \text{as} \quad t \to \infty,
   \]
   \[
   \lim_{t \to \infty} \frac{g(t)}{\phi(t)t} = \infty \quad \text{as} \quad t \to \infty,
   \]
2. \( G(t) \frac{\Phi^*(t)}{\Phi(t)} \to 0 \quad \text{as} \quad t \to \infty,
3. there exists a constant \( \theta > m \) such that
   \[
   \frac{\theta G(t) - g(t)t}{\Phi(t)} \to 0 \quad \text{as} \quad t \to \infty,
   \]
4. \( a(x) > 0 \) in \( \Omega \), and there exists \( x_0 \in \Omega \) such that \( b(x_0) > 0 \).

Then, there exist at least two positive solutions \( u_{\lambda}, v_{\lambda} \) in \( W_0^1L_\Phi(\Omega) \) of (1.1) for any \( \lambda \in (0, \Lambda_0) \) which satisfy \( u_{\lambda} \leq \neq v_{\lambda} \).

### 5 Existence of a second solution - critical case

In this section we assume that there exists \( p > 1 \) such that
\[
\phi(t)t = t^{p-1} + o(t^{p-1}) \quad \text{as} \quad t \to \infty.
\]
Put
\[ \phi(t) = t^{r-1} + \psi(t), \]
\[ \Phi(t) = \frac{1}{p}t^p + \Psi(t), \quad \Psi(t) = \int_0^t \psi(s) ds . \]

Moreover, we put the assumptions as follows:

(C1) there exist \( x_0 \in \Omega \), a ball \( B \) with center \( x_0 \) in \( \Omega \), and constants \( M > 0 \) and \( \gamma > \frac{N(N-p)}{p^2 + N - p} \) such that the inequality
\[ 0 \leq \| b \|_{L^\infty(B)} - b(x) \leq M |x - x_0|^{\gamma} \]
holds for \( x \in B \),

(C2) \( g(t) = t^{p^* - 1} + h(t), \quad \frac{h(t)}{t^{p^* - 1}} \to 0 \) as \( t \to \infty \),

and

(C3) (a) \( \frac{3N}{N+1} < p < 3 \), the inequality
\[ \sum_{i,j=1}^N \frac{\partial^2}{\partial \eta_i \partial \eta_j} [\Psi(|\eta|)] \xi_i \xi_j \leq c|\xi|^2 \]
holds with some constant \( c > 0 \) for any \( \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}^N \), or

(b) \( p \geq 3 \), and the following (i), (ii) are satisfied.

(i) the inequality
\[ \sum_{i,j=1}^N \frac{\partial^2}{\partial \eta_i \partial \eta_j} [\Psi(|\eta|)] \xi_i \xi_j \leq c(1 + |\eta|^{p-2-\delta})|\xi|^2 \]
holds with some constant \( c > 0 \) and \( \delta = \frac{2(N-p)}{N(p-1)} (\leq 1) \) for any \( \xi \in \mathbb{R}^N \),

(ii) \( \liminf_{t \to \infty} \frac{f(t)}{t^{p^* - 1}} = \infty \) or \( \liminf_{t \to \infty} \frac{h(t)}{t^{p^* - 1}} = \infty \) for \( r = p^* - \frac{2}{p-1} \).

Note that
\[ \sum_{i,j=1}^N \frac{\partial^2}{\partial \eta_i \partial \eta_j} [\Psi(|\eta|)] \xi_i \xi_j = \left( \psi(|\eta|) + |\eta| \psi'(|\eta|) \left( \frac{\xi}{|\xi|} \frac{\eta}{|\eta|} \right)^2 \right) |\xi|^2 . \]

Then we have

**Theorem 5.1** Let (1) \( \sim \) (3) below be satisfied.

(1) \( f_0 \), \( f_\infty \), \( g_0 \), \( g_\infty \) and (C1), (C2), (C3),
(2) \( f(t) \) and \( h(t) \) are nondecreasing,
(3) \( a(x) > 0 \) in \( \Omega \) and \( b(x) \not\equiv 0 \).

Then, there exist at least two positive solutions \( u_\lambda, v_\lambda \) in \( W_0^1 L_\Phi(\Omega) \) of (1.1) for any \( \lambda \in (0, \Lambda_\Omega) \) which satisfy \( u_\lambda \leq \not\equiv v_\lambda \).

Here we give a modified mean curvature equation as an example.

For \( \frac{3N}{N+1} < p < \frac{N}{2} \) and \( 1 < q < p \), consider an equation

\[
\begin{aligned}
-\text{div} \left( \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) &= \lambda a(x) u^{q-1} + b(x) u^{p-1} \quad \text{in } \Omega, \\
0 &= \text{on } \partial \Omega,
\end{aligned}
\]

the energy function attached to this equation is given by

\[
I_\lambda(u) = \frac{1}{p} \int_\Omega \left( \sqrt{1+|\nabla u|^2} - 1 \right) dx - \frac{\lambda}{q} \int_\Omega a(x) (u^+)^q dx - \frac{1}{p^*} \int_\Omega b(x) (u^+)^{p^*} dx.
\]

Thus,

\[
\Phi(t) = \frac{1}{p} \left( \sqrt{1+t^{2p}} - 1 \right), \quad \Psi(t) = \frac{1}{p} \left( \sqrt{1+t^{2p}} - 1 \right) - \frac{1}{p^*} t^{p^*}.
\]

It is easy to see that the conditions \( (\phi_1) \sim (\phi_3) \) are satisfied. Further, the functions

\[
\psi(t) \left( \frac{\Psi'(t)}{t} \right) = \frac{t^{2p-2} - \sqrt{1+1^{2p}}}{t^{2p-2}}
\]

and \( t \psi'(t) \) are bounded. Hence both the conditions \( (C3)(a) \) and \( (i) \) in \( (C3)(b) \) are satisfied. In this case, since \( h(t) \equiv 0 \), the condition \( (ii) \) in \( (C3)(b) \) is trivially satisfied. Since

\[
\phi(t)t \sim \begin{cases} 
1/t^{2p-1} & \text{as } t \to 0, \\
1/t^{p-1} & \text{as } t \to \infty,
\end{cases}
\]

all the conditions \( (f_0), (f_\infty), (g_0) \) and \( (g_\infty) \) are satisfied. Note here \( p^* > 2p \) by the assumption \( p < \frac{N}{2} \). Thus, if \( a(x) > 0, b(x) \not\equiv 0 \) in \( \Omega \) and \( b(x) \) satisfies \( (C1) \), then the conclusion in Theorem 5.1 holds for this equation.

6 Outline of the proofs of Theorems 4.1 and 5.1

In this section we only give the outline of the proofs of Theorems 4.1 and 5.1.

1st step: Take any \( \mu \in (0, \Lambda_\Omega) \). By the definition of \( \Lambda_\Omega \), there exists \( \bar{\lambda} \in (\mu, \Lambda_\Omega) \) such that (1.1) has a positive solution \( u_{\bar{\lambda}} \) for \( \lambda = \bar{\lambda} \). The function \( u_{\bar{\lambda}} \), which is denoted \( u_{\bar{\lambda}} \), is a supersolution of (1.1) with \( \lambda = \mu \). From Theorem 3.1, there exists a positive solution \( u_\lambda \) for small \( \lambda > 0 \). Since \( \|u_\lambda\|_{W_0^1 L_\Phi(\Omega)} \to 0 \) as \( \lambda \to 0 \), \( \|u_\lambda\|_{L^{q_0}(\Omega)} \to 0 \) as \( \lambda \to 0 \) by a priori estimate. Thus, small \( \lambda > 0, u_\lambda \), which is also \( u_\lambda \), is a subsolution of (1.1) with \( \lambda = \mu \) and \( u_\lambda \leq \pi_\mu \). Let \( u_\mu \) be the minimizer of \( I_\mu \) on the set \( \{ u \in W_0^1 L_\Phi(\Omega) \mid u \leq \pi_\mu \} \). Then, using Perron’s method, we see that \( u_\mu \) is a solution (1.1) with \( \lambda = \mu \).\( \square \)
2nd step: For degenerate quasilinear equations of the form (1.1), the strong comparison principle holds as in the following:

**Strong Comparison Principle:** Let $M \geq 0$ and consider the equations

\[
\begin{align*}
-\text{div}(\phi(|\nabla u|)\nabla u) + M\phi(u)u &= f & \text{in } \Omega, \\
-\text{div}(\phi(|\nabla v|)\nabla v) + M\phi(v)v &= g & \text{in } \Omega, \\
u = v = 0 & \text{ on } \partial\Omega.
\end{align*}
\]

Further, let

\[
f, \ g \in L^\infty(\Omega), \ f \geq 0 \text{ in } \Omega,
\]

and, for any compact set $K$ in $\Omega$, there exists $\varepsilon > 0$ such that

\[f(x) + \varepsilon < g(x) \quad \text{on } K.
\]

Then,

\[u < v \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} > \frac{\partial v}{\partial \nu} \quad \text{on } \partial\Omega,
\]

where $\nu$ is the outward unit normal on $\partial\Omega$.

Using the strong comparison principle, we see that $u_\mu$ is a local minimizer of $I_\mu$ in $C^1(\Omega)$-topology. In the followings, let denote $u_\mu$ by $u_0$.

3rd step: Modifying the arguments of “$H^1$ versus $C^1$ local minimizer” by Brezis and Nirenberg [5], we see that $u_0$ is a local minimizer of $I_\mu$ in $W_0^1 L_\Phi(\Omega)$.

4th step: By putting $u = u_0 + w$, consider the equation

\[
\begin{align*}
-\text{div}(\phi(|\nabla (u_0 + w)|)\nabla (u_0 + w)) &= \mu a(x)f(u_0 + w^+) + b(x)g(u_0 + w^+) & \text{in } \Omega, \\
w &= 0 & \text{on } \partial\Omega,
\end{align*}
\]

By maximum principle, nonzero solution of (6.1) is nonnegative. Thus, if $w$ is a nonzero solution of (6.1), then $u = u_0 + w$ is a second positive solution of (1.1) for $\lambda = \mu$. Solutions of (6.1) are critical points of the functional

\[J_\mu(u) = \int_\Omega \Phi(|\nabla (u_0 + w)|)dx - \int_\Omega H_\mu(x, w)dx,
\]

where

\[H_\mu(x, t) = \begin{cases} 
\mu a(x)f(u_0 + t) + b(x)g(u_0 + t) - \mu a(x)f(u_0) + b(x)g(u_0) & \text{if } t \geq 0, \\
\mu a(x)f(u_0)t + b(x)g(u_0)t & \text{if } t \leq 0.
\end{cases}
\]

5th step: By using mountain pass lemma, we look for a critical point of $J_\mu$.

From 3rd step, $w = 0$ is a local minimum point of $J_\mu$ in $W_0^1 L_\Phi(\Omega)$. For a function $w_0 \in C_0^\infty(\Omega)$, $w_0 \geq 0$, $w_0 \neq 0$ with $\text{supp } w_0 \subset \text{supp } b$, $J_\mu(tw_0) \to -\infty$ as $t \to \infty$. Using these facts, we easily see that $J_\mu$ satisfies the mountain pass geometry. In the subcritical case, we obtain a nontrivial critical point of $J_\mu$ by usual arguments. Thus Theorem 4.1 is shown.

6th step: In the critical case, the lack of compactness occurs. Thus the Palais-Smale sequence given by the mountain pass lemma does not converge in general.
However we can show the locally compactness of a Palais-Smale sequence as follows:

**Locally Compactness of Palais-Smale Sequence:** Let $0$ be the only critical point $J_{\mu}$. Then, $J_{\mu}$ satisfies (PS) condition for the value $c$ satisfying

\[
c < \int_{\Omega} \Phi(|u_0|)dx + \frac{S^{\frac{p}{p-1}}}{N||b||_{L^{\frac{p}{p-1}}}^{\frac{p}{p-1}}},\]

Here $S$ is the usual Sobolev constant, that is,

\[
S = \inf_{v \in L^p(\mathbb{R}^N) \setminus \{0\}} \frac{||\nabla v||_{L^p}^p}{||v||_{L^p}^p}.
\]

7th step: For any $\varepsilon > 0$, take

\[
w_0 = u_\varepsilon = \rho(x) \frac{\varepsilon^{\frac{N-p}{p}}}{\varepsilon^{\frac{N-p}{p}} + |x-x_0|^{\frac{p}{p-1}}},
\]

where $x_0 \in \Omega$ is the point stated in (C1) and $\rho(x)$ is a smooth, nonnegative function satisfying $\rho(x) = 1$ in $B_\delta(x_0)$ and $\text{supp } \rho \subset B_{2\delta}(x_0)$ with sufficiently small $\delta > 0$. For large $T > 0$, $J_{\mu}(Tu_\varepsilon) < 0$ holds. Then, from mountain pass lemma, there exists a sequence $\{w_n\}$ in $W^1_0 L^p(\Omega)$ such that

\[
J_{\lambda}(w_n) \to c \equiv \inf_{\gamma \in C^0([0,1]; W^1_0 L^p(\Omega))} \max_{t \in [0,1]} J_{\lambda}(\gamma(t))
\]

\[
J_{\lambda}'(w_n) \to 0 \quad \text{in } W^1_0 L^p(\Omega)^*,
\]

where

\[
\Gamma = \{ \gamma \in C^0([0,1]; W^1_0 L^p(\Omega)) \mid \gamma(0) = 0, \gamma(1) = Tu_\varepsilon \}.
\]

Through subtle estimates, we can show that the inequality

\[
\max_{t \geq 0} J_{\mu}(tu_\varepsilon) < \int_{\Omega} \Phi(|u_0|)dx + \frac{S^{\frac{p}{p-1}}}{N||b||_{L^{\frac{p}{p-1}}}^{\frac{p}{p-1}}}
\]

holds for small $\varepsilon > 0$. Since $c \leq \max_{t \geq 0} J_{\mu}(tu_\varepsilon)$, the conclusion in 6th step holds. Thus, if $0$ is the only critical point, then there exists a convergent subsequence of $\{w_n\}$. The limit function is a nonzero critical point of $J_{\mu}$. This contradicts the assumption that $0$ is the only critical point. Thus we have a nonzero critical point $w$ of $J_{\mu}$. As was stated in 4th step, $u = u_0 + w$ is a second positive solution of (1.1) for $\lambda = \mu$. This completes the proof of Theorem 6.1.

**References**


Construction of fundamental solution of
degenerate parabolic differential equations

Hiroki Tanabe

Let $Ω ⊂ \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial Ω$. Let $L(x, D_x)$ be a strongly elliptic linear differential operator of second order with real valued smooth coefficients defined in $Ω$:

$$L(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left( a_{i,j}(x) \frac{\partial}{\partial x_i} \right) + \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i} + a_0(x),$$

and $B(x, D_x)$ be a boundary operator of Robin type:

$$B(x, D_x) = \sum_{i,j=1}^{n} a_{i,j}(x) \nu_j(x) \frac{\partial}{\partial x_i} - b(x),$$

where $b$ is a real-valued continuous function and $\nu = (\nu_1, \ldots, \nu_n)$ is the outer normal unit vector to $\partial Ω$. Assume that

$$\frac{1}{p} \sum_{i=1}^{n} a_i(x) \nu_i(x) + b(x) \geq 0 \quad \text{for} \quad x ∈ \partial Ω × [0,T], \quad 1 < p < \infty.$$

Let $L_p$ be the realization of $L(x, D_x)$ in $L^p(Ω)$ with the boundary condition $B(x, D_x)u(x) = 0$, $x ∈ \partial Ω$. It is known that $-L_p$ generates an analytic semigroup in $L^p(Ω)$, and $0 ∈ ρ(L_p)$.

Let $m$ be a nonnegative function belongong to $L^∞(Ω)$. The notation $M_p$ denotes the multiplication operator by $m$:

$$(M_p u)(x) = m(x) u(x) \quad \text{for} \quad u ∈ L^p(Ω).$$

It is known that there exists a sector

$$\Sigma = \{ \lambda ∈ \mathbb{C}; -\theta_0 ≤ \arg \lambda ≤ \theta_0 \}, \quad \frac{\pi}{2} < \theta_0 < \pi,$$

such that for $\lambda ∈ Σ$ the operator $\lambda M_p + L_p$ has a bounded inverse and

$$\|M_p(\lambda M_p + L_p)^{-1}\|_{L^p(Ω), L^p(Ω)} \leq C|\lambda|^{-1/p}. \quad (1)$$

This was proved in A. Favini and A. Yagi [3; Example 3.6] in case of the Dirichlet boundary condition for $L(x, D_x) = -Δ$, and the same proof applies in our case.

Let

$$a(u, v) = \int_Ω \left\{ \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i} v + a_0(x) uv \right\} dx$$

$$+ \int_{\partial Ω} b(x) u \nu dS, \quad u ∈ W^{1,p}(Ω), v ∈ W^{1,p}(Ω),$$
be the sesquilinear form associated with the boundary value problem
\[ \mathcal{L}(x, D_x)u(x) = f(x), \quad x \in \Omega, \]
\[ B(x, D_x)u(x) = 0, \quad x \in \partial \Omega. \]

Identifying \( L^p(\Omega) \) with \( L^{p'}(\Omega)^* \) we consider
\[ W^{1,p}(\Omega) \subset L^p(\Omega) \subset W^{1,p'}(\Omega)^*. \]

Let \( u \in W^{1,p}(\Omega) \). Then, mapping \( v \mapsto a(u, v) \) is a continuous conjugate linear functional from \( W^{1,p'}(\Omega) \) to \( \mathbb{C} \). Hence there exists an element which is denoted by
\[ \tilde{L}_p \in W^{1,p'}(\Omega)^* \]
such that
\[ a(u, v) = (\tilde{L}_p u, v)_{W^{1,p'}(\Omega)^*, W^{1,p'}(\Omega)}, \quad \forall v \in W^{1,p'}(\Omega), \]
where \( (\cdot, \cdot)_{W^{1,p'}(\Omega)^*, W^{1,p'}(\Omega)} \) is the pairing between \( W^{1,p'}(\Omega)^* \) and \( W^{1,p'}(\Omega) \). \( \tilde{L}_p \) is a bounded linear operator from \( W^{1,p}(\Omega) \) to \( W^{1,p'}(\Omega)^* \). It is easy to see that
\[ \tilde{L}_p|_{D(L_p)} = L_p. \]

The object of this talk is to establish the estimates of the following form:
\[
\| M_p(\lambda M_p + \tilde{L}_p)^{-1} \|_{L(W^{1,p'}(\Omega)^*, W^{1,p'}(\Omega)^*)} \leq C|\lambda|^{-\delta_1}, \quad (2)
\]
\[
\| M_p(\lambda M_p + \tilde{L}_p)^{-1} \|_{L(W^{1,p'}(\Omega)^*, L^p(\Omega))} \leq C|\lambda|^{-\delta_2}, \quad (3)
\]
for \( \lambda \in \Sigma \), where \( \delta_1, \delta_2 \in (0, 1) \).

The motivation is as follows. Consider the case where the coefficients of \( \mathcal{L}(x, D_x) \), \( B(x, D_x) \) and the function \( m(x) \) are dependent also on the time variable \( t \in [0, T] \). The corresponding operators are denoted by \( L_p(t) \), \( \tilde{L}_p(t) \) and \( M_p(t) \) respectively. Consider the initial value problem
\[
d(M_p(t)u(t))/dt + L_p(t)u(t) = f(t), \quad t \in [0, T], \quad (4)
\]
\[ M_p(0)u(0) = M_p u_0. \quad (5)\]

Introducing the new unknown function \( v(t) = M_p(t)u(t) \) we transform this problem to
\[
dv(t)/dt + A_p(t)v(t) \geq f(t), \quad t \in [0, T], \quad (6)
\]
\[ v(0) = M_p(0)u_0, \quad (7)\]
where \( A_p(t) = L_p(t)M_p(t)^{-1} \) is a possibly multivalued operator. We try to solve problem (4)-(5) by constructing a fundamental solution \( U(t,s) \) to problem (6)-(7) by the method of T. Kato and H. Tanabe [4]. Then we need to establish an estimate of the form
\[
\left\| \frac{\partial}{\partial t} (\lambda + A_p(t))^{-1} \right\|_{L(L^p(\Omega), L^p(\Omega))} \leq C|\lambda|^{-\delta}, \quad \lambda \in \Sigma,
\]
for some \( \delta \in (0, 1) \). Using
\[
(\lambda + A_p(t))^{-1} = M_p(t)(\lambda M_p(t) + L_p(t))^{-1}
\]
and making a formal calculation one observes
\[
\frac{\partial}{\partial t}(\lambda + A_p(t))^{-1} = \hat{M}_p(t)(\lambda M_p(t) + L_p(t))^{-1}
\]
\[-\lambda M_p(t)(\lambda M_p(t) + L_p(t))^{-1}\hat{M}_p(t)(\lambda M_p(t) + L_p(t))^{-1}.
\]
\[-M_p(t)(\lambda M_p(t) + L_p(t))^{-1}\hat{L}_p(t)(\lambda M_p(t) + L_p(t))^{-1}.
\]
(8)

The last term of the above equality does not make sense, since \(D(L_p(t))\) depends on \(t\) in general, and hence \(\hat{L}_p(t)\) cannot be defined. However, its extension \(\hat{L}_p(t)\) has a constant domain \(W^{1,p}(\Omega)\). So, we replace \(L_p(t)\) by \(\hat{L}_p(t)\) in (8). Then the last term becomes
\[
M_p(t)(\lambda M_p(t) + \hat{L}_p(t))^{-1}\hat{L}_p(t)(\lambda M_p(t) + \hat{L}_p(t))^{-1}.
\]
Since
\[
\hat{L}_p(t)(\lambda M_p(t) + \hat{L}_p(t))^{-1} = \hat{L}_p(t)\hat{L}_p(t)^{-1} = \hat{L}_p(t)(\lambda M_p(t) + \hat{L}_p(t))^{-1},
\]
we need to obtain estimates for \(M_p(t)(\lambda M_p(t) + \hat{L}_p(t))^{-1}\).

In case \(p = 2\) it is known that (2) and (3) hold with \(\delta_1 = 1, \delta_2 = 1/2\):
\[
\|M_2(\lambda M_2 + \hat{L}_2)^{-1}\|_{\mathcal{L}(W^{1,2}(\Omega)^*,W^{1,2}(\Omega)^*)} \leq C|\lambda|^{-1}, \tag{9}
\]
\[
\|M_2(\lambda M_2 + \hat{L}_2)^{-1}\|_{\mathcal{L}(W^{1,2}(\Omega)^*,L^2(\Omega))} \leq C|\lambda|^{-1/2}. \tag{10}
\]

This is also shown in [3; Example 3.3] in case of the Dirichlet boundary condition for \(\mathcal{L}(x,t,D_x) = -\Delta\), and the same proof applies in the present case. The method is the same as the non-degenerate case \(m(x) \equiv 1\). By using this result the fundamental solution to problem (6)-(7) was constructed in case \(p = 2\) in A. Favini, A. Lorenzi and H. Tanabe [2].

If we try to apply the method of the proof of (9) and (10), we get
\[
a(u,|u|^{p-2}u) \geq c_0 \int |u|^{p-2} |\nabla u|^2 dx + \cdots.
\]
Hence we cannot obtain an estimate for \(|\nabla u|^p\).

Let \(L'_p\) be the realization of the formal adjoint \(\mathcal{L}'(x,D_x)\) of \(\mathcal{L}(x,D_x)\) in \(L^{p'}(\Omega)\) with the adjoint boundary condition
\[
\sum_{i,j=1}^{n} a_{i,j}(x)\nu_j(x) \frac{\partial u(x)}{\partial x_i} + \sum_{i=1}^{n} a_i(x)\nu_i(x)u(x) - b(x)u(x) = 0, \quad x \in \partial \Omega.
\]
It is known that \(L'_p = L'_p\). Let \(M'_p\) be the multiplication operator by \(m(x): (M'_pu)(x) = m(x)u(x)\) for \(u \in L^{p'}(\Omega)\). Then correspondingly to (1) the operator \(\lambda M'_p + L'_p\) has a bounded inverse for \(\lambda \in \Sigma\) and
\[
\|M'_p(\lambda M'_p + L'_p)^{-1}\|_{\mathcal{L}(L^{p'}(\Omega),L^{p'}(\Omega))} \leq C|\lambda|^{-1/p'}.
\]
(11)

Let \((L'_p)'\) be the adjoint operator of \(L'_p\) considered as a bounded linear operator from \(D(L'_p)\) to \(L^{p'}(\Omega)\). Then
\[
(L'_p)' \in \mathcal{L}(L^p(\Omega),D(L'_p)^*).
It is not difficult to show that $D(L_p)$ and $D(L_p')$ are dense in $W^{1,p}(\Omega)$ and $W^{1,p'}(\Omega)$ respectively. Hence we may consider

$$D(L_p) \subset W^{1,p}(\Omega) \subset L^p(\Omega) \subset W^{1,p'}(\Omega)^* \subset D(L_p').$$

Let $u \in D(L_p)$. Then, for $\forall v \in D(L_p')$

$$(L_p u, v)_{L^p(\Omega), L^p(\Omega)} = (u, L_p' v)_{L^p(\Omega), L^p(\Omega)} = ((L_p')' u, v)_{D(L_p')^*, D(L_p')}.$$ 

This implies $L_p u = (L_p')' u$. Hence

$$(L_p')'|_{D(L_p)} = L_p \in \mathcal{L}(D(L_p), L^p(\Omega)).$$

Application of a result of R. Seeley [6] or H. Triebel [7; 4.3.3] yields

$$[L^p(\Omega), D(L_p)]_{1/2} = W^{1,p}(\Omega). \quad (12)$$

Hence

$$(L_p')': W^{1,p}(\Omega) = [L^p(\Omega), D(L_p)]_{1/2} \to [D(L_p')^*, L^p(\Omega)]_{1/2}. \quad (13)$$

By virtue of a result of A. P. Calderon [1] one has

$$[D(L_p')^*, L^p(\Omega)]_{1/2} = [D(L_p'), L^p(\Omega)]_{1/2} = [L^p(\Omega), D(L_p')]_{1/2}. \quad (14)$$

Analogously to (12)

$$[L^p(\Omega), D(L_p')]_{1/2} = W^{1,p'}(\Omega). \quad (15)$$

From (13), (14) and (15) it follows that

$$(L_p')' \in \mathcal{L}(W^{1,p}(\Omega), W^{1,p'}(\Omega)^*). \quad (16)$$

From (16) it readily follows that

$$(L_p')'|_{W^{1,p}(\Omega)} = \bar{L}_p. \quad (17)$$

Suppose that $M$ and $L$ are a couple of linear operators such that $D(L) \subset D(M)$, $0 \in \rho(L)$ and $\lambda M + L$ has a bounded inverse for some $\lambda \in \mathbb{C}$. Let $v \in D(L)$. Then, using

$$(\lambda M + L)^{-1} = ((\lambda ML^{-1} + I)L)^{-1} = L^{-1}(\lambda ML^{-1} + I)^{-1}.$$ 

we obtain

$$L(\lambda M + L)^{-1} M v = (\lambda ML^{-1} + I)^{-1} ML^{-1} \cdot L v$$

$$= ML^{-1}(\lambda ML^{-1} + I)^{-1} \cdot L v = M(\lambda M + L)^{-1} L v.$$ 

Applying this to $L = L_p'$, $M = M_p'$ and $v \in D(L_p')$ and using (11) one gets for $\lambda \in \Sigma$ that

$$\|L_p' (\lambda M_p' + L_p')^{-1} M_p' v\|_{L^p(\Omega)}$$

$$= \|M_p' (\lambda M_p' + L_p')^{-1} L_p' v\|_{L^p(\Omega)} \leq C|\lambda|^{-1/p'}\|L_p' v\|_{L^p(\Omega)}.$$
This is rewritten as
\[ \| (\overline{\lambda} M_p + L'_p)^{-1} M_{p'} v \|_{D(L'_p)} \leq C|\lambda|^{-1/p'} \| v \|_{D(L'_p)}. \]

Hence
\[ \| (\overline{\lambda} M_p + L'_p)^{-1} M_{p'} \|_{L(D(L'_p), D(L'_p))} \leq C|\lambda|^{-1/p'}. \]

Taking adjoint
\[ \| M_p (\overline{\lambda} M_p + (L'_p)^\prime)^{-1} \|_{L(D(L'_p), \mathcal{L}(L_p, L_p(\Omega)))} \leq C|\lambda|^{-1/p'}. \] (18)

On the other hand in view of (1)
\[ \| M_p (\overline{\lambda} M_p + L_p)^{-1} \|_{L(L_p(\Omega), L_p(\Omega))} \leq C|\lambda|^{-1/p}. \] (19)

From
\[ [D(L'_p), L_p(\Omega)]_{1/2} = W^{1, p'}(\Omega)^*, \quad (L'_p)^\prime|_{D(L_p)} = L_p, \quad (L'_p)^\prime|_{W^{1, p}(\Omega)} = \tilde{L}_p \] (20)

it follows that
\[ M_p (\overline{\lambda} M_p + (L'_p)^\prime)^{-1}|_{L_p(\Omega)} = M_p (\overline{\lambda} M_p + L_p)^{-1}, \]
\[ M_p (\overline{\lambda} M_p + (L'_p)^\prime)^{-1}|_{W^{1, p'}(\Omega)} = M_p (\overline{\lambda} M_p + \tilde{L}_p)^{-1}. \] (21)

It follows from (18), (19), (20) and (21) that
\[ \| M_p (\overline{\lambda} M_p + \tilde{L}_p)^{-1} \|_{L(W^{1, p}(\Omega)^*, W^{1, p'}(\Omega)^*)} \leq C|\lambda|^{-1/2}. \]

Let \( \lambda \in \Sigma, |\lambda| \geq 1 \) and \( v \in L^p'(\Omega) \). Then, using (11)
\[
\| (\overline{\lambda} M_p' + L'_p)^{-1} M_{p'} v \|_{D(L'_p)} = \| (\overline{\lambda} M_p' + L'_p)^{-1} M_{p'} v \|_{L^{p'}(\Omega)} \\
= \| \{ I - \overline{\lambda} M_p' (\overline{\lambda} M_p' + L'_p)^{-1} \} M_{p'} v \|_{L^{p'}(\Omega)} \\
\leq \| M_{p'} v \|_{L^{p'}(\Omega)} + |\lambda| \| M_{p'} (\overline{\lambda} M_p' + L'_p)^{-1} M_{p'} v \|_{L^{p'}(\Omega)} \\
\leq \| M_{p'} v \|_{L^{p'}(\Omega)} + C|\lambda| |\lambda|^{-1/p'} \| M_{p'} v \|_{L^{p'}(\Omega)} \\
\leq C|\lambda|^{-1/p'} \| M_{p'} v \|_{L^{p'}(\Omega)} \leq C|\lambda|^{-1/p'} \| v \|_{L^{p'}(\Omega)}. 
\]

Namely
\[ \| (\overline{\lambda} M_p' + L'_p)^{-1} M_{p'} \|_{L(L^{p'}(\Omega), D(L'_p))} \leq C|\lambda|^{1/p}. \] (22)

On the other hand in view of (1)
\[ \| (\overline{\lambda} M_p' + L'_p)^{-1} M_{p'} \|_{L(L^{p'}(\Omega), L^{p'}(\Omega))} = \| (\overline{\lambda} M_p' + L'_p)^{-1} M_{p'} \|_{L(L^{p'}(\Omega), L^{p'}(\Omega))} \\
= \| M_p (\overline{\lambda} M_p + L_p)^{-1} \|_{L(L^{p}(\Omega), L^{p}(\Omega))} \leq C|\lambda|^{-1/p}. \]
i.e.
\[ \| (\overline{\lambda} M_p' + L'_p)^{-1} M_{p'} \|_{L(L^{p'}(\Omega), L^{p'}(\Omega))} \leq C|\lambda|^{-1/p}. \] (23)

Interpolating (22) and (23)
\[ \| (\overline{\lambda} M_p' + L'_p)^{-1} M_{p'} \|_{L(L^{p'}(\Omega), W^{1, p'}(\Omega))} \leq C. \]
Taking adjoint and using (21) one observes
\[ \| M_p(\lambda M_p + \bar{L}_p)^{-1}\|_{L(W^{1,p'}(\Omega)^*,L^p(\Omega))} \leq C. \]
Thus it has been proved that the following estimates hold:
\[ \| M_p(\lambda M_p + \bar{L}_p)^{-1}\|_{L(W^{1,p'}(\Omega)^*,W^{1,p'}(\Omega)^*)} \leq C|\lambda|^{-1/2}, \quad (24) \]
\[ \| M_p(\lambda M_p + \bar{L}_p)^{-1}\|_{L(W^{1,p'}(\Omega)^*,L^p(\Omega))} \leq C. \quad (25) \]
As it was mentioned before in case \( p = 2 \) the following better estimates hold:
\[ \| M_2(\lambda M_2 + \bar{L}_2)^{-1}\|_{L(W^{1,2}(\Omega)^*,W^{1,2}(\Omega)^*)} \leq C|\lambda|^{-1}, \quad (9) \]
\[ \| M_2(\lambda M_2 + \bar{L}_2)^{-1}\|_{L(W^{1,2}(\Omega)^*,L^2(\Omega))} \leq C|\lambda|^{-1/2}. \quad (10) \]
We are going to interpolate (24) and (9), and (25) and (10). We use the following result
\[ S(p_0, \xi_0, L^{p_0}; p_1, \xi_1, L^{p_1}) = L^p, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \theta = \frac{\xi_0}{\xi_0 - \xi_1}, \quad (26) \]
which is Colloray 1.1 in Chapter 7 of J. L. Lions and J. Peetre [5] to deduce
\[ S(p_0, \xi_0, W^{1,p_0'}(\Omega)^*; p_1, \xi_1, W^{1,p_1'}(\Omega)^*) = W^{1,p'}(\Omega)^*. \quad (27) \]
where \( p_0, p_1, p, \xi_0, \xi_1 \) are the ones given in (26).
First we consider the case \( 1 < p < 2 \). Let \( p_0 \) be such that \( 1 < p_0 < p \). Then \( 1 < p_0 < 2 \). Applying (27) with \( p_1 = 2 \) one observes
\[ S(p_0, \xi_0, W^{1,p_0'}(\Omega)^*; 2, \xi_1, W^{1,2}(\Omega)^*) = W^{1,p'}(\Omega)^*, \quad (28) \]
where for \( \theta \) determined by \( \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{2} \) the numbers \( \xi_0 \) and \( \xi_1 \) are chosen so that
\[ \theta = \frac{\xi_0}{\xi_0 - \xi_1}, \quad \text{say } \xi_0 = \theta, \xi_1 = \theta - 1. \quad \text{One has} \]
\[ \theta = \frac{2(p - p_0)}{p(p - p_0)} = \frac{2(p - 1)}{p} - \frac{2(p_0 - 1)(2 - p)}{p(2 - p_0)}. \quad (29) \]
Applying Theorem 3.1 in Chapter I of [5] one obtains in view of (28) and
\[ \| M_{p_0}(\lambda M_{p_0} + \bar{L}_{p_0})^{-1}\|_{L(W^{1,p_0'}(\Omega)^*,W^{1,p_0'}(\Omega)^*)} \leq C|\lambda|^{-1/2}, \]
which is (24) with \( p = p_0 \), and
\[ \| M_2(\lambda M_2 + \bar{L}_2)^{-1}\|_{L(W^{1,2}(\Omega)^*,W^{1,2}(\Omega)^*)} \leq C|\lambda|^{-1}, \quad (9) \]
that
\[ \| M_p(\lambda M_p + \bar{L}_p)^{-1}\|_{L(W^{1,p'}(\Omega)^*,W^{1,p'}(\Omega)^*)} \leq (C|\lambda|^{-1/2})^{1-\theta}(C|\lambda|^{-1})^\theta = C|\lambda|^{-1/2-\theta/2}. \]
In view of (29) \( \theta < 2(p - 1)/p \). However, by choosing \( p_0 \) sufficiently close to 1, \( \theta \) may be arbitrarily close to \( 2(p - 1)/p \). Analogously we can show
\[ \| M_p(\lambda M_p + \bar{L}_p)^{-1}\|_{L(W^{1,p'}(\Omega)^*,L^p(\Omega))} \leq C|\lambda|^{-\theta/2}. \]
In case \( p > 2 \) choosing \( p_1 > p \) and reasoning analogously we can show that

\[
\| M_p(\lambda M_p + \tilde{L}_p)^{-1} \|_{L^p(W^{1,p'}(\Omega),W^{1,p'}(\Omega)^*)} \leq C|\lambda|^{\tilde{\theta}/2 - 1},
\]

\[
\| M_p(\lambda M_p + \tilde{L}_p)^{-1} \|_{L^p(W^{1,p'}(\Omega)^*,L^p(\Omega))} \leq C|\lambda|^{\tilde{\theta}/2 - 1/2},
\]

where

\[
\tilde{\theta} = \frac{p - 2}{p} + \frac{2(p - 2)}{p(p_1 - 2)}.
\]

Hence \( \tilde{\theta} > (p - 2)/p \). However, \( \tilde{\theta} \) can be arbitrarily close to \((p - 2)/p\) by choosing \( p_1 \) sufficiently large.

Using the above results we can construct the fundamental solution to (6)-(7) under some smoothness hypotheses for the coefficients. However, at the present stage the admissible values of \( p \) is limited to the range \( 4/3 < p < 8/3 \).

References


EXPONENTIAL ATTRACTORS FOR NON-AUTONOMOUS DYNAMICAL SYSTEMS

ATSUSHI YAGI

1. INTRODUCTION

In this Note we will introduce a version of exponential attractor for non-autonomous equations as a time dependent set with uniformly bounded finite fractal dimension which is positively invariant and attracts every bounded set at an exponential rate. This is a natural generalization of the existent notion for autonomous equations. A generation theorem will be proved under the assumption that the evolution operator is a compact perturbation of a contraction. These results are applicable for various non-autonomous dissipative systems. The full paper will be published by [17].

This is a joint work with Professors Messoud Efendiev (Helmholtz-Zentrum München, Institute of Biomathematics and Biometry, 85764, Neuherberg, Germany) and Yoshitaka Yamamoto (Department of Information Physical Science, Graduate School of Information Science and Technology, Osaka University, Suita, Osaka 565-0871, Japan).

2. NON-AUTONOMOUS DISSIPATIVE SYSTEMS

Our aim is to discuss the behavior as time goes to infinity of ordinary differential equations of the form

\[ \frac{dU}{dt} = F(t, U) \]

in a Banach space \( X \).

When the system is autonomous, i.e., when the time does not appear explicitly in (2.1) \((F(t, U) = F(U))\), then, very often, the long time behavior of the system can be described in terms of the global attractor \( \mathcal{A} \). More precisely, assuming that the system is well-posed, we can define the family of solving operators

\[ S(t): U_0 \mapsto U(t), \quad t \geq 0, \]

acting on \( X \), which maps the initial datum \( U_0 \) onto the solution at time \( t \). This family of operators satisfies

\[ S(0) = I, \]

\[ S(t + s) = S(t) \circ S(s), \quad \forall t, \ s \geq 0, \]

\( I \) denoting the identity operator on \( X \), and we say that it forms a semigroup on the phase space \( X \).

**Definition 2.1.** We then say that a set \( \mathcal{A} \) is the global attractor for \( S(t) \) in \( X \) if:

(i) It is a compact set of \( X \).

(ii) It is a strictly invariant set, i.e., \( \forall t \geq 0, \ S(t)\mathcal{A} = \mathcal{A} \).
(iii) It attracts (uniformly) the bounded sets of initial data in the following sense:

\[ \forall B \subset X \text{ bounded}, \lim_{t \to +\infty} h(S(t)B, A) = 0, \]

where \( h(\cdot, \cdot) \) denotes the Hausdorff semidistance between sets, defined by

\[ h(A, B) = \sup_{a \in A} \inf_{b \in B} \| a - b \|_H. \]

This is equivalent to the following: \( \forall B \subset X \text{ bounded}, \forall \varepsilon > 0, \exists t_0 = t_0(B, \varepsilon) \text{ such that} \]
\( t \geq t_0 \) implies \( S(t)B \subset U_\varepsilon \), where \( U_\varepsilon \) denotes the \( \varepsilon \)-neighborhood of \( A \).

We note that it follows from (ii) and (iii) that the global attractor, if it exists, is unique. Furthermore, it follows from (i) that it is essentially thinner than the original phase space \( X \); indeed, here, in general, \( X \) is an infinite-dimensional function space and, in infinite dimensions, a compact set cannot contain a ball and is nowhere dense. It is not difficult to prove that the global attractor is the smallest (for the inclusion) closed set enjoying the attraction property (iii); it is also the largest bounded invariant set. Finally, in most (if not all) cases, one can prove that the global attractor has finite dimension (in the sense of covering dimensions, such as the Hausdorff and the fractal dimensions; the global attractor is not a smooth manifold in general, but it can have a very complicated geometric structure), so that, even though the initial phase space is infinite-dimensional, the dynamics, reduced to the global attractor, is, in some proper sense, finite-dimensional and can be described by a finite number of parameters. It thus follows that the global attractor appears as a suitable object in view of the study of the long time behavior of the system. We refer the reader to [5, 12, 21, 26, 28, 30] for extensive reviews on this subject.

Now, the global attractor may present some defaults. Indeed, it may attract the trajectories slowly (see, e.g., [24]). Furthermore, in general, it is very difficult, if not impossible, to express the convergence rate in terms of the physical parameters of the problem. A second drawback, which can also be seen as a consequence of the first one, is that the global attractor may be sensitive to perturbations; a given system is only an approximation of reality and it is thus essential that the objects that we study must be robust under small perturbations. Actually, in general, the global attractor is outer semicontinuous with respect to perturbations, i.e.,

\[ h(A_\varepsilon, A_0) \to 0 \text{ as } \varepsilon \to 0, \]

where \( A_0 \) is the global attractor associated with the nonperturbed system and \( A_\varepsilon \) that associated with the perturbed one, \( \varepsilon > 0 \) being the perturbation parameter. Now, the inner semicontinuity, i.e.,

\[ h(A_0, A_\varepsilon) \to 0 \text{ as } \varepsilon \to 0, \]

is much more difficult to prove (see, e.g., [28]). Furthermore, this property may not hold. This is in particular the case when the perturbed and nonperturbed problems do not have the same equilibria (stationary solutions). Furthermore, in many situations, the global attractor may not be observable in experiments or in numerical simulations. This can be due to the fact that it has a very complicated geometric structure, but not necessarily. Indeed, we can consider for instance the following Chafee-Infante equation in one space
Then, due to the boundary conditions, \( \mathcal{A} = \{-1\} \). Now, this problem possesses many metastable “almost stationary” equilibria which live up to a time \( t_* \sim e^{-2/\nu} \). Thus, for \( \nu \) small, one will not see the global attractor in numerical simulations. Finally, in some situations, the global attractor may fail to capture important transient behaviors. This can be observed, e.g., on some models of one-dimensional Burgers equations with a weak dissipation term (see [6]). In that case, the global attractor is trivial, it is reduced to one exponentially attracting point, but the system presents very rich and important transient behaviors, which resemble some modified version of the Kolmogorov law. We can also mention models of pattern formation equations in autonomous chemotaxis model for which one observes important transient behaviors which are not contained in the global attractor (see [2, 3, 20, 29]).

So, it follows from the above considerations that it should be useful to have a (possibly) larger object which contains the global attractor, attracts the trajectories at a fast rate, is still finite-dimensional and is more robust under perturbations.

The first attempt to study such an object, i.e., an exponential attractor for an autonomous system, was made by A. Eden, C. Foias, B. Nicolaenko and R. Temam in [14]. Indeed, let \( S(t), \ t \geq 0 \), be the semigroup associated with the problem

\[
\begin{cases}
\frac{dU}{dt} = F(U), & 0 < t < \infty, \\
U(0) = U_0,
\end{cases}
\]

in a Banach space \( X \) (in particular, we assume that (2.2) is well-posed for \( u_0 \in X \)). We have the following definition.

**Definition 2.2.** A set \( \mathcal{M} \) is an exponential attractor for \( S(t) \) in \( X \) if:

(i) It is a compact set of \( X \) with finite fractal dimension.

(ii) It is a positively invariant set, i.e., \( \forall t \geq 0, \ S(t)\mathcal{M} \subseteq \mathcal{M} \).

(iii) It attracts exponentially fast the bounded sets of initial data in the following sense:

There exist a constant \( \alpha > 0 \) and a monotonic function \( Q \) such that

\[
\forall B \subset X \text{ bounded}, \ h(S(t)B, \mathcal{M}) \leq Q(\|B\|_{X})e^{-\alpha t}, \ t \geq 0.
\]

It follows from this definition that an exponential attractor always contains the global attractor (actually, it follows from the definition that, if \( S(t) \) possesses an exponential attractor \( \mathcal{M} \), then it also possesses the global attractor \( \mathcal{A} \subseteq \mathcal{M} \); indeed, \( \mathcal{M} \) is a compact attracting set (see, e.g., [5]; the continuity of \( S(t), \forall t \geq 0 \), generally holds)).

**Remark 2.1.** (i) Actually, proving the existence of an exponential attractor is also one way of proving the finite (fractal) dimensionality of the global attractor.

(ii) The choice of the fractal dimension over other dimensions, e.g., the Hausdorff dimension, in Definition 1.2 is related, with the Mané theorem which gives some indications on the existence of a reduced finite-dimensional system which is H"older continuous (but, unfortunately, not Lipschitz continuous) with respect to the initial data, see [14].
The main drawback of exponential attractors is however that an exponential attractor, if it exists, is not unique. Therefore, the question of the best choice, if it makes sense, of an exponential attractor is a crucial one.

The first construction of exponential attractors was due to A. Eden, C. Foias, B. Nicolaenko and R. Temam [14]. This construction is based on the so-called squeezing property which, roughly speaking, says that either the higher modes are dominated by the lower ones or that the flow is contracted exponentially. It is only valid in Hilbert spaces (since it makes an essential use of orthogonal projectors with finite rank). Furthermore, based on this construction, it is possible to prove the inner semicontinuity of proper exponential attractors under perturbations, but only up to some time shift, so that, essentially, one only proves that

\[ h(A_0, M_\varepsilon) \to 0 \text{ as } \varepsilon \to 0, \]

where \( A_0 \) is the global attractor associated with the nonperturbed system and \( M_\varepsilon \) an exponential attractor associated with the perturbed one, which is not satisfactory.

In [16], was proposed a second construction, valid in Banach spaces also (see also [13] for another construction of exponential attractors valid in Banach spaces; this second construction consists in adapting that of [14] to a Banach setting and has thus some of the drawbacks mentioned above). The key point in this construction is a smoothing property on the difference of two solutions which generalizes in some sense (and, in particular, to a Banach setting) techniques proposed by O.A. Ladyzhenskaya in order to prove the finite dimensionality of the global attractor, see, e.g., [25] of the form

\[ \|S(\tau^*) U_0 - S(\tau^*) V_0\|_Z \leq c \|U_0 - V_0\|_Z, \]

where \( Z \) is a second Banach space which is compactly embedded into \( X \), which has to hold for some \( \tau^* > 0 \) and on some bounded positively invariant subset of \( X \) (see [16] for generalizations and other forms of the smoothing property (2.3)). We can note that, in a Hilbert setting, i.e., when \( X \) and \( Z \) are Hilbert spaces, then (2.3) implies the squeezing property, see [15]. Furthermore, based on this construction, it is possible to construct robust (i.e., inner and outer semicontinuous with respect to perturbations) families of exponential attractors (see [18]) which satisfy in particular an estimate of the form

\[ d(M_\varepsilon, M_0) \leq c\varepsilon^\kappa, \quad c > 0, \quad \kappa \in (0, 1), \]

where the constants \( c \) and \( \kappa \) are independent of \( \varepsilon \) and can be computed explicitly in terms of the physical parameters of the problem and where \( d(\cdot, \cdot) \) denotes the symmetric Hausdorff distance between (closed) sets

\[ d(A, B) = \max \{h(A, B), h(B, A)\}. \]

Of course, such constructions are obtained having in mind the nonuniqueness problem.

**Remark 2.2.** (i) It is in general very difficult, if not impossible, to prove an estimate of the form (2.4) for global attractors. This is possible, for instance, when the stationary solutions enjoy some hyperbolicity assumption. In that case, the global attractor is regular (see [5]) and exponential and one has an estimate of the form (2.4). However, even in that case, one cannot compute in general the constants \( c \) and \( \kappa \) in terms of the physical parameters of the problem.

(ii) We also refer to [4] for results on the stability of exponential attractors under numerical approximations.
Now, let us consider the non-autonomous problem

\[
\begin{aligned}
    \frac{dU}{dt} &= F(t, U), & s < t < \infty, \\
    U(s) &= U_s, & -\infty < s < \infty,
\end{aligned}
\]  

(2.5)

in a Banach space $X$. Assuming that (2.5) is well-posed for $U_s \in X$, we have the family of solving operators

\[
U(t, s)U_s : U_s \mapsto U(t), \quad -\infty < s \leq t < \infty.
\]

The family of operators has the properties

\[
(2.6) \quad U(s, s) = I, \quad -\infty < s < \infty,
\]

\[
(2.7) \quad U(t, r) \circ U(r, s) = U(t, s), \quad -\infty < s \leq r \leq t < \infty.
\]

It is then said that $U(t, s)$ forms an evolution operator or a process on the phase space $X$. We especially emphasize that the theory of attractors for non-autonomous systems is less understood than that for autonomous systems. We have essentially two approaches.

The first one, initiated by A. Haraux (see [22]) and further studied and developed by V.V. Chepyzhov and M.I. Vishik (see, e.g., [11, 12]), is based on the notion of a uniform attractor. The major drawback of this approach is that it leads, for general (translation-compact, see [11]) time dependences, to an artificial infinite dimensionality of the uniform attractor. This can already be seen for the following simple linear equation:

\[
\frac{\partial u}{\partial t} - \Delta u = h(t), \quad u|_{\partial \Omega} = 0,
\]

in a bounded smooth domain $\Omega$, whose dynamics is simple, namely, one has one exponentially attracting trajectory. However, the uniform attractor has infinite dimension and infinite topological entropy (see [12]). However, for periodic and quasiperiodic time dependences, one has in general finite-dimensional uniform attractors (i.e., if the same is true for the corresponding autonomous system, see [10, 19]). Furthermore, one can derive sharp upper and lower bounds on the dimension of the uniform attractor, so that this approach is quite relevant in that case. We can note that, as in the autonomous case, an exponential attractor in this setting always contains the uniform attractor and, again, one has, for general time dependences, an artificial infinite dimensionality.

The second approach is based on the notion of a pullback attractor (see, e.g. [8, 23] and the references therein). In that case, one has a time dependent attractor $\{A(t)\}_{t \in \mathbb{R}}$, contrary to the uniform attractor which is time independent.

**Definition 2.3.** A family $\{A(t)\}_{t \in \mathbb{R}}$ is a pullback attractor for the evolution operator $U(t, s)$ on $X$ if:

(i) Each $A(t)$ is a compact set of $X$.

(ii) It is strictly invariant, i.e., $-\infty < s \leq t < \infty, \quad U(t, s)A(s) = A(t)$.

(iii) It satisfies the following pullback attraction property:

\[
\forall B \subset X \text{ bounded, } \lim_{s \to -\infty} h(U(t, t - s)B, A(t)) = 0.
\]

One can prove that, in general, $A(t)$ has finite dimension for every $t \in \mathbb{R}$. We also note that it follows from the above definition that the pullback attractor, if it exists, is unique. Furthermore, if the system is autonomous, then one recovers the global attractor. Now, the attraction property essentially means that, at time $t$, the attractor $A(t)$ attracts
the bounded sets of initial data coming from the past (i.e., from \(-\infty\)). However, in (iii), the rate of attraction is not uniform in \(t\), so that the forward convergence is not true in general (see nevertheless [7, 9] for cases where the forward convergence can be proven).

In this Note, we want to introduce a version of exponential attractor for non-autonomous equations as a time dependent set satisfying certain natural assumptions. Our definition is stated as follows.

**Definition 2.4.** A family \(\{\mathcal{M}(t)\}_{t \in \mathbb{R}}\) is an exponential attractor for the evolution operator \(U(t, s)\) on \(X\) if:

(i) Each \(\mathcal{M}(t)\) is a compact set of \(X\) and its fractal dimension is finite and uniformly bounded, i.e., \(\sup_{t \in \mathbb{R}} \dim \mathcal{M}(t) < \infty\).

(ii) It is positively invariant, i.e., \(-\infty < s \leq t < \infty\), \(U(t, s)\mathcal{M}(s) \subset \mathcal{M}(t)\).

(iii) There exist an exponent \(\alpha > 0\) and two monotonic functions \(Q\) and \(\tau\) such that

\[
\forall B \subset X \text{ bounded, } h(U(t, s)B, \mathcal{M}(t)) \leq Q(\|B\|_X)e^{-\alpha(t-s)},
\]

\[
s \in \mathbb{R}, \ s + \tau(\|B\|_X) \leq t < \infty.
\]

We can show construction of exponential attractors for non-autonomous systems. To this end, we will assume existence of a family of bounded sets \(\mathcal{X}(t), t \in \mathbb{R}\), which is positively invariant and absorbs all bounded sets, and will generalize (2.3) into the form (2.8) \(\|U(\tau^* + s, s)U_0 - U(\tau^* + s, s)V_0\|_Z \leq c\|U_0 - V_0\|_X\), \(U_0, V_0 \in \mathcal{X}(s)\), for all \(s \in \mathbb{R}\), where \(\tau^* > 0\) is some fixed constant. (Actually our assumption will be of the more general form, see (3.1) and (3.2).) This condition together with some minor ones in fact enables us to generalize the method of construction for autonomous systems (due to [16]) for non-autonomous ones. Our exponential attractor \(\mathcal{M}(t)\) then depends on \(t\) continuously if \(t \neq n\tau^*, n \in \mathbb{Z}\), and is right continuous at \(t = n\tau^*, n \in \mathbb{Z}\). Left discontinuity of \(\mathcal{M}(t)\) at time \(n\tau^*\) comes completely from a technical reason. We notice in applications that (2.8) is actually verified for any \(\tau^*\) contained in some interval \((\tau_0, \tau_1)\), where \(0 < \tau_0 < \tau_1\), which means that, even if \(\mathcal{M}(t)\) is left discontinuous at \(n\tau^*\), it is possible to choose another \(\tau^*\) in order to construct another exponential attractor \(\overline{\mathcal{M}}(t)\) which is now continuous at the \(n\tau^*\).

This construction is applicable to many non-autonomous diffusion systems. For example, consider a non-autonomous chemotaxis system

\[
\begin{cases}
\frac{\partial u}{\partial t} = a\Delta u - \nabla \cdot [u\nabla \chi(t, \rho)] + f(t, u) & \text{in } \Omega \times (s, \infty), \\
\frac{\partial \rho}{\partial t} = b\Delta \rho - c\rho + \nu u & \text{in } \Omega \times (s, \infty), \\
\frac{\partial \rho}{\partial n} = 0 & \text{on } \partial \Omega \times (s, \infty), \\
u(x, s) = u(x, s), \ \rho(x, s) = \rho_s(x) & \text{in } \Omega,
\end{cases}
\]

in a bounded domain \(\Omega \subset \mathbb{R}^2\) with initial time \(s \in \mathbb{R}\). For autonomous chemotaxis systems, we have already constructed exponential attractors in the papers [1, 27] (cf. also [31, Chapter 12]). In [2] we estimated their fractal dimensions from below and showed that, if the chemotaxis parameter becomes large, then the fractal dimensions also increase and finally tend to infinity. Meanwhile, in [18] we proved that the exponential attractor can depend continuously with respect to the chemotaxis parameter. We then
consider a time dependent sensitivity function and a time dependent growth function as in (2.9). Under reasonable assumptions on the functions, our general result can applied for constructing exponential attractors as before. Our result seems to be in good agreement with the former ones in the sense that the dimension of $\mathcal{M}(t)$ is uniformly bounded and is continuous with respect to the variable $t$.

3. CONSTRUCTION OF EXPONENTIAL ATTRACTORS

Let $X$ be a Banach space with norm $\| \cdot \|_X$. Let $\mathcal{K}$ be a subset of $X$ which is a metric space equipped with the distance $d(U, V) = \| U - V \|_X$. We consider a family of nonlinear operators $U(t, s)$ acting on $\mathcal{K}$ defined for

$$(t, s) \in \Delta = \{(t, s); -\infty < s \leq t < \infty\}.$$ 

We assume that $U(t, s)$ has the properties (2.6) and (2.7) on $\mathcal{K}$. A family of $U(t, s)$ having these properties is called an evolution operator or a process on the space $\mathcal{K}$. We assume also that $U(t, s)$ is continuous in the sense that

the mapping $G: \Delta \times \mathcal{K} \to \mathcal{K}, ((t, s), U_0) \mapsto U(t, s)U_0$ is continuous.

Such an evolution operator is said simply to be continuous on $\mathcal{K}$. When $U(t, s)$ is a continuous evolution operator on $\mathcal{K}$, the triplet $(U(t, s), \mathcal{K}, X)$ is called a non-autonomous dynamical system, and $\mathcal{K}$ and $X$ are called the phase space and the universal space, respectively. The trace of a function $U(\cdot, s)U_0$ for $t \in [s, \infty)$ in the space $\mathcal{K}$ is called a trajectory starting from $U_0 \in \mathcal{K}$ at initial time $s \in \mathbb{R}$.

We now restate the definition of exponential attractors. (Note that in Definition 1.4, $\mathcal{K}$ coincides with $X$).

**Definition 3.1.** A family $\{M(t)\}_{t \in \mathbb{R}}$ of subsets of $\mathcal{K}$ is called an exponential attractor for $(U(t, s), \mathcal{K}, X)$ if:

(i) Each $M(t)$ is a compact set of $X$ and its fractal dimension is finite and uniformly bounded, i.e., $\sup_{t \in \mathbb{R}} \dim M(t) < \infty$.

(ii) It is positively invariant, i.e., $U(t, s)M(s) \subset M(t)$ for all $(t, s) \in \Delta$.

(iii) There exist an exponent $\alpha > 0$ and two monotonic functions $Q$ and $\tau$ such that

$$\forall B \subset \mathcal{K} \text{ bounded}, \quad h(U(t, s)B, M(t)) \leq Q(\| B \|_X)e^{-\alpha(t-s)},$$

$$s \in \mathbb{R}, \quad s + \tau(\| B \|_X) \leq t < \infty.$$ 

In order to construct exponential attractors, we have to assume existence of a family $\{\mathcal{X}(t)\}_{t \in \mathbb{R}}$ of bounded closed subsets of $\mathcal{K}$ with the following properties:

1. The diameter $\| \mathcal{X}(t) \|_X$ of $\mathcal{X}(t)$ is uniformly bounded, i.e., $\sup_{t \in \mathbb{R}} \| \mathcal{X}(t) \|_X = R < \infty$.

2. It is positively invariant, i.e., $U(t, s)\mathcal{X}(s) \subset \mathcal{X}(t)$ for all $(t, s) \in \Delta$.

3. It is absorbing in the sense that there is a monotonic function $\sigma$ such that

$$\forall B \subset \mathcal{K} \text{ bounded}, \quad U(t, s)B \subset \mathcal{X}(t), \quad s \in \mathbb{R}, \quad s + \sigma(\| B \|_X) \leq t < \infty.$$
(4) There is $\tau^* > 0$ such that, for every $s \in \mathbb{R}$, $U(\tau^* + s, s)$ is a compact perturbation of contraction on $X(s)$ in the sense that

$$
\|U(\tau^* + s, s)U_0 - U(\tau^* + s, s)V_0\|_X \leq \delta \|U_0 - V_0\|_X \quad + \quad \|K(s)U_0 - K(s)V_0\|_X, \quad U_0, V_0 \in X(s),
$$

where $\delta$ is a constant such that $0 \leq \delta < \frac{1}{2}$ and where $K(s)$ is an operator from $X(s)$ into another Banach space $Z$ which is embedded compactly in $X$ and satisfies a Lipschitz condition

$$
\|K(s)U_0 - K(s)V_0\|_Z \leq L_1 \|U_0 - V_0\|_X, \quad U_0, V_0 \in X(s),
$$

with some constant $L_1 > 0$ independent of $s$.

(5) It holds for any $s \in \mathbb{R}$ and any $\tau \in [0, \tau^*]$ that

$$
\|U(\tau + s, s)U_0 - U(\tau + s, s)V_0\|_X \leq L_2 \|U_0 - V_0\|_X, \quad U_0, V_0 \in X(s),
$$

with some constant $L_2 > 0$ independent of $s$ and $\tau$.

**Theorem 3.1.** Let $(U(t, s), X, \mathcal{X})$ be a non-autonomous dynamical system in $X$. Assume that the conditions (1)~(5) be satisfied. Then, one can construct an exponential attractor $\{\mathcal{M}(t)\}_{t \in \mathbb{R}}$ for $(U(t, s), X, \mathcal{X})$.

We are now concerned with continuity of $\mathcal{M}(t)$ with respect to the variable $t$. And we make the following assumptions. For each fixed $-\infty < t < \infty$,

$$
\lim_{t' \to t} \sup_{U_0 \in X(t)} \|U(t', t) - 1|U_0\|_X = 0.
$$

For each fixed $-\infty < t < \infty$,

$$
\lim_{t' \to t} \sup_{U_0 \in X(t')} \|U(t', t) - 1|U_0\|_X = 0.
$$

**Theorem 3.2.** Let $\{\mathcal{M}(t)\}_{t \in \mathbb{R}}$ be the exponential attractor constructed in Theorem 3.1. Let $U(t, s)$ satisfy (3.4) and (3.5), too. Then, $\mathcal{M}(t)$ is right continuous at any $t \in \mathbb{R}$, i.e., $\lim_{t' \to t^+} d(\mathcal{M}(t'), \mathcal{M}(t)) = 0$. If $t \neq nt^*$ for any $n \in \mathbb{Z}$, then $\mathcal{M}(t)$ is left continuous, too, i.e., $\lim_{t' \to t^-} d(\mathcal{M}(t'), \mathcal{M}(t)) = 0$. If $t = nt^*$ with some $n \in \mathbb{Z}$, then $\mathcal{M}(t)$ is at least left outer continuous, i.e., $\lim_{t' \to t^-} h(\mathcal{M}(t'), \mathcal{M}(t)) = 0$.

**REFERENCES**


DEPARTMENT OF APPLIED PHYSICS, GRADUATE SCHOOL OF ENGINEERING, OSAKA UNIVERSITY, SITA, OSAKA 565-0871, JAPAN.
Structural Health Monitoring of Nuclear Power Plants using Inverse Analysis in Measurements

Fumio Kojima
Organization of Advanced Science and Technology, Kobe University
1-1, Rokkodai, Nada-ku Kobe 657-8501 Japan
Phone/FAX: +81-78-803-6493
E-mail : kojima@koala.kobe-u.ac.jp

Abstract. Recently, interest has been grown for structural health monitoring related to ageing management of nuclear power plants (NPPs) in Japan. Especially, material failures in reactor containment have become a critical issue for safety of NPPs. Various kinds of nondestructive testing (NDT) techniques such as ultrasonic, eddy current testing, thermal testing, are applied to detecting and characterizing material damages like as stress corrosion cracking, piping wastage, etc. In this article, we show that modeling and simulation of NDT should be treated as a key component of the future structured monitoring technologies in NPPs. Our aim of this lecture is to show that the interaction between measurements and simulations of inspection process is an indispensable problem-solving methodology for implementing high performance monitoring of NPPs.

1. Introduction

Japanese utilities are currently operating fifty five nuclear power plants and cover about thirty percent of the total electricity generated in Japan. Nuclear power generation has been considered as a clean energy source with no carbon dioxide emissions. However increasing ageing plants need keeping a high level of safety and improving the safety of older nuclear power plants. Periodic safety review (PSR) is to assess the cumulative effects of plant ageing and plant modifications, operating experience, technical developments [1]. The reviews include an assessment of plant design and operation against current safety standards and practices, and they have the objective of ensuring a high level of safety throughout the plant’s operating lifetime.

Most Japanese nuclear power plants consist of the pressurized water reactor (PWR) and the boiling water reactor (BWR) as shown in Fig. 1. PWRs are the most common type of power producing nuclear reactor and there are two separate coolant loops. Heating the water in the primary coolant loop by thermal conduction through the fuel cladding and it is pumped into the steam generator, where heat is transferred to the lower pressure secondary coolant. On the contrary, BWR has only one coolant loop. Heat is produced by nuclear fission in the reactor core and the producing steam is directly used to drive a turbine. In PWR, the steam generator tubes constitute one of the primary barriers between the radioactive and non-radioactive sides of the plant. Therefore, in-service inspections of the steam generator tubes are essential in keeping safety of operations. In BWR, the core shroud is a large stainless steel cylinder of circumferentially welded plates surrounding the reactor fuel core. The shroud provides for the core geometry of the fuel bundles. Extensive cracking of circumferential welds on the core shroud has been discovered in a increasing number of Japanese BWRs plants since 2002. In such a situation, it is becoming clear that the aging of reactor components poses serious safety risks at NPPs. In Japan, a committee on ageing
management was established in the Nuclear and Industrial Safety Agency (NISA) which is a reorganization of central government ministries. The committee has confirmed the NISA report on improved ageing management in 2005 [2]. Technical evaluation review manuals for ageing management published by Japan Nuclear Energy Safety Organization (JNES) include the following eight degradation phenomena:

1. Neutron irradiation embrittlement of reactor vessels
2. Stress corrosion cracking, such as intergranular stress corrosion cracking (IGSCC), primary water stress corrosion cracking (PWSCC), irradiation-assisted stress corrosion cracking (IASCC), etc
3. Fatigue
4. Thinning of pipeing, such as flow accelerated corrosion (FAC), Liquid droplet impingement erosion (LDI), flashing erosion, cavitation erosion, etc
5. Insulation degradation of electrical cables of instruments and control facilities
6. Strength and shielding capability degradation of concrete
7. Seismic safety evaluation
8. Viewpoint and understanding of approaches to prevention for organization culture degradation

Stress corrosion cracks, such as IGSCC, PWSCC, and IASCC, are the critical phenomena of ageing management and NISA has performed their technical evaluations at every ten years after thirty years operation. To accomplish PSR for those components mentioned above, it is more important to verify the sizing accuracy of a material damage with the technology of non-destructive inspection applied to real plants and the feasible guideline will be developed to judge the adequacy of inspection.

In this paper, we discuss structural health monitoring of NPPs. Structural health monitoring (SHM) is an upcoming technology in keeping safety of large scale complex systems, such as nuclear power plants, airplanes, etc. SHM involves the broad concept of assessing ongoing and in-service performance of structures using variety of measurements. Those elements include sensors in structures, data acquisition, data management, data interpretation, diagnosis, etc. For convenience of discussions, the problem is focused into the characterization of stress corrosion cracking (SCC) of...
stainless steel used in recirculating pipe and in shroud in BWR plants. This paper is organized as follows. First, direct and inverse problems are mathematically formulated based on advanced electromagnetic nondestructive evaluation. Secondly, data interpretation and diagnosis are discussed within the framework of parameter identification method. Finally, current achievements are summarized with laboratory experiments.

2. Role of Simulation and Modeling

To ensure a high level of safety throughout the plant’s operating lifetime, in-service inspection (ISI) had been performed at every thirteen months of operation until 2008. Recently, Japanese utilities must implement a new safety regulation related to ageing management. In that process, an inspection cycle might be varied corresponding to operational conditions of each plant. This implies that the precise management for material failures are required. For instance, as shown in Fig. 2, if a crack has been detected in the current ISI, it is necessary to predict the crack progress in the next ISI and to judge whether the predicted progress exceeds the safety standards. The feasibility and reliability of quantitative non-destructive evaluation must be studied. Simulation of inspection procedures and modeling of material failures of reactor plants are crucial parts at the course of this investigation. In this sequel, we deal with the mathematical framework of our SHM system. For convenience of discussions, the problem is focused into the characterization of stress corrosion cracking (SCC) of stainless steel used in recirculating pipe and in shroud in BWR plants.

![Fig. 2 Illustration of structured integrity evaluation](image-url)
3. General Framework of Model Based Approach

Electromagnetic nondestructive testing is to find material flaws by evaluating structure sensitive electromagnetic properties from measurement data using appropriate sensors. Mathematical descriptions of NDT can be formulated as a direct and an inverse problem in electromagnetic fields [3-5]. A direct problem is to design a real NDT system mathematically using the input and output relation with the appropriate admissible class of material flaws, while an inverse problem is to construct a method for recovering and/or visualizing material flaw information under the mathematical formulation of the direct problem. Figure 3 demonstrates the overall configuration of our proposed system. In order to implement the system, we need the following four steps:

Step 1: Mathematical Modeling of NDT and Defect Profiles
Step 2: Numerical Scheme for Direct Problem
Step 3: Inverse Analysis for Model-based NDT
Step 4: Performance Test

4. Mathematical Model of Inspection

Eddy current analysis can be implemented by measuring voltage of detecting coil corresponding to the applied current of the exciting coil. Multiple transmitter-receiver coils have the capabilities of detecting crack orientations and distinguishing the adjacent cracks by choosing the transmitter-receiver pairs [6]. This new type of probes makes it possible to capture natural cracks, such as stress corrosion cracks (SCC), fatigue cracks

![Fig. 3 Overall configuration of model based diagnosis system](image)
Let \( B = \nabla \times A \) be magnetic flux density that has a complex phasor representation in three dimensions. The magnetic vector potential \( A = (A_1, A_2, A_3) \) at the neighborhood of the exciting coil is governed by an Euler equation

\[
- \frac{1}{\mu_0} \nabla^2 A = \chi_c \circ J_s ,
\]

where \( \mu_0, \chi_c, \) and \( J_s \) denote the magnetic permeability, the characteristic function with respect to the exciting coil, and the amplitude of the applied current, respectively. The eddy current generated in a conducting material can be derived from the diffusion equations:

\[
- \frac{1}{\mu_0} \nabla^2 A + j \omega \sigma (A + \nabla \Phi) = 0
\]

\[
\nabla \cdot j \omega \sigma (A + \nabla \Phi) = 0
\]

where \( \sigma \) and \( \omega \) denote the electrical conductivity and the angular frequency of the applied current. By virtue of the Biot-Savart's law, the detecting voltage can be obtained by

\[
V_d \propto - j \omega N c \int A_s \cdot dl
\]

where \( N_c \) and \( A_s \) denote the number of turns of the detecting coil and the magnetic vector potential at the detecting coil given by

\[
A_s(x) = j \omega \int \int \int_V \sigma(A + \nabla \Phi)(x') \frac{dx'}{|x - x'|}
\]

From the numerical points of views, the finite element method is a simple scheme and is easy to implement to solve the above problem. However it requires re-meshing at each measurement point. This re-meshing procedures result in considerable amount of computational efforts for the problem considered here. In our approach, the hybrid scheme of the finite element and the boundary element method [7] is adopted to the forward problem. The hybrid scheme of the finite element and the boundary element method is an effective method for the computational cost. Thus our numerical scheme is written by

\[
\begin{pmatrix}
(P) + j \omega (Q)(\sigma_h) + [K] \end{pmatrix} A_h = \begin{pmatrix}
F(J_s) \\
0
\end{pmatrix}
\]

(6)
where \( P \) and \( Q \) denote the finite element matrices corresponding to conducting region and where \( K \) denotes the boundary element matrix associated with the air region, respectively. In the numerical model, the existence of cracks is characterized by the intensity of electrical conductivities at each finite element as illustrated in Fig. 4.

5. **Inverse Analysis for Model-based NDT**

One possible way to solve such inverse problem is shown in Fig. 5 [8]. In the procedure, the database stores knowledge about ECT signals of cracks with different sizes and various scanning parameters, such as driving frequency, scanning patterns and scanning positions. Pre-processing examines measured images to determine crack length and position and to prescribe possible crack depth. After this step, a set of crack with initial size and position is created to be the input for the next step. A set of crack with possible size and position is created and finding peak value of measurement voltage and comparing with measurement results allow us to determine possible depth of crack created from process. Thus the proposed algorithm can set up possible the size, position, and depth of cracks. The output of Pre-process is a set of cracks with possible size and position, to be the input for the next process. Figure 6 depicts the real image for the natural crack, the recovered image, and the recovered crack shape.
6. Conclusions

In this article, it was shown that modeling and simulation of NDT is indispensable problem-solving methodology for implementing high performance monitoring of NPPs. Electromagnetic inverse methodologies for sizing stress corrosion crack were considered for boiled water reactor (BWR) plants. It is very crucial to characterize target cracking of SUS304 and SUS316L materials used in BWR plants since those involve various kind of complexities, such as orientation, multiple deep-lying branching, partially conducting, etc. The model based health monitoring system was outlined for sizing natural crack in three dimensions. The feasibility and validity of our inverse algorithm using laboratory data were demonstrated.

References

ファウドステッピング法による並流型熱交換器の
境界フィードバック安定化

Boundary Feedback Stabilization of Parallel-Flow
Heat Exchanger Process Using a
Forwardstepping Method

中桐 信一　神戸大学大学院工学研究科

1 はじめに

本研究は、佐野英樹氏との共同研究である。本研究では、ファウドステッピング法、言い換え
ると変形公式に基づいて境界制御により指定された安定指数をもつ系に変換する手法、を用いて
並流型熱交換器モデルの境界フィードバック安定化をはかる。
並流型熱交換器プロセスを下図で示す。

並流型 2 層流熱交換器

この並流型熱交換器モデルを記述する方程式系は

\[
\begin{align*}
\frac{\partial z_1}{\partial t} &= D \frac{\partial^2 z_1}{\partial x^2} - \alpha \frac{\partial z_1}{\partial x} + h_1(z_2 - z_1) + b(x)u_1(t), \\
\frac{\partial z_2}{\partial t} &= D \frac{\partial^2 z_2}{\partial x^2} - \alpha \frac{\partial z_2}{\partial x} + h_2(z_1 - z_2) + b(x)u_2(t), \quad (t, x) \in (0, \infty) \times (0, 1) \\
D \frac{\partial z_1}{\partial x}(t, 0) - \alpha z_1(t, 0) &= -\alpha v_1(t), \quad D \frac{\partial z_2}{\partial x}(t, 0) - \alpha z_2(t, 0) = -\alpha v_2(t), \quad (1.1) \\
\frac{\partial z_1}{\partial x}(t, 1) &= 0, \quad \frac{\partial z_2}{\partial x}(t, 1) = 0, \quad t \in (0, \infty) \\
z_1(0, x) &= \varphi_1(x), \quad z_2(0, x) = \varphi_2(x), \quad x \in [0, 1]
\end{align*}
\]

である。ここでは
・$z_1(t, x), z_2(t, x) \in R$ はそれぞれ時刻 $t$, 位置 $x \in [0, 1]$ における平行流の温度分布

・$u_1(t), u_2(t), v_1(t), v_2(t) \in R$ は制御入力

・$D > 0, \alpha > 0, \text{および} h_1, h_2$ は反応プロセスに依存した結合定数で $h_1 + h_2 \neq 0$

・$b(x)$ は反応動作分布関数

とする。制御系 (1.1) に、ゲイン $k > 0$ として出力フィードバック制御

$$u_1(t) = -kz_1(t, 1), \quad u_2(t) = -kz_2(t, 1) \quad (1.2)$$

を行うと制御系 (1.1) は次の境界制御系になる：

$$\frac{\partial z_1}{\partial t} = D \frac{\partial^2 z_1}{\partial x^2} - \alpha \frac{\partial z_1}{\partial x} + h_1(z_2 - z_1) - k\overline{b}(x)z_1(t, 1),$$

$$\frac{\partial z_2}{\partial t} = D \frac{\partial^2 z_2}{\partial x^2} - \alpha \frac{\partial z_2}{\partial x} + h_2(z_1 - z_2) - k\overline{b}(x)z_2(t, 1), \quad (t, x) \in (0, \infty) \times (0, 1)$$

$$D \frac{\partial z_1}{\partial x}(t, 0) - \alpha z_1(t, 0) = -\alpha v_1(t), \quad D \frac{\partial z_2}{\partial x}(t, 0) - \alpha z_2(t, 0) = -\alpha v_2(t), \quad (1.3)$$

$$\frac{\partial z_1}{\partial x}(t, 1) = 0, \quad \frac{\partial z_2}{\partial x}(t, 1) = 0, \quad t \in (0, \infty)$$

$$z_1(0, x) = \varphi_1(x), \quad z_2(0, x) = \varphi_2(x), \quad x \in [0, 1].$$

ここでの目的は、境界制御系 (1.3) に対してフォワードステッピング法による指数安定化のための境界制御則を構成することである。加えて、フォワードステッピング法とは、本質的には逆問題における変形公式に他ならないうことを注意する。またこのフォワードステッピング法は、理論のみならず数値解析的にも有効な手法になる得ることを注意しておこう。ここでの解析スキームを図示すると次のようになる。

フォワードステッピング法

\[
v_f(t) = -\frac{1}{\alpha} \int_0^t (DK_f(0, y) + (\alpha/2 - \alpha)K(0, y)) \exp(-\alpha y/2D)z_f(t, y) dy
\]

\[
v_f(t) = -\frac{1}{\alpha} \int_0^t (DK_f(0, y) + (\alpha/2 - \alpha)K(0, y)) \exp(-\alpha y/2D)z_f(t, y) dy
\]
系 (1.3) は、非局所項を持つ移流拡散方程式系である。非局所項がない移流拡散方程式系の可到達性・可視測性を扱った結果として Winkin et al [19], Sano and Nakagiri [8], [9] を挙げておく。

2 変形公式とは何か？

つきの熱方程式の初期境界値問題を考える：

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} - p(x)u, \quad (t, x) \in (0, \infty) \times (0, 1) \\
- \frac{\partial u(t, 0)}{\partial x} + hu(t, 0) &= \frac{\partial u(t, 1)}{\partial x} + Hu(t, 1) = 0, \quad t \in (0, \infty) \\
u(0, x) &= a(x), \quad x \in [0, 1].
\end{aligned}
$$

(2.1)

Gel'fand-Levitan 理論、言い換えると、熱方程式 (2.1) のスペクトル密度関数から $p \in C^1[0, 1], h \in R, H \in R$ を再構成する理論において、いわゆる変型公式が本質的に使われる。これを説明する。$p \in C^1[0, 1], h \in R, \lambda \in R$ に対し $\Phi = \Phi(x; p, h, \lambda)$ を初期値問題

$$
\begin{aligned}
- \frac{d^2 \Phi}{dx^2} + p(x)\Phi = \lambda \Phi, \quad x \in (0, 1) \\
\Phi(0) = 1, \quad \Phi'(0) = h
\end{aligned}
$$

(2.2)

の解とする。さらに $q \in C^1[0, 1], j \in R$ とする。このとき変型公式

$$
\Phi(x; q, j, \lambda) = \Phi(x; p, h, \lambda) + \int_0^x K(x, y)\Phi(y; p, h, \lambda)dy, \quad x \in [0, 1]
$$

(2.3)

がなりたつ。ここで、$D = \{(x, y) : 0 < y < x < 1\}$ として $K = K(x, y) \in C^2(D)$ は次の双曲型方程式の唯一解である：

$$
\begin{aligned}
K_{xx} - K_{yy} + p(y)K &= q(x)K, \quad (x, y) \in D \\
K(x, x) &= (j - h) + \frac{1}{2} \int_0^x (q(s) - p(s))ds, \quad x \in [0, 1] \\
K_y(x, 0) &= hK(x, 0), \quad x \in [0, 1].
\end{aligned}
$$

(2.4)

この変形公式は、一般に熱方程式 (2.1) を別の 1 次元熱方程式

$$
\begin{aligned}
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} - q(x)v(t, x), \quad (t, x) \in (0, \infty) \times (0, 1) \\
- \frac{\partial v(t, 0)}{\partial x} + ju(t, 0) &= \frac{\partial v(t, 1)}{\partial x} + Ju(t, 1) = 0, \quad t \in (0, \infty) \\
v(0, x) &= b(x), \quad x \in [0, 1]
\end{aligned}
$$

(2.5)

に変形する核関数 $K(x, y)$ そのものである。ここで

- $p \in C^1[0, 1], h \in R, H \in R$ は未知パラメータ
- $q \in C^1[0, 1], j \in R, J \in R$ は既知パラメータ

としている。この変形核 $K = K(x, y)$ を用いて Suzuki [14], [15], [16] は様々なタイプの逆問題を解いた。
放物型逆問題
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} - p(x)u(t, x), \quad (t, x) \in (0, \infty) \times (0, 1) \\
- \frac{\partial u(t, 0)}{\partial x} + hu(t, 0) &= \frac{\partial u(t, 1)}{\partial x} + Hu(t, 1) = 0, \quad t \in (0, \infty) \\
u(x, 0) &= a(x), \quad x \in [0, 1]
\end{aligned}
\]
の未知パラメータ \( p \in C^1[0, 1], h \in \mathbb{R}, H \in \mathbb{R} \) を境界観測から一意的に決定する問題

双曲型逆問題
\[
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} - p(x)u, \quad (t, x) \in (0, \infty) \times (0, 1) \\
- \frac{\partial u(t, 0)}{\partial x} + hu(t, 0) &= \frac{\partial u(t, 1)}{\partial x} + Hu(t, 1) = 0, \quad t \in (0, \infty) \\
u(x, 0) &= a(x), \quad u_t(x, 0) = b(x), \quad x \in [0, 1]
\end{aligned}
\]
の未知パラメータ \( p \in C^1[0, 1], h \in \mathbb{R}, H \in \mathbb{R} \) を境界観測から一意的に決定する問題

スペクトル逆問題
\[
\begin{aligned}
- \frac{d^2 \varphi}{dx^2} + p(x) \varphi &= \lambda \varphi(x), \quad x \in (0, 1) \\
- \frac{d \varphi(0)}{dx} + h \varphi(0) &= \frac{d \varphi(1)}{dx} + H \varphi(1) = 0
\end{aligned}
\]
の未知パラメータ \( p \in C^1[0, 1] \) をスペクトルデータ

\[
\sigma(A_P) = \{ \lambda_n \}_{n=0}^{\infty}, \quad P = (p, h, H) \\
\sigma(A_{P^*}) = \{ \lambda_n^* \}_{n=0}^{\infty}, \quad P^* = (p, h, H^*)
\]
から一意的に決定する問題 (Borg 理論)

さらには、より一般な双曲型方程式系の固有値問題
\[
\begin{aligned}
\frac{d \varphi_2}{dx} + p_{11}(x)\varphi_1(x) + p_{12}(x)\varphi_2 &= \lambda \varphi_1(x), \quad x \in (0, 1) \\
\frac{d \varphi_1}{dx} + p_{21}(x)\varphi_1 + p_{22}(x)\varphi_2 &= \lambda \varphi_2, \quad x \in (0, 1) \\
\varphi_2(0) + h \varphi_1(0) &= \varphi_2(1) + H \varphi_1(1) = 0
\end{aligned}
\]
に対して Yamamoto [20], [21], [22], Trooshin and Yamamoto [18] は 変形核を適切に構成することにより, (2.3) と類似の変形公式を導き, 対応する Gel’fand-Levitan 理論を構築し, その結果を関連するスペクトル逆問題および放物型逆問題に適用し, 数々の興味ある結果を導いている. 関連する逆問題についての結果の概説については, Nakagiri [5] を参照されたい.
3 非局所項をもつ拡散方程式に対する変形公式

本研究のきっかけになったのは、Smyshlyaev and Krstic [12], [13] の論文である。この論文において、彼らは次元非局所型拡散方程式

\[
\begin{cases}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - p(x)u + g(x)u(t, 0), & (t, x) \in (0, \infty) \times (0, 1) \\
u_x(t, 0) - \alpha u(t, 0) = 0, & u_x(t, 1) = U(t), \ t \in (0, \infty) \\
u(0, x) = u_0(x), & x \in [0, 1]
\end{cases}
\] (3.1)

を \( c \) を正の定数とする別の非常に単純な局所項のない1次元拡散方程式

\[
\begin{cases}
\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - cv, & (t, x) \in (0, \infty) \times (0, 1) \\
v_x(t, 0) - \alpha v(t, 0) = 0, & v(t, 1) := V(t) = U(t) - \int_0^1 k(1, y)u(t, y)dy, \ t \in (0, \infty) \\
v(0, x) = v_0(x), & x \in [0, 1]
\end{cases}
\] (3.2)

に変形する核関数 \( k(x, y) \) を求め、その変換を用いて非局所型拡散方程式 (3.1) の境界制御による安定化の問題を解いた。ここで、\( k(x, y) \) は次の2次元双曲型偏微分方程式の境界値問題の解として定義される:

\[
\begin{cases}
k_{xx}(x, y) - k_{yy}(x, y) = (p(y) + c)k(x, y), & (x, y) \in D \\
k_y(x, 0) = \alpha k(x, 0) + g(x) - \int_0^x k(x, y)g(y)dy, & x \in [0, 1] \\
k(x, x) = -\frac{1}{2} \int_0^x (p(y) + c)dy, & x \in [0, 1]
\end{cases}
\] (3.3)

Smyshlyaev and Krstic は、データ

- \( g \in C^1[0, 1], \ p \in C^1[0, 1], \ c, \alpha \in \mathbb{R} \)

の条件の下で、(3.3) の解を逐次近似の手法により構成している。

(注意 3.1) 方程式 (3.3) には境界条件に未知関数が含まれているので解の構成はそれほど容易ではない。

この節では、1次元非局所型拡散方程式

\[
\begin{cases}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - p(x)u + g(x)u(t, 0), & (t, x) \in (0, \infty) \times (0, 1) \\
-\frac{\partial u(t, 0)}{\partial x} + hu(t, 0) = \frac{\partial u(t, 1)}{\partial x} + Hu(t, 1) = 0, & t \in (0, \infty) \\
u(0, x) = u_0(x), & x \in [0, 1]
\end{cases}
\] (3.4)

を別の（同じタイプ）1次元非局所型拡散方程式

\[
\begin{cases}
\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - P(x)v(t, 0) + G(x)v(t, 0), & (t, x) \in (0, \infty) \times (0, 1) \\
-\frac{\partial v(t, 0)}{\partial x} + jv(t, 0) = \frac{\partial v(t, 1)}{\partial x} + Jv(t, 1) = 0, & t \in (0, \infty) \\
v(0, x) = v_0(x), & x \in [0, 1]
\end{cases}
\] (3.5)
に変形する核関数 \( K(x, y) \) が存在することを示す。さらに、その変換を用いて Smyshlyaev and Krstic [12] では取り扱われていない 非局所型拡散方程式系 (1.3) の境界制御問題を解く。得られた結果はより精密なものになっている。

【定理 3.1】 \( p, g \in C^1[0, 1], P, G \in C^1[0, 1], h, j \in \mathbb{R} \) とする。このとき 2 次元双曲型微分方程式の境界値問題

\[
\begin{align*}
K_{xx}(x, y) - K_{yy}(x, y) &= (P(x) - p(y))K(x, y), \quad (x, y) \in D \\
K_y(x, 0) &= hK(x, 0) - g(x) - \int_0^x K(x, y)g(y)dy + G(x), \quad x \in [0, 1] \\
K(x, x) &= (j - h) + \frac{1}{2} \int_0^x (P(y) - p(y))dy, \quad x \in [0, 1]
\end{align*}
\]

（3.6）

の解 \( k(x, y) \in C^2(D) \) が唯一存在し、評価

\[
\|K\|_{C^2(D)} \leq M(\|p\|_{C^1[0, 1]} + \|p\|_{C^1[0, 1]} + \|g\|_{C^1[0, 1]} + |h|)(\|p - P\|_{C^1[0, 1]} + \|g - G\|_{C^1[0, 1]} + |h - j|)
\]

が成り立つ。ここで \( M(r), r \geq 0 \) は \( r \) に関する単調増加関数である。

定理 3.1 の証明はかなり面倒なので、ここでは省略する。

【定理 3.2】 \( p, g \in C^1[0, 1], h, j \in \mathbb{R}, P, G \in C^1[0, 1], H \in \mathbb{R} \) とする。変型公式

\[
v(t, x) = u(t, x) + \int_0^x K(x, y)u(t, y)dy, \quad x \in [0, 1]
\]

（3.7）

により 1 次元非局所型拡散方程式

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} - p(x)u + g(x)u(t, 0), \quad (t, x) \in (0, \infty) \times (0, 1) \\
- \frac{\partial u(t, 0)}{\partial x} + hu(t, 0) &= 0, \quad \frac{\partial u(t, 1)}{\partial x} + Hu(t, 1) = U(t), \quad t \in (0, \infty) \\
u(0, x) &= u_0(x), \quad x \in [0, 1]
\end{align*}
\]

（3.8）

は 1 次元非局所型拡散方程式

\[
\begin{align*}
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} - P(x)v + G(x)v(t, 0), \quad (t, x) \in (0, \infty) \times (0, 1) \\
- \frac{\partial v(t, 0)}{\partial x} + jv(t, 0) &= 0, \quad \frac{\partial v(t, 1)}{\partial x} + Jv(t, 1) = V(t), \quad t \in (0, \infty) \\
v(0, x) &= v_0(x), \quad x \in [0, 1]
\end{align*}
\]

（3.9）

に変形される。ここで 定数 \( J \) は

\[
J = H - (j - h) - \int_0^1 (P(y) - p(y))dy
\]

（3.10）

で与えられ、制御変数 \( V(t) \) は、次で与えられる

\[
V(t) = U(t) + \int_0^1 \left[ JK(1, y) + K_x(1, y) \right] u(t, y)dy,
\]

（3.11）
（証明）変形公式（3.7）を \( t \) で微分して，\( u \) の方程式を使うと，

\[
v_t(t, x) = u_t(t, x) + \int_0^x K(x, y)\{u_{yy}(t, y) - p(y)u(t, y) + g(y)u(t, 0)\}dy
\]

\[
= u_{xx}(t, x) - p(x)u(t, x) + u(t, 0)\{g(x) + \int_0^x K(x, y)g(y)dy\}
\]

\[
+ \int_0^x u(t, y)[-p(y)K(x, y)]dy + \int_0^x K(x, y)u_{yy}(t, y)dy.
\]

部分積分を用いると，

\[
\int_0^x K(x, y)u_{yy}(t, y)dy = K(x, x)u_x(t, x) - K(x, 0)u_x(t, 0) - \int_0^x K_y(x, y)u_y(t, y)dy.
\]

\[
\int_0^x K_y(x, y)u_y(t, y)dy = K_y(x, x)u(t, x) - K_y(x, 0)u(t, 0) - \int_0^x K_{yy}(x, y)u(t, y)dy
\]

なので，

\[
\int_0^x K(x, y)u_{yy}(t, y)dy = K(x, x)u_x(t, x) - K_y(x, x)u(t, x)
\]

\[
- (K(x, 0)u_x(t, 0) - K_y(x, 0)u(t, 0)) + \int_0^x K_{yy}(x, y)u(t, y)dy.
\]

ここでさらに \( v(t, 0) = u(t, 0) \) を用いると

\[
v_t(t, x) = u_{xx}(t, x) - p(x)u(t, x) + v(t, 0)\{K_y(x, 0) + g(x) + \int_0^x K(x, y)g(y)dy\}
\]

\[
+ \int_0^x u(t, y)[K_{yy}(x, y) - p(y)K(x, y)]dy
\]

\[
+ K(x, x)u_x(t, x) - K_y(x, x)u(t, x) - K(x, 0)u_x(t, 0).
\]

（3.12）

変形公式（3.7）を \( x \) で微分して

\[
v_x(t, x) = u_x(t, x) + K(x, x)u(t, x) + \int_0^x K_x(x, y)u(t, y)dy.
\]

もう一度 \( x \) で微分すると

\[
v_{xx}(t, x) = u_{xx}(t, x) + \frac{d}{dx}K(x, x)u(t, x) + K(x, x)u_x(t, x) + K_x(x, x)u(t, x)
\]

\[
+ \int_0^x K_{xx}(x, y)u(t, y)dy.
\]

（3.13）

境界条件 \( u_x(t, 0) = hu(t, 0) = hv(t, 0) \) と式（3.13）を式（3.12）に代入して，

\[
v_t(t, x) = v_{xx}(t, x) - p(x)u(t, x) + v(t, 0)\{K_y(x, 0) - hK(x, 0) + g(x) + \int_0^x K(x, y)g(y)dy\}
\]

\[
+ \int_0^x u(t, y)[-K_{xx}(x, y) + K_{yy}(x, y) - p(y)K(x, y)]dy
\]

\[
+ K(x, x)u_x(t, x) - K_y(x, x)u(t, x) - \frac{d}{dx}K(x, x)u(t, x) + K(x, x)u_x(t, x) - K_x(x, x)u(t, x)
\]
\begin{align*}
&= v_{xx}(t, x) + v(t, 0)\{K_y(x, 0) - hK(x, 0) + g(x) + \int_0^x K(x, y)g(y)dy \}\nonumber \\
&+ \int_0^x u(t, y)[-K_{xx}(x, y) + K_{yy}(x, y) - p(y)K(x, y)]dy
\nonumber \\
&- p(x)u(t, x) - (d_{dx}K(x, x) + K_x(x, x) + K_y(x, x))u(t, x). \\
&= \frac{1}{2}(P(x) - p(x)) \\
\end{align*}

一方
\begin{equation}
\frac{d}{dx}K(x, x) = K_x(x, x) + K_y(x, x) = \frac{1}{2}(P(x) - p(x))
\end{equation}

なので
\begin{equation}
-p(x)u(t, x) - (d_{dx}K(x, x) + K_x(x, x) + K_y(x, x))u(t, x) = -P(x)u(t, x).
\end{equation}

従ってこれらを式 (3.14) に代入すると
\begin{align*}
v_t(t, x) &= v_{xx}(t, x) + v(t, 0)\{K_y(x, 0) - hK(x, 0) + g(x) + \int_0^x K(x, y)g(y)dy \} \\
&+ \int_0^x u(t, y)[-K_{xx}(x, y) + K_{yy}(x, y) - p(y)K(x, y)]dy - P(x)u(t, x).
\end{align*}

ここで、\( k(x, y) \) の満たすべき方程式 (3.6) を使うと
\begin{align*}
v_t(t, x) &= v_{xx}(t, x) + v(t, 0)G(x) + \int_0^x u(t, y)[-P(x)K(x, y)]dy - P(x)u(t, x) \\
&= v_{xx}(t, x) + v(t, 0)G(x) - P(x)\left[u(t, x) + \int_0^x K(x, y)u(t, y)dy \right].
\end{align*}

すなわち
\begin{equation}
v_t(t, x) = v_{xx}(t, x) - P(x)v(t, x) + G(x)v(t, 0) + \int_0^x F(x, y)v(t, y)dy \tag{3.15}
\end{equation}

が示される。\( v \) についての \( x = 0 \) での境界条件を見ていく。
\begin{equation}
v_x(t, x) = u_x(t, x) + K(x, x)u(t, x) + \int_0^x K_x(x, y)u(t, y)dy \nonumber
\end{equation}

であったから、(3.6) の境界条件より、\( K(0, 0) = (j - h) \) よって

\begin{equation}
v_x(t, 0) = u_x(t, 0) + K(0, 0)u(t, 0) = hu(t, 0) + (j - h)u(t, 0) = jv(t, 0).
\end{equation}

つまり、

\begin{equation}
-\frac{\partial}{\partial x}v(t, 0) + jv(t, 0) = 0.
\end{equation}

つぎに \( v \) についての \( x = 1 \) での境界条件を見ていく。
\begin{align*}
v(t, 1) &= u(t, 1) + \int_0^1 K(1, y)u(t, y)dy, \\
&= u_x(t, 1) + K(1, 1)u(t, 1) + \int_0^1 K_x(1, y)u(t, y)dy \\
&= u_x(t, 1) + [j - h + \int_0^1 (P(y) - p(y))dy]u(t, 1) + \int_0^1 K_x(1, y)u(t, y)dy.
\end{align*}
したがって

\[ v_x(t, 1) + Jv(t, 1) = u_x(t, 1) + [(j - h) + \int_0^1 (P(y) - p(y)) dy] u(t, 1) + \int_0^1 K_x(1, y) u(t, y) dy 
+ Ju(t, 1) + J \int_0^1 K(1, y) u(t, y) dy, \]

\[ = u_x(t, 1) + [(j - h) + J + \int_0^1 (P(y) - p(y)) dy] u(t, 1) 
+ \int_0^1 [JK(1, y) + K_x(1, y)] u(t, y) dy. \]

今与えられた \( H \) に対して, \( J \) を

\[ J = J(H, j, h, p, P) = H - (j - h) - \int_0^1 (P(y) - p(y)) dy \]

なるように選ぶと

\[ v_x(t, 1) + Jv(t, 1) = V(t) 
= U(t) + \int_0^1 [JK(1, y) + K_x(1, y)] u(t, y) dy \]

がいえる. 残りの \( v \) についての初期条件

\[ v(0, x) = u_0(x) + \int_0^x K(x, y) u_0(y) dy, \quad x \in [0, 1] \]

は変形公式を代入すれば確かめられる. \( \square \)

(注意 3.2) 定理 3.1 と 定理 3.2 は 方程式 (3.8) に Volterra 積分項 \( \int_0^t f(x, y) u(t, y) dy \) が含まれ、さらに境界条件も非局所型になる場合に拡張できる。

4 単独系のフォワードステッピング法による安定化

\( D > 0, \alpha > 0, b \in C^1[0, 1] \) として, 次の非局所項をも含む単独拡散方程式の境界制御系を考える。

\[ \frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2} - b(x) v(t, 1), \quad (t, x) \in (0, \infty) \times (0, 1) \]

\[ D \frac{\partial v}{\partial x}(t, 0) - \alpha v(t, 0) = -\alpha V(t), \quad \frac{\partial v}{\partial x}(t, 1) + \frac{\alpha}{2D} v(t, 1) = 0, \quad t \in (0, \infty) \]

\[ v(0, x) = v_0(x), \quad x \in [0, 1]. \]

この方程式を変形核 \( k(x, y) \) を用いて, 局所項のない次の指数安定な拡散方程式に変換しよう。

\[ \frac{\partial \psi}{\partial t} = D \frac{\partial^2 \psi}{\partial x^2} - c \psi, \quad (t, x) \in (0, \infty) \times (0, 1) \]

\[ \frac{\partial \psi}{\partial x}(t, 0) + \left( \frac{c}{2} - \alpha \right) \psi(t, 0) = 0, \quad \frac{\partial \psi}{\partial x}(t, 1) = 0, \quad t \in (0, \infty) \]

\[ \psi(0, x) = \psi_0(x), \quad \text{a.e. } x \in [0, 1]. \]
ここで，定数 \( c > 0 \) および \( \psi_0 \in L^2(0,1) \) とする。パラメータの変形は，

\[
g(x) := -b(x) \to 0, \quad p(x) := 0 \to -c, \quad h := \frac{\alpha}{2D} \to \frac{c}{2} - \alpha, \quad H := \frac{\alpha}{2D} \to 0
\]

であり制御変数 \( V(t) \) をうまく選んでやることにより

\[
\frac{\partial \psi}{\partial x}(t, 0) + \left( \frac{c}{2} - \alpha \right) \psi(t, 0) = 0
\]

となるように変数核 \( k(x,y) \) を構成したい。新たに \( D \) と対称な領域

\[
\Omega = \{(x,y) : 0 < x < y < 1\}
\]

を導入する。このとき変数核 \( k(x,y) \) のみたすべき方程式は

\[
\begin{aligned}
& D_k_{xx}(x,y) - D_k_{yy}(x,y) = -ck(x,y), \quad (x,y) \in \Omega \\
& D_k_y(x,1) = -\frac{\alpha}{2} k_x(x,1) + b(x) - \int_x^1 k(x,y)b(y)dy, \quad x \in [0,1] \\
& k(x,x) = -\frac{c}{2D}(1-x) + \frac{\alpha}{2D}, \quad x \in [0,1]
\end{aligned}
\]

となる。ちなみに，方程式系 (4.2) については次の指数安定性がいえる。

【補題 4.1】任意の初期値 \( \psi_0 \in L^2(0,1) \) に対し，方程式 (4.2) の解 \( \psi = \psi(t,\cdot) \in C([0,\infty); L^2(0,1)) \) は唯一つ存在して Fourier 級数展開

\[
\psi(t,x) = \sum_{n=1}^{\infty} e^{-\mu_n^c t} \langle \psi_0, \phi_n \rangle \phi_n(c;x), \quad \text{a.e. } x \in [0,1], \quad \forall t \geq 0
\]

で表される。ここで \( \{\mu_n^c, \phi_n(c;x)\} \) は

\[
\begin{cases}
\mu_n^c = D s_n^2 + c, \quad \phi_n(c;x) = K_n \left[ \cos s_n x - \left( \frac{c}{2} - \alpha \right) \frac{1}{s_n} \sin s_n x \right], \quad \forall n \geq 1, \quad \text{if } c \neq 2\alpha \\
\mu_n^c = D(n-1)^2 \pi^2 + c, \quad \phi_1(c;x) \equiv 1, \quad \phi_n(c;x) = \frac{1}{\sqrt{2}} \cos(n-1)\pi x, \quad \forall n \geq 2, \quad \text{if } c = 2\alpha
\end{cases}
\]

で与えられ，\( \{s_n : n \geq 1\} \) は超越方程式

\[
s \tan s = \left( \frac{c}{2} - \alpha \right), \quad s > 0
\]

の \( 0 < s_n < s_{n+1}, \forall n \geq 1 \) なる解であり，\( K_n \) は \( L^2(0,1) \) における正規化のための定数である。さらに，解 \( \psi \) は次の指数安定性をもつ：

\[
\|\psi(t,\cdot)\|_{L^2} \leq M_0 \exp(-\mu_1^c t) \|\psi_0\|_{L^2}, \quad \forall t \geq 0.
\]

（注意 4.1）もし初期値 \( \psi_0 \in H^m(0,1) \) ならば，(4.6) の代わりに方程式 (4.2) の解 \( \psi \) は,

\[
\|\psi(t,\cdot)\|_{H^m} \leq M_m \exp(-\mu_1^c t) \|\psi_0\|_{H^m}, \quad \forall t \geq 0
\]

を満たす。ここで \( M_m \geq 1 \) は \( \psi_0 \) に関係しない定数である。
変形核 \( k(x, y) \) の存在は次の補題で示される。

【補題 4.2】 \( c > 0 \) および \( b \in C^1[0,1] \) とする。このとき次の双曲型境界値問題の一つの解
\[ k(x, y) = k(c, b; x, y) \in C^2(\Omega) \] が存在する。

\[
\begin{align*}
Dk_{xx}(x, y) - Dk_{yy}(x, y) &= -ck(x, y), \quad (x, y) \in \Omega \\
Dk_y(x, 1) &= \frac{\alpha}{2}k(x, 1) + b(x) - \int_x^1 k(x, y)b(y)dy, \quad x \in [0, 1] \\
k(x, x) &= -\frac{c}{2D}(1 - x) + \frac{\alpha}{2D}, \quad x \in [0, 1].
\end{align*}
\]

さらに (4.7) の解 \( k(x, y) \) は次の評価をもつ:
\[
|k(x, y)| \leq Me^{\alpha y}, \quad (x, y) \in \Omega.
\]
ここで \( M \) は \( D, \alpha, c \in \mathbb{R}, \quad \|b\|_{C} = \max\{|b(x)| : x \in [0, 1]\} \) にのみ依存する正の定数である。

（証明） 変数変換
\[
(x, y) \mapsto (1 - x, 1 - y)
\]
により (4.7) は

\[
\begin{align*}
k_{xx}(x, y) - k_{yy}(x, y) &= -\frac{c}{2D}k(x, y), \quad (x, y) \in D \\
k_y(x, 0) &= \frac{\alpha}{2D}k(x, 0) - \frac{1}{2D}b(x) + \int_0^x k(x, y)\frac{1}{2D}b(y)dy, \quad x \in [0, 1] \\
k(x, x) &= \frac{c}{2D}x - \frac{1}{2D}(c - \alpha), \quad x \in [0, 1]
\end{align*}
\]
に変換される。よって定理 3.1 を適用できて解の一意的性質が成り立つ。

【補題 4.3】 \( c > 0 \) および \( b \in C^1[0,1] \) とする。\( k(x, y) = k(c, b; x, y) \in C^2(\Omega) \) を補題 2 の核関数,
\( \psi \in L^2(0,1) \) とする。このとき, Volterra 積分方程式
\[
\phi(x) = \psi(x) - \int_x^1 k(x, y)\phi(y)dy, \quad \text{a.e. } x \in [0, 1]
\]
の解 \( \phi(x) \in L^2(0,1) \) が唯一存在して,
\[
\phi(x) = \psi(x) - \int_x^1 r(x, y)\psi(y)dy, \quad \text{a.e. } x \in [0, 1]
\]
で与えられる。ここで, (4.12) におけるレゾルベント核 \( r \in C^2(\Omega) \) は, レゾルベント方程式
\[
r(x, y) = -k(x, y) + \int_x^y r(x, \xi)k(\xi, y)d\xi, \quad (x, y) \in \Omega
\]
の唯一解である。

（証明） 変数変換 (4.9) により方程式 (4.11) は
\[
\phi(x) = \psi(x) - \int_0^x k(x, y)\phi(y)dy, \quad \text{a.e. } x \in [0, 1]
\]
に変換される。よって Volterra 方程式に関する良く知られた結果からレゾルベント核の存在と解の公式（4.11）が従う。詳しい証明は Miller [4] を参照されたい。

変形公式により、非局所型拡散方程式の境界制御系

\[
\frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2} - b(x) v(t, 1), \quad (t, x) \in (0, \infty) \times (0, 1)
\]

\[
D \frac{\partial v}{\partial x}(t, 0) - \frac{\alpha}{2} v(t, 0) = -\alpha V(t), \quad \frac{\partial v}{\partial x}(t, 1) + \frac{\alpha}{2D} v(t, 1) = 0, \quad t \in (0, \infty)
\]

\[
v(0, x) = v_0(x), \quad x \in [0, 1]
\]

は，\( k(x, y) \) を補題 4.2 の核関数として，変換

\[
\psi(t, x) := v(t, x) - \int_x^1 k(x, y)v(t, y)dy
\]

により，局所項のない指数安定な拡散方程式に変換される:

\[
\frac{\partial \psi}{\partial t} = D \frac{\partial^2 \psi}{\partial x^2} - c \psi, \quad (t, x) \in (0, \infty) \times (0, 1)
\]

\[
\frac{\partial \psi}{\partial x}(t, 0) + \left(\frac{c}{2} - \alpha\right) \psi(t, 0) = -\alpha V(t) - \int_0^1 \left\{Dk_x(0, y) + \left(\frac{c}{2} - \alpha\right)k(0, y)\right\} v(t, y)dy,
\]

\[
\frac{\partial \psi}{\partial x}(t, 1) = 0, \quad t \in (0, \infty)
\]

\[
\psi(0, x) = \psi_0(x), \quad x \in [0, 1].
\]

これを確かめよう。初期値 \( v_0(x) \) を

\[
W_c := \left\{ \varphi \in H^2(0, 1); \ D\varphi'(0) - \frac{\alpha}{2} \varphi(0) = -\frac{1}{\alpha} \int_0^1 \left\{Dk_x(0, y) + \left(\frac{c}{2} - \alpha\right)k(0, y)\right\} \varphi(y)dy \right\}
\]

から選び，制御項 \( V(\cdot) \in H^1(0, T), \forall T > 0 \) とすれば 解 \( v(t, x) \) は強い解になる。従って以下の計算は意味をもつ。変換 (4.16) を微分して，

\[
\psi_x(t, x) = v_x(t, x) + k(x, x)v(t, x) - \int_x^1 k_x(x, y)v(t, y)dy.
\]

これと式 (4.10) で \( x = 0 \) を代入して \( k(0, 0) \) を求めるとき

\[
\psi(t, 0) = v(t, 0) - \int_0^1 k(0, y)v(t, y)dy,
\]

\[
\psi_x(t, 0) = v_x(t, 0) + \frac{\alpha - c}{2D} v(0, 0) - \int_0^1 k_x(0, y)v(t, y)dy
\]

が得られる。上の 2 式より，

\[
D \psi_x(t, 0) + \left(\frac{c}{2} - \alpha\right) \psi(t, 0)
= \ D v_x(t, 0) - \frac{\alpha}{2} v(t, 0) - \int_0^1 \left\{Dk_x(0, y) + \left(\frac{c}{2} - \alpha\right)k(0, y)\right\} v(t, y)dy
\]

\[
\quad = -\alpha V(t) - \int_0^1 \left\{Dk_x(0, y) + \left(\frac{c}{2} - \alpha\right)k(0, y)\right\} v(t, y)dy, \quad t \in (0, \infty).
\]
同時に
\[ \psi(t, 1) = v(t, 1), \quad \psi_x(t, 1) = v_x(t, 1) + k(1, 1)v(t, 1) = v_x(t, 1) + \frac{\alpha}{2D}v(t, 1) = 0. \]
すなわち
\[ \frac{\partial \psi}{\partial x}(t, 1) = 0, \quad t \in (0, \infty) \]
が示された。

\[ x = 0 \] での解 \( \psi(t, x) \) の境界条件が 0 になるように制御項 \( V(t) \) を決めるのが、フォワードステッピング法の基本になる。即ち次の定理が証明されたことになる。

【定理 4.1】 令 \( c > 0 \) および \( b \in C^1[0, 1] \) とする。単独の境界制御系
\[ \frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2} - b(x)v(t, 1), \quad (t, x) \in (0, \infty) \times (0, 1) \]
\[ D \frac{\partial v}{\partial x}(t, 0) - \frac{\alpha}{2}v(t, 0) = -\alpha V(t), \quad \frac{\partial v}{\partial x}(t, 1) + \frac{\alpha}{2D}v(t, 1) = 0, \quad t \in (0, \infty) \] (4.19)
\[ v(0, x) = v_0(x), \quad x \in [0, 1] \]
を考える。\( k(x, y) = k(c, b; x, y) \) を補題 4.2 の核関数として、集合
\[ W_c := \left\{ \varphi \in H^2(0, 1); \right. \quad D\varphi'(0) - \frac{\alpha}{2}\varphi(0) = -\frac{1}{\alpha} \int_0^1 \left\{ Dk_x(0, y) + \left( \frac{c}{2} - \alpha \right)k(0, y) \right\} \varphi(y)dy \right\} \]
導入する。このときフォワードステッピング法による制御則
\[ V(t) = -\frac{1}{\alpha} \int_0^1 \left\{ Dk_x(0, y) + \left( \frac{c}{2} - \alpha \right)k(0, y) \right\} v(t, y)dy \] (4.20)
により閉ループ系 (4.19), (4.20) は、任意の初期値 \( v_0 \in W_c \) に対して唯一解 \( v \in C([0, \infty); W_c) \cap C^1((0, \infty); L^2(0, 1)) \) を持ち、次の結果安定性が導入できる：
\[ \|v(t, \cdot)\|_{H^2} \leq Me^{-\nu_1 t}\|v_0\|_{H^2}, \quad \forall t \geq 0. \] (4.21)
さらに、\( v(t) \) は \( r(x, y) \) をのレゾルベント核として
\[ v(t, x) = \psi(t, x) - \int_x^1 r(x, y)\psi(t, y)dy \] (4.22)
で与えられる。ここで \( \psi(t, x) \) は補題 4.1 で与えられる方程式系の解である。

(注 4.2) 任意の \( v_0 \in L^2(0, 1) \) に対しても、閉ループ系 (4.19), (4.20) の唯一の一般化された解 (mild solution) \( v(t, \cdot) \) が存在し
\[ \|v(t, \cdot)\|_{L^2} \leq Me^{-\nu_1 t}\|v_0\|_{L^2}, \quad \forall t \geq 0 \]
がなりたつ。

5 結合された非局所型境界制御系の安定化

この節の目的は、1 節で述べた並流型熱交換器モデルを境界フィードバックにより安定化をはかることである。非局所項をもつ境界制御系 (1.3) において、以下簡単に

\[ \tilde{b}(x) := k\varphi(x) \] (5.1)

とおく。すなわち考察する境界制御系は、

\[ \begin{align*}
\frac{\partial z_1}{\partial t} &= D \frac{\partial^2 z_1}{\partial x^2} - \alpha \frac{\partial z_1}{\partial x} + h_1(z_2 - z_1) - \tilde{b}(x)z_1(t, 1), \\
\frac{\partial z_2}{\partial t} &= D \frac{\partial^2 z_2}{\partial x^2} - \alpha \frac{\partial z_2}{\partial x} + h_2(z_1 - z_2) - \tilde{b}(x)z_2(t, 1), \quad (t, x) \in (0, \infty) \times (0, 1) \\
D \frac{\partial z_1}{\partial x}(t, 0) - \alpha z_1(t, 0) &= -\alpha v_1(t), \\
D \frac{\partial z_2}{\partial x}(t, 0) - \alpha z_2(t, 0) &= -\alpha v_2(t),
\end{align*} \] (5.2)

で与えられる。ここで、\( h_1, h_2 \) は反応プロセスに依存した定数で \( h_1 + h_2 \neq 0 \) と仮定する。

我々の最終目的は、制御入力 \( v_1(t), v_2(t) \in \mathbb{R} \) をうまく選んでやり、境界制御系 (5.2) を任意指数で安定化をはかることである。制御系 (5.2) に変換

\[ \begin{align*}
\tilde{z}_1(t, x) &= z_1(t, x)e^{-\frac{\alpha v_1(t)}{2}x}, \\
\tilde{z}_2(t, x) &= z_2(t, x)e^{-\frac{\alpha v_2(t)}{2}x}
\end{align*} \] (5.3)

を施すと、次の制御系が得られる:

\[ \begin{align*}
\frac{\partial \tilde{z}_1}{\partial t} &= D \frac{\partial^2 \tilde{z}_1}{\partial x^2} + h_1(\tilde{z}_2 - \tilde{z}_1) - \tilde{b}(x)\tilde{z}_1(t, 1), \\
\frac{\partial \tilde{z}_2}{\partial t} &= D \frac{\partial^2 \tilde{z}_2}{\partial x^2} + h_2(\tilde{z}_1 - \tilde{z}_2) - \tilde{b}(x)\tilde{z}_2(t, 1), \quad (t, x) \in (0, \infty) \times (0, 1) \\
D \frac{\partial \tilde{z}_1}{\partial x}(t, 0) - \frac{\alpha}{2} \tilde{z}_1(t, 0) &= -\alpha v_1(t), \\
D \frac{\partial \tilde{z}_2}{\partial x}(t, 0) - \frac{\alpha}{2} \tilde{z}_2(t, 0) &= -\alpha v_2(t), \quad (5.4)
\end{align*} \]

\[ \begin{align*}
D \frac{\partial \tilde{z}_1}{\partial x}(t, 1) + \frac{\alpha}{2} \tilde{z}_1(t, 1) &= 0, \\
D \frac{\partial \tilde{z}_2}{\partial x}(t, 1) + \frac{\alpha}{2} \tilde{z}_2(t, 1) &= 0, \quad t \in (0, \infty) \\
\tilde{z}_1(0, x) = \varphi_1(x), \quad \tilde{z}_2(0, x) = \varphi_2(x), \quad x \in [0, 1].
\end{align*} \]

ここで

\[ b(x) := e^{\frac{\alpha v_1(t)}{2}(1-x)}\tilde{b}(x), \quad \varphi_1(x) = e^{-\frac{\alpha v_1(t)}{2}x}\varphi_1(x), \quad \varphi_2(x) = e^{-\frac{\alpha v_2(t)}{2}x}\varphi_2(x). \] (5.5)

結合系 (5.4) を線形変換により 2 つのサブシステムに分解する。

サブシステムへの分解 I

変換

\[ f(t, x) := \tilde{z}_1(t, x) - \tilde{z}_2(t, x) \] (5.6)
により，システム（5.4）からサブシステム
\[
\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2} - (h_1 + h_2)f - b(x)f(t, 1), \quad (t, x) \in (0, \infty) \times (0, 1)
\]
\[
D \frac{\partial f}{\partial x}(t, 0) - \frac{\alpha}{2} f(t, 0) = -\alpha(v_1(t) - v_2(t)), \quad D \frac{\partial f}{\partial x}(t, 1) + \frac{\alpha}{2} f(t, 1) = 0, \quad t \in (0, \infty)
\]
\[
f(0, x) = \tilde{\varphi}_1(x) - \tilde{\varphi}_2(x) := f_0(x), \quad x \in [0, 1]
\]
が構成される。

サブシステムへの分解 II
変換
\[
g(t, x) := h_2\tilde{z}_1(t, x) + h_1\tilde{z}_2(t, x)
\]
により，システム（5.4）からサブシステム
\[
\frac{\partial g}{\partial t} = D \frac{\partial^2 g}{\partial x^2} - b(x)g(t, 1), \quad (t, x) \in (0, \infty) \times (0, 1)
\]
\[
D \frac{\partial g}{\partial x}(t, 0) - \frac{\alpha}{2} g(t, 0) = -\alpha(h_2v_1(t) + h_1v_2(t)), \quad D \frac{\partial g}{\partial x}(t, 1) + \frac{\alpha}{2} g(t, 1) = 0, \quad t \in (0, \infty)
\]
\[
g(0, x) = h_2\tilde{\varphi}_1(x) + h_1\tilde{\varphi}_2(x) := g_0(x), \quad x \in [0, 1]
\]
が構成される。

任意の減衰指数 \( \nu > 0 \) に対し，
\[
c_\nu := \begin{cases} \nu - Ds^2_1, & \text{if } \nu \neq 2\alpha \\ 2\alpha, & \text{if } \nu = 2\alpha \end{cases}
\]
とおく。今 サブシステム（5.7），（5.9）において
\[
b(x) := ke^{\frac{\omega}{2\omega}}(1-x)b(x)
\]
なることを注意しておく。フィードバックシステム（5.4）に対して 核関数 \( K(x, y) \) と集合 \( W \) を次で定義する。
\[
K(x, y) := k(c_\nu, b; x, y),
\]
\[
W := \left\{ \varphi \in H^2(0, 1); \quad D\varphi'(0) - \frac{\alpha}{2} \varphi(0) = -\frac{1}{\alpha} \int_0^1 \left\{ DK_y(0, y) + \left( c_\nu \frac{\alpha}{2} - \alpha \right) K(0, y) \right\} e^{-\frac{\omega}{2\omega}} \varphi(y) dy \right\}
\]
このとき，任意の \( \nu > 0 \) に対し，\( \mu_1^{c_\nu} = \nu \) である。さらに \( R(x, y) \)を \( k(x, y) = K(x, y) := k(c_\nu, b; x, y) \) に対する補題 4.3 のレゾルベント核とする。
【定理 5.1】 境界制御系

\[
\frac{\partial z_1}{\partial t} = D \frac{\partial^2 z_1}{\partial x^2} - \alpha \frac{\partial z_1}{\partial x} + h_1(z_2 - z_1) - b(x)z_1(t, 1),
\]

\[
\frac{\partial z_2}{\partial t} = D \frac{\partial^2 z_2}{\partial x^2} - \alpha \frac{\partial z_2}{\partial x} + h_2(z_1 - z_2) - b(x)z_2(t, 1), \quad (t, x) \in (0, \infty) \times (0, 1)
\]

\[
D \frac{\partial z_1}{\partial x}(t, 0) - \alpha z_1(t, 0) = -\alpha v_1(t), \quad D \frac{\partial z_2}{\partial x}(t, 0) - \alpha z_2(t, 0) = -\alpha v_2(t), \quad t \in (0, \infty)
\]

\[
\frac{\partial z_1}{\partial x}(t, 1) = 0, \quad \frac{\partial z_2}{\partial x}(t, 1) = 0, \quad t \in (0, \infty)
\]

\[
z_1(0, x) = \varphi_1(x), \quad z_2(0, x) = \varphi_2(x), \quad x \in [0, 1]
\]

を考える．境界制御則を

\[
\left\{\begin{array}{l}
\nu_1(t) = -\frac{1}{\alpha} \int_0^1 \left\{ DK_x(0, y) + \left( \frac{c_\nu}{2} - \alpha \right) K(0, y) \right\} e^{-\frac{\alpha}{2} y} z_1(t, y) dy \\
\nu_2(t) = -\frac{1}{\alpha} \int_0^1 \left\{ DK_x(0, y) + \left( \frac{c_\nu}{2} - \alpha \right) K(0, y) \right\} e^{-\frac{\alpha}{2} y} z_2(t, y) dy
\end{array}\right.
\]

により定義する．さらに

\[
W := \left\{ \varphi \in H^2(0, 1); D\varphi'(0) - \frac{\alpha}{2} \varphi(0) = -\frac{1}{\alpha} \int_0^1 \left\{ DK_x(0, y) + \left( \frac{c_\nu}{2} - \alpha \right) K(0, y) \right\} e^{-\frac{\alpha}{2} y} \varphi(y) dy \right\}
\]

とおく．このとき，任意の初期値 \([\varphi_1, \varphi_2]^T \in W \times W\) に対し閉ループ系 (5.14), (5.15) の唯一つの解

\[
[z_1(\cdot), z_2(\cdot)]^T \in C([0, \infty); W \times W) \cap C^1((0, \infty); L^2(0, 1) \times L^2(0, 1))
\]

が存在して

\[
||[z_1(t), z_2(t)]^T||_{H^2 \times H^2} \leq M e^{-\nu t} ||[\varphi_1, \varphi_2]^T||_{H^2 \times H^2}, \quad \forall t \geq 0.
\]

さらに \(z_1(t, x), z_2(t, x)\) は具体的に

\[
z_1(t, x) = \frac{h_1 e^{\frac{c_\nu}{2} x}}{h_1 + h_2} e^{-(h_1 + h_2)t} \left[ \psi_1(t, x) - \int_x^1 R(x, y) \psi_1(t, y) dy \right]
\]

\[
+ \frac{h_2 e^{\frac{c_\nu}{2} x}}{h_1 + h_2} \left[ \psi_2(t, x) - \int_x^1 R(x, y) \psi_2(t, y) dy \right],
\]

\[
z_2(t, x) = \frac{-h_2 e^{\frac{c_\nu}{2} x}}{h_1 + h_2} e^{-(h_1 + h_2)t} \left[ \psi_1(t, x) - \int_x^1 R(x, y) \psi_1(t, y) dy \right]
\]

\[
+ \frac{h_1 e^{\frac{c_\nu}{2} x}}{h_1 + h_2} \left[ \psi_2(t, x) - \int_x^1 R(x, y) \psi_2(t, y) dy \right]
\]

と表現される．ここで \(\psi_1(t, x)\) と \(\psi_2(t, x)\) はそれぞれ

\[
\psi_1(t, x) = \sum_{n=1}^{\infty} e^{-\mu_n t} \langle \psi_0, \phi_n \rangle \phi_n(c_\nu; x), \quad x \in [0, 1],
\]

\[
\psi_2(t, x) = \sum_{n=1}^{\infty} e^{-\mu_n t} \langle \psi_0, \phi_n \rangle \phi_n(c_\nu; x), \quad x \in [0, 1]
\]
で与えられ、初期値 \( \psi_0^1(x) \) と \( \psi_0^2(x) \) はそれぞれ
\[
\psi_0^1(x) = e^{-\frac{x}{\alpha}}(\varphi_1(x) - \varphi_2(x)) - \int_x^1 K(x, y)e^{-\frac{x}{\alpha}y}(\varphi_1(y) - \varphi_2(y))dy,
\]
\[
\psi_0^2(x) = e^{-\frac{x}{\alpha}}(h_2\varphi_1(x) + h_1\varphi_2(x)) - \int_x^1 K(x, y)e^{-\frac{x}{\alpha}y}(h_2\varphi_1(y) + h_1\varphi_1(y))dy
\]
と与えられる。

(証明) システム (5.14) は 2 つのサブシステム (5.7), (5.9) へ分解できた。変換
\[
\tilde{f}(t, x) := e^{(h_1+h_2)t}f(t, x)
\]
により、系 (5.7) は
\[
\frac{\partial \tilde{f}}{\partial t} = D \frac{\partial^2 \tilde{f}}{\partial x^2} - b(x)\tilde{f}(t, 0), \quad (t, x) \in (0, \infty) \times [0, 1]
\]
\[
D\tilde{f}_x(t, 0) - \frac{\alpha}{2} \tilde{f}(t, 0) = -\alpha e^{(h_1+h_2)t}(v_1(t) - v_2(t)), \quad D\tilde{f}_x(t, 1) + \frac{\alpha}{2} f(t, 1) = 0, \quad t \in (0, \infty)
\]
\[
\tilde{f}(0, x) = f(0, x), \quad x \in [0, 1]
\]
に変換される。集合 \( \tilde{W} \) を
\[
\tilde{W} := \left\{ \varphi \in H^2(0, 1); D\varphi'(0) - \frac{\alpha}{2} \varphi(0) = -\frac{1}{\alpha} \int_0^1 \left\{ DK_x(0, y) + \left( \frac{c_\nu}{2} - \alpha \right) K(0, y) \right\} \varphi(y) dy \right\}
\]
により定義する。このとき、任意の初期値 \( [\tilde{\varphi}_0^1, \tilde{\varphi}_0^2]^T \in \tilde{W} \times \tilde{W} \) に対して \( \tilde{f}(0, x) = \tilde{f}_0(x) = \tilde{\varphi}_0^1(x) - \tilde{\varphi}_0^2(x) \in \tilde{W} \) であるから、定理 4.1 よりサブシステム (5.7) に制御則
\[
V_1(t) := e^{(h_1+h_2)t}(v_1(t) - v_2(t)) = -\frac{1}{\alpha} \int_0^1 \left\{ DK_x(0, y) + \left( \frac{c_\nu}{2} - \alpha \right) K(0, y) \right\} \tilde{f}(t, y) dy,
\]
\[
= -\frac{1}{\alpha} e^{(h_1+h_2)t} \int_0^1 \left\{ DK_x(0, y) + \left( \frac{c_\nu}{2} - \alpha \right) K(0, y) \right\} f(t, y) dy
\]
を加えた閉ループ系は唯一つの解 \( \tilde{f} \in C([0, \infty); \tilde{W}) \cap C^1([0, \infty); L^2(0, 1)) \) を持つ指数的減衰
\[
\| \tilde{f}(t, \cdot) \|_{H^2} \leq Me^{-\nu t} \| \tilde{f}_0 \|_{H^2}, \quad \forall t > 0
\]
がいえる。さらに \( \tilde{f}(t, x) \) は次のように表現される。
\[
\tilde{f}(t, x) = \psi_1(t, x) - \int_x^1 R(x, y)\psi_1(t, y) dy.
\]
ここで \( \psi_1(t, x) \) は 補題 4.1 で与えた解離数である。\( f(t, x) = e^{-(h_1+h_2)t}\tilde{f}(t, x) \) であったから、\( f \in C([0, \infty); W) \cap C^1([0, \infty); L^2(0, 1)) \) かつ
\[
f(t, x) = e^{-(h_1+h_2)t} \left[ \psi_1(t, x) - \int_x^1 R(x, y)\psi_1(t, y) dy \right],
\]
\[ Df_x(t,0) - \frac{\alpha}{2} f(t,0) = -\frac{1}{\alpha} \int_0^1 \left\{ DK_x(0,y) + \left( \frac{c_y}{2} - \alpha \right) K(0,y) \right\} f(t,y) dy \]  

（5.30）

がえる。全く同様にしてサブシステム (5.9) に制御則

\[ V_2(t) := h_2 v_1(t) + h_1 v_2(t) = \int_0^1 \left\{ DK_x(0,y) + \left( \frac{c_y}{2} - \alpha \right) K(0,y) \right\} g(t,y) dy, \]  

すなわち

\[ Dg_x(t,0) - \frac{\alpha}{2} g(t,0) = -\alpha \int_0^1 \left\{ DK_x(0,y) + \left( \frac{c_y}{2} - \alpha \right) K(0,y) \right\} g(t,y) dy \]  

（5.32）

を加えた閉ループ系は唯一の解 \( g \in C([0,\infty); \tilde{W}) \cap C^1((0,\infty); L^2(0,1)) \) を持つ指数的減衰

\[ \|g(t,\cdot)\|_{H^2} \leq M e^{-\nu t} \|g_0\|_{H^2}, \quad \forall t > 0 \]  

（5.33）

が得られる。\( g(0,x) := g_0 = h_2 \tilde{\varphi}_1(x) + h_1 \tilde{\varphi}_2(x) \in \tilde{W} \) であり、\( g(t,x) \) は次のように表現される。

\[ g(t,x) = v_2(t,x) - \int_x^1 R(x,y) v_2(t,y) dy. \]  

（5.34）

ここで \( v_2(t, x) \) は補題 4.1 で与えた解級数である。

最後に、求める制御則を元々の制御変数 \( v_1(t), v_2(t) \) と状態変数 \( z_1(t, x), z_2(t, x) \) で表現しよう。サブシステム (5.7), (5.9) に対する制御則より

\[ v_1(t) - v_2(t) = -\frac{1}{\alpha} \int_0^1 \left\{ DK_x(0,y) + \left( \frac{c_y}{2} - \alpha \right) K(0,y) \right\} (\tilde{z}_1(t,y) - \bar{z}^2(t,y)) dy, \]

\[ h_2 v_1(t) + h_1 v_2(t) = -\frac{1}{\alpha} \int_0^1 \left\{ DK_x(0,y) + \left( \frac{c_y}{2} - \alpha \right) K(0,y) \right\} (h_2 \tilde{z}_1(t,y) + h_1 \bar{z}_2(t,y)) dy \]

が成り立ち、この関係から

\[ v_1(t) = -\frac{1}{\alpha} \int_0^1 \left\{ DK_x(0,y) + \left( \frac{c_y}{2} - \alpha \right) K(0,y) \right\} \tilde{z}_1(t,y) dy, \]

\[ v_2(t) = -\frac{1}{\alpha} \int_0^1 \left\{ DK_x(0,y) + \left( \frac{c_y}{2} - \alpha \right) K(0,y) \right\} \bar{z}_2(t,y) dy \]

が従う。この制御則のもとでシステム (5.14) は唯一の解

\[
[z_1(t, x), z_2(t, x)]^T \quad = \quad \left[ \frac{1}{h_1 + h_2} (h_1 e^{\frac{c_y}{2}x} f(t, x) + e^{\frac{c_y}{2}x} g(t, x)), \quad \frac{1}{h_1 + h_2} (h_2 e^{\frac{c_y}{2}x} f(t, x) + e^{\frac{c_y}{2}x} g(t, x)) \right]^T
\]

\( \in C([0,\infty); W \times W) \cap C^1((0,\infty); L^2(0,1) \times L^2(0,1)) \)

をもつ。以上により定理が証明された。 □

非局所項をもつ平行 3 層流拡散型熱交換プロセスにも拡張できることを注意しておく（cf. Bielski and Malinowski [1], Sano [7], Nakagiri and Sano [6])。

\[
\begin{align*}
\frac{\partial z_1}{\partial t} &= D \frac{\partial^2 z_1}{\partial x^2} - \alpha \frac{\partial z_1}{\partial x} + h_{12}(z_2 - z_1) - \bar{b}(x)z_1(t, 1), \\
\frac{\partial z_2}{\partial t} &= D \frac{\partial^2 z_2}{\partial x^2} - \alpha \frac{\partial z_2}{\partial x} + h_{21}(z_1 - z_2) + h_{23}(z_3 - z_2) - \bar{b}(x)z_2(t, 1), \\
\frac{\partial z_3}{\partial t} &= D \frac{\partial^2 z_3}{\partial x^2} - \alpha \frac{\partial z_3}{\partial x} + h_{32}(z_2 - z_3) - \bar{b}(x)z_3(t, 1), \quad (t, x) \in (0, \infty) \times (0, 1) \\
D \frac{\partial z_1}{\partial x}(t, 0) - \alpha z_1(t, 0) &= -\alpha v_1(t), \quad D \frac{\partial z_2}{\partial x}(t, 0) - \alpha z_2(t, 0) = -\alpha v_2(t), \\
D \frac{\partial z_3}{\partial x}(t, 0) - \alpha z_3(t, 0) &= -\alpha v_3(t), \\
\frac{\partial z_1}{\partial x}(t, 1) = \frac{\partial z_2}{\partial x}(t, 1) &= \frac{\partial z_3}{\partial x}(t, 1) = 0, \quad t \in (0, \infty) \\
z_1(0, x) &= \varphi_1(x), \quad z_2(0, x) = \varphi_2(x), \quad z_3(0, x) = \varphi_3(x), \quad x \in [0, 1].
\end{align*}
\]

ここで \( h_{12}, h_{21}, h_{23}, h_{32} \) は反応プロセスに依存した正の定数である。

References


Abstract

Lagrangian swarm models consider long-range attraction and short-range repulsion between individuals, moving with the velocity of the swarm’s centroid, as a seed in the formation of the swarm itself and its behavior. By constructing a Lyapunov function based on this heuristic rule, we create a relatively simple gradient system which surprisingly exhibits complex emergent or self-organized motions in the absence of fixed or moving obstacles. The Lyapunov function contains an inter-individual collision-avoidance component; hence the component is bounded, yet it guarantees collision avoidance. Three parameters are utilized, and which we call cohesion parameter, coupling parameter, and convergence parameter. They are, respectively, a measure of the strengths of the cohesion of the swarm, the interaction between any two individuals and the instantaneous velocity of an individual with respect to the swarm centroid. By varying these parameters in a precise way, computer simulations show that for a sufficiently large number of individuals, our proposed model generates four types of swarming-like behaviors. They are (1) the cruise formation (linear or nonlinear) reminiscent of a cruising and leaderless school of fish, or a moving herd of land animals with a leader (leader-following), (2) random walks similar to the swarming behavior of fruit flies *Drosophila melanogaster*, (3) constant arrangements requiring individuals to aggregate and stop, as in fruiting body formation by bacteria, and (4) circular motions reminiscent of the behavior of a school of fish when threatened by a predator.

1 Introduction

With the amount of work carried out over the last three decades on studying and modelling swarms, beginning with the work of Okubo (1980), it is now possible to group different modelling approaches into at least two; the Eulerian and the Lagrangian approaches. In the Eulerian approach, the swarm is considered a continuum described by its density. In the Lagrangian approach, the state (position, instantaneous velocity and instantaneous acceleration) of each individual and its relationship with other individuals in the swarm is studied; it is an individual-based modelling. A question, posed by Edelstein-Keshet (2001) in her descriptive survey of mathematical models of swarming and social aggregation, vividly elucidates the dichotomy between the two; “are we following a given individual to see how it is affected by its neighbors, or are we watching the herd move past us as a density wave?” Edelstein-Keshet and colleagues indeed provided two separate illuminating papers on a continuum model and a Lagrangian model for swarms [Mogilner and Edelstein-Keshet (1999); Mogilner et al (2003)]. Excellent reviews of these approaches and their advantages and disadvantages can be found in Gazi and Passino (2004b) and Merrifield (2006).

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*School of Computing, Information & Mathematical Sciences, University of the South Pacific, Suva, Fiji. E-mail: vanualailai@usp.ac.fj
†Same as above. E-mail: bibhya.sharma@usp.ac.fj
Our interest lies in utilizing the key component of the Lagrangian approach, and that is, the use of attraction and repulsion functions to model the swarming behavior in which there is a long-range attraction and a short-range repulsion between individuals in the swarm [Edelstein-Keshet (2001), Gazi and Passino (2004b)]. This behavior leads to aggregation and formation, which are important for the survival of the members of the swarm [Brown and Warburton (1999); Ekanayaka and Pathirana (2009)]. If constructed appropriately, these attraction and repulsive functions, can be expressed as a gradient of some artificial or social potential function. This means that there is a Lyapunov function, a minimum of which corresponds to a stable stationary state of the individual-based Lagrangian system [Edelstein-Keshet (2001)]. As noted in Gazi and Passino (2003, 2004b), the use of a gradient system ensures there is an element of distribution of tasks among the members of the swarm and that the swarm members are performing distributed optimization. Indeed, because of the existence of the Lyapunov function, each individual in the swarm is individually and optimally searching a minimum. The stability conditions provided by the Lyapunov function can also provide the cohesiveness of the swarm in which the distances between individuals in the swarm are bounded from above [Mogilner et al (2003)].

The model by Mogilner et al (2003) uses a class of attraction and repulsion functions that are formed using both exponential and power laws. A recent stream of modelling that utilizes the same basic form of this class of interaction functions is traceable to the work of Gazi and Passino (2003). The attraction-repulsion function has an attraction component that dominates for large distances and a repulsion component that dominates for small distances. The stability conditions, provided by a Lyapunov function, are used to estimate the size of the swarms. In 2004, the authors extended their 2003 results by also considering interactions between individuals and their environment [Gazi and Passino (2004b)]. Specifically, they considered a swarm that is moving in a profile of nutrients or toxic substances, i.e., an attractant/repellent profile. Also in 2004, the authors provided another type of attraction-repulsion function [Gazi and Passino (2004a)].

The 2003 Gazi-Passino model is isotropic; there is uniformity in attraction or repulsion between all members of the swarm. Moreover, it is reciprocal; every member moves toward every other member exactly the same amount as the other member moves toward it. In 2003, Chu and colleagues generalized the Gazi-Passino model to include anisotropy [Chu et al (2003)]. Their anisotropic model contains a coupling matrix that allows the interaction strength between individuals in a swarm to vary. They assumed that the interactions between only at least two individuals, and not all, were reciprocal. In 2004, Wang and colleagues removed this reciprocity argument by adopting an asymmetric coupling matrix [Wang et al (2004)].

A shortcoming of the the Gazi-Passino model and its variants mentioned above is that they do not have a collision-avoidance capability between members of the swarm because the attraction-repulsion functions does not grow to infinity when individuals collide. The effect of their attraction-repulsion functions is only enough for each individual to move towards the center of the swarm and stop without collapsing to a tight cluster. To resolve this issue, Liu and colleagues, in 2005, introduced a repulsion term which is inversely proportional to the forth-power of the distance between two individuals [Liu et al (2005)]. In 2009, they expanded their work to obtain swarm models that are non-reciprocal and exhibit self-organized oscillations [Liu et al (2009)].

In 2006, Chen and Fang (2006a,b) added a component to a Geza-Passino-like model, and produced a system that is practically scalable, in the sense that regardless of how large the size of the swarm is, there is no or limited cost associated with any increase in size. This is in contrast to other Geza-Passino-like models in which every member knows the state of every other member in the swarm. However, Gazi and Passino (2004b) has argued that sensing limitations in engineering applications, like controlling robots, could be solved with technologies such as the Global Positioning System. Indeed, as reported in Martinoli et al (2004), distributed control principles had been successfully applied to a series
of case studies in collective robotics (aggregation and segregation, foraging, collaborative
stick pulling, cooperative transportation, flocking and navigation in formation, odor source
localization, cooperative mapping, and soccer tournaments) for which several approaches
extensively exploited global communication capabilities.

Another stream of modelling within the Lagrangian framework uses algebraic graph
theory, potential functions and the Lyapunov method to study flocking. The work of Olfati-
Saber (2006) provides distributed and decentralized algorithms with obstacle avoidance ca-
pabilities. In 2007, Tanner and colleagues used, in addition, non-smooth analysis to con-
struct discontinuous controllers that ensure a robust flock model [Tanner et al (2007)]. In
2008, Gu and Hu used fuzzy logic and the algebraic graph theory, together with non-smooth
analysis, to create functions for collision avoidance between members in a flock [Gu and Hu
(2008)]. These three scalable models express the three well-known heuristic flocking rules of
Reynolds (1987) into precise mathematical statements.

Our approach is also Lagrangian; hence, we consider spacing between individuals, which
moves with the velocity of the swarm centroid, of primary importance. We create functions
to measure the distance between individuals, and use them to move the individuals toward
the swarm centroid and for collision-avoidance between the members of the swarm, and for
avoidance of any fixed obstacle in the swarm's path. We do this by having these functions as
part of a Lyapunov function, which in turn generates the appropriate forms of the velocity
of each individual. These velocity components, in turn, are used to construct a gradient
system of first-order ODEs that govern the motion of the swarm. The system is a gradient
system because its component is the gradient of the Lyapunov function. Therein lies a major
contribution of this paper, and that is, because the Lyapunov function is non-increasing in
time, every solution of the system is bounded, yet collision avoidance will occur. This is a
depart from the model of Liu et al (2009) which requires an unbounded attraction-repulsion
function to guarantee collision avoidance. As explained later in some detail in subsection 5.3,
our deterministic system will always have a sufficiently large value of the Lyapunov function
at the initial state. This ensures sufficient control efforts for collision avoidance. In other
words, during collision avoidance, it is not the Lyapunov function that increases in time,
but rather its instantaneous rate of change, with respect to time, that increases in absolute
value. This corresponds to sufficient control efforts required for collision avoidance. Another
major contribution of this paper is the precise use of three parameters – the convergence,
coupling and cohesive parameters – to predict the behavior of the swarm, and to allow for
an isotropic and a reciprocal swarm model, or anisotropic and non-reciprocal swarm model,
a generality missing in other Lagrangian models discussed above.

Our approach is a result of a development of a Lyapunov-based robot control technique
that was proposed by Stonier Stonier (1990), whose work is an application of the Lyapunov
method to qualitative differential games that involve dynamical systems subject to control
by one or more players [Leitmann and Skowronski (1977), Getz and Leitmann (1979), Stonier
(1983)]. Using these differential games concepts, Stonier constructs Lyapunov-like functions
that provide nonlinear controllers for collision-avoidance between robot arms, and between
robot arms and stationary objects. The main advantage of this global potential approach,
as classified by Lee (2004), is the ease at which it can be used to extract control laws based
on velocity or acceleration. Stonier's work was later expanded and improved by Vannualailai
et al (1995, 1998). Their paper was the basis of further improvements by Ha and Shim
(2001). In addition, Vannualailai et al (2008) applied their method to the point stabilization
of nonholonomic vehicles. Further, Sharma et al (2009, 2010) applied the method to a flock
of nonholonomic vehicles.

In this paper, we show that our approach, which utilizes three parameters, not only
captures the basic feature of aggregation, cohesion and stability of a swarm, but also exhibits
more complex dynamics such as random walks, and self-organized oscillatory motions.

We begin in the next section by re-calling the Direct Method of Lyapunov and the
2 The Lyapunov Function and Gradient Systems

Here, we briefly recall some of the important Lyapunov stability concepts, and properties of gradient systems. The book by Hirsch et al (2004) is a good reference material.

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space with the Euclidean norm $\| \cdot \|$. Let $X = (x_1, x_2, \ldots, x_n)$ denote an element of $\mathbb{R}^n$. Consider an autonomous nonlinear system

$$\dot{X} = F(X), \quad X(t_0) = X_0, \quad t_0 \geq 0,$$

where $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be smooth enough to guarantee the existence, uniqueness and continuous dependence of solutions $X(t) = X(t; t_0, X_0)$ of (1) in $\Omega$, an open set in $\mathbb{R}^n$.

For the purpose of considering stability concept in the sense of Lyapunov, we assume there is a point $X^* \in \mathbb{R}^n$ such that $F(X^*) = 0$. Then $X(t) \equiv X^*$ is trivially a solution of (1) through $X^* \in \Omega$ for all $t \geq t_0$. We call $X^*$ an equilibrium point of system (1).

The equilibrium point $X^*$ of (1) is stable if, for each $\epsilon > 0$ and $t_0 \geq 0$, there is a $\delta = \delta(t_0, \epsilon) > 0$ such that $\|X_0 - X^*\| < \delta$ implies $\|X(t) - X^*\| < \epsilon$ for all $t \geq t_0$. The equilibrium point $X^*$ of (1) is said to be asymptotically stable if it is stable and there exists $\delta(t_0) > 0$ such that $\|X_0 - X^*\| < \delta$ implies $\lim_{t \to \infty} \|X(t) - X^*\| = 0$.

Lyapunov’s Direct Method (also called the Second Method of Lyapunov) is summarized in the following theorem, where $\mathbb{R}^+ := [0, \infty)$:

**Theorem 1 (Lyapunov Theorem)** Let $X^*$ be an equilibrium point of (1) and let $V : \Omega \to \mathbb{R}^+$ be a $C^1$ function defined on some neighborhood $\Omega$ of $X^*$ such that (i) $V(X^*) = 0$, (ii) $V(X) > 0$ for $X \in \Omega \setminus \{X^*\}$ and (iii) $\dot{V}(X) \leq 0$ for all $X \in \Omega$. Then $X^*$ is stable. If (iii) is replaced by (iii)' $\dot{V}(1)(X) < 0$ for all $X \in \Omega \setminus \{X^*\}$, then $X^*$ is asymptotically stable.

The well-known LaSalle’s Invariance Principle gives an alternative asymptotic stability condition:

**Theorem 2 (LaSalle’s Invariance Principle)** Let $V : \Omega \subset \mathbb{R}^n \to \mathbb{R}^+$, $\Omega_c = \{X \in \Omega : V(X) \leq c\}$, and suppose $\dot{V}(1)(X) \leq 0$ on $\Omega_c$. Let $E = \{X \in \Omega_c : \dot{V}(1)(X) = 0\}$. Then every solution of (1) tends to the largest invariant set in $E$ as $t \to \infty$. In particular, if $E$ contains no invariant set other than $\{X^*\}$, then $X^*$ is asymptotically stable.

We refer to $V$ in Theorem 1 and Theorem 2 as a Lyapunov function for system (1). Its gradient is the vector field

$$\nabla V = \left( \frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n} \right).$$

A gradient system on $\mathbb{R}^n$ is a system of differential equations of the form

$$\dot{X} = -\nabla L(X),$$

where $L$ is a Lyapunov function for system (2), and $\dot{L}(2)(X^*) = 0$ if and only if $X^*$ is an equilibrium point of system (2). That is, the critical points of $L$ are the equilibrium points of the system. Moreover, as discussed in Hirsch et al (2004), if a critical point is an isolated minimum of $L$, then this point is an asymptotically stable equilibrium point of system (2).
3 A Two-Dimensional Swarm Model

A general swarm model, formulated by Mogilner et al (2003), considers a swarm of \( n \) individuals being viewed in a coordinate system moving with the velocity of the swarm’s centroid. We shall follow this formulation in this paper, but with a divergent approach to the construction of the attraction-repulsion function.

We confine ourselves to constructing a 2-dimensional version of the model, as it will be a simple matter to extend it to 3-dimensional.

At time \( t \geq 0 \), let \((x_i(t), y_i(t))\), \( i = 1, 2, \ldots, n \), be the planar position of the \( i \)th individual, which we shall define as a point mass residing in a disk of radius \( r_i > 0 \),

\[
\begin{align*}
  b_i = \{(z_1, z_2) \in \mathbb{R}^2 : (z_1 - x_i)^2 + (z_2 - y_i)^2 \leq r_i^2\}.
\end{align*}
\]

(3)

The disk, incidently, is described in Mogilner et al (2003) as a bin, and in Gazi and Passino (2004b) as a private or safety area of each individual. Also, as discussed in Gazi and Passino (2004a), there are some good reasons why individuals in a swarm could be considered as a point mass; for instance, when considering some organisms such as bacteria.

Let us define the centroid of the swarm as

\[
(x_c, y_c) = \left( \frac{1}{n} \sum_{k=1}^{n} x_k, \frac{1}{n} \sum_{k=1}^{n} y_k \right).
\]

At time \( t \geq 0 \), let \((v_i(t), w_i(t)) := (x'_i(t), y'_i(t))\) be its instantaneous velocity, which we will need to show that is relative to the swarm centroid.

Using the above notations, we have thus a system of first-order ODEs for the \( i \)th individual, assuming the initial condition at \( t = t_0 \geq 0 \):

\[
\begin{align*}
  x'_i(t) &= v_i(t) \\
  y'_i(t) &= w_i(t) \\
  x_{i0} := x_i(t_0), y_{i0} := y_i(t_0).
\end{align*}
\]

(4)

Suppressing \( t \), we let \( x_i = (x_i, y_i) \in \mathbb{R}^2 \) and \( x = (x_1, \ldots, x_n) \in \mathbb{R}^{2n} \) be our state vectors. Also, let

\[
x_0 = x(t_0) = \underbrace{\{x_{10}, y_{10}, \ldots, x_{n0}, y_{n0}\}}_{2n \text{ terms}}.
\]

If \( f_i(x) := (v_i, w_i) \in \mathbb{R}^2 \) and \( V(x) := (f_1(x), \ldots, f_n(x)) \in \mathbb{R}^{2n} \), then our swarm system of \( n \) individuals is

\[
\dot{x} = V(x), \quad x_0 = x(t_0).
\]

(5)

An equilibrium point of system (5) for which (4) is the \( i \)th component will be denoted \( x^*_i = (x^*_1, \ldots, x^*_n) \in \mathbb{R}^{2n} \).

We will use the following two terms from Mogilner et al (2003):

1. A cohesive group is a group in which the distances between individuals are bounded from above (members of a cohesive group tend to stay together and avoid dispersing).

2. A well-spaced group is a group which does not collapse into a tight cluster, i.e., where some minimal bin size exists such that each bin contains at most one individual. Moreover, the size of such a bin is independent of the number of individuals in a group.
4 A Lyapunov Function with Attraction and Repulsion Components

4.1 Attraction to the Centroid

We can ensure that individuals are attracted to each other and also form a cohesive group by having a measurement of the distance from the \( i \)th individual to the swarm centroid. This is the concept behind flock centering, which is one of the well-known three heuristic flocking rules of Reynolds (1987). The rule stipulates that individuals stay close to nearest flock mates. It is therefore a form of attraction between individuals. Centering necessitates a measurement of the distance from the \( i \)th individual to the swarm centroid. Thus, we define the following function:

\[
R_i(x) := \frac{1}{2} \left[ \left( x_i - \frac{1}{n} \sum_{i=1}^{n} x_i \right)^2 + \left( y_i - \frac{1}{n} \sum_{i=1}^{n} y_i \right)^2 \right].
\]

It will be part of a Lyapunov function for system (4), and as we shall see later, its role is to ensure that \( i \)th individual is attracted to the swarm centroid.

4.2 Inter-individual Collision Avoidance

The short-range repulsion between individuals necessitates a measurement of the distance between the \( i \)th and the \( j \)th individuals, \( j \neq i \in \mathbb{N} \). With the definition (3) of the \( i \)th individual in mind, we define the following function for this purpose:

\[
R_{ij}(x) := \frac{1}{2} \left[ (x_i - x_j)^2 + (y_i - y_j)^2 - (r_i + r_j)^2 \right].
\]

It will also be part of the same Lyapunov function we mentioned above.

5 Swarming in the Absence of Obstacles

We first consider the scenario where there are no obstacles in the environment. We formally construct the Lyapunov function and then discuss its form and its relationship to swarming.

5.1 Lyapunov Function

Let there be real numbers \( \gamma_i > 0, \beta_{ij} > 0 \), and define, for \( i, j = 1, \ldots, n \),

\[
L_i(x) = \gamma_i R_i(x) + \sum_{j=1, j \neq i}^{n} \beta_{ij} \frac{R_i(x)}{R_{ij}(x)}.
\]

Consider as a tentative Lyapunov function for system (5),

\[
L(x) = \sum_{i=1}^{n} L_i(x_i).
\]

It is clear that \( L \) is continuous and locally positive definite on the domain

\[
D_1(L) := \{ x \in \mathbb{R}^n : R_{ij}(x) > 0, i, j \in \mathbb{N} \}.
\]

This means that if \( x^* \) is an equilibrium point of system (5) for which \( L \) is a Lyapunov function, then \( L(x) > 0 \) for all \( x \in D_1(L) \setminus \{x^*\} \) and \( L(x^*) = 0 \), with \( x^* \in D_1(L) \).
The time-derivative of \( L \) along a solution of system (5) is the dot product of the gradient of \( L \),

\[
\nabla L = \left( \frac{\partial L}{\partial x_1}, \frac{\partial L}{\partial y_1}, \ldots, \frac{\partial L}{\partial x_n}, \frac{\partial L}{\partial y_n} \right),
\]

and the time-derivative of the state vector \( \mathbf{x} = (x_1, y_1, \ldots, x_n, y_n) \). That is,

\[
\dot{L}_{(4)}(\mathbf{x}) = \nabla L(\mathbf{x}) \cdot \dot{\mathbf{x}} = \sum_{i=1}^{n} \left( \dot{R}_i(\mathbf{x}) + \sum_{j=1, \ j \neq i}^{n} \beta_{ij} \dot{R}_{ij}(\mathbf{x}) - \sum_{j=1, \ j \neq i}^{n} \beta_{ij} R_i(\mathbf{x}) \dot{R}_{ij}(\mathbf{x}) \right),
\]

where

\[
\dot{R}_i(\mathbf{x}) = -\sum_{i=1}^{n} \left[ \left( x_i - \frac{1}{n} \sum_{k=1}^{n} x_k \right) - \frac{1}{n} \sum_{m=1}^{n} \left( x_m - \frac{1}{n} \sum_{k=1}^{n} x_k \right) \right] x'_i
\]

\[
+ \sum_{i=1}^{n} \left[ \left( y_i - \frac{1}{n} \sum_{k=1}^{n} y_k \right) - \frac{1}{n} \sum_{m=1}^{n} \left( y_m - \frac{1}{n} \sum_{k=1}^{n} y_k \right) \right] y'_i,
\]

and

\[
\sum_{i=1}^{n} \sum_{j=1, \ j \neq i}^{n} \dot{R}_{ij}(\mathbf{x}) = 2 \sum_{i=1}^{n} \sum_{j=1, \ j \neq i}^{n} (x_i - x_j) x'_i + 2 \sum_{i=1}^{n} \sum_{j=1, \ j \neq i}^{n} (x_i - x_j) y'_i.
\]

Noting that \( \sum_{m=1}^{n} \left( u_m - \frac{1}{n} \sum_{k=1}^{n} u_k \right) = 0 \) for any \( u_i \in \mathbb{R}, \ i = 1, 2, \ldots, n \), we simplify the former expression to

\[
\dot{R}_i(\mathbf{x}) = \sum_{i=1}^{n} \left[ \left( x_i - \frac{1}{n} \sum_{k=1}^{n} x_k \right) x'_i + \left( y_i - \frac{1}{n} \sum_{k=1}^{n} y_k \right) y'_i \right].
\]

Now, collecting terms with \( x'_i \) and \( y'_i \), and substituting \( x'_i = v_i \) and \( y'_i = w_i \) from system (4), we have

\[
\dot{L}_{(5)}(\mathbf{x}) = \sum_{i=1}^{n} \left( \gamma_i + \sum_{j=1, \ j \neq i}^{n} \frac{\beta_{ij}}{R_{ij}(\mathbf{x})} \right) \left( x_i - \frac{1}{n} \sum_{k=1}^{n} x_k \right) - 2 \sum_{j=1, \ j \neq i}^{n} \frac{\beta_{ij} R_i(\mathbf{x})}{R_{ij}(\mathbf{x})} (x_i - x_j) \right) \dot{x}_i
\]

\[
+ \sum_{i=1}^{n} \left( \gamma_i + \sum_{j=1, \ j \neq i}^{n} \frac{\beta_{ij}}{R_{ij}(\mathbf{x})} \right) \left( y_i - \frac{1}{n} \sum_{k=1}^{n} y_k \right) - 2 \sum_{j=1, \ j \neq i}^{n} \frac{\beta_{ij} R_i(\mathbf{x})}{R_{ij}(\mathbf{x})} (y_i - y_j) \right) \dot{y}_i
\]

\[
= \sum_{i=1}^{n} \left[ \frac{\partial L}{\partial x_i} \dot{x}_i + \frac{\partial L}{\partial y_i} \dot{y}_i \right] = \sum_{i=1}^{n} \left[ \frac{\partial L}{\partial x_i} \cdot v_i + \frac{\partial L}{\partial y_i} \cdot w_i \right].
\]

Let there be real numbers \( \alpha_i^1 > 0 \) and \( \alpha_i^2 > 0 \) such that

\[
v_i = -\alpha_i^1 \frac{\partial L}{\partial x_i}, \quad \text{and} \quad w_i = -\alpha_i^2 \frac{\partial L}{\partial y_i}.
\]

Then

\[
\dot{L}_{(5)}(\mathbf{x}) = -\sum_{i=1}^{n} \left[ \alpha_i^1 \left( \frac{\partial L}{\partial x_i} \right)^2 + \alpha_i^2 \left( \frac{\partial L}{\partial y_i} \right)^2 \right] = -\sum_{i=1}^{n} \left[ \frac{\alpha_i^2}{\alpha_i^1} + \frac{w_i^2}{\alpha_i^2} \right] \leq 0,
\]
for all \( \mathbf{x} \in D_1 \).

For the \( i \)th individual, system (4) therefore becomes
\[
\begin{aligned}
x_i'(t) &= v_i(t) = -\alpha_i^1 \frac{\partial L}{\partial x_i}, \\
y_i'(t) &= w_i(t) = -\alpha_i^2 \frac{\partial L}{\partial y_i},
\end{aligned}
\]
\( x_{i0} = x_i(t_0), y_{i0} = y_i(t_0), \quad t_0 \geq 0, \)
where
\[
\frac{\partial L}{\partial x_i} = \left( \gamma_i + \sum_{j=1, j \neq i}^n \beta_{ij} \frac{R_{ij}(\mathbf{x})}{R_{ij}(\mathbf{x})} \right) \left( x_i - \frac{1}{n} \sum_{k=1}^n x_k \right) - 2 \sum_{j=1, j \neq i}^n \beta_{ij} R_{ij}(\mathbf{x}) (x_i - x_j),
\]
\[
\frac{\partial L}{\partial y_i} = \left( \gamma_i + \sum_{j=1, j \neq i}^n \beta_{ij} \frac{R_{ij}(\mathbf{x})}{R_{ij}(\mathbf{x})} \right) \left( y_i - \frac{1}{n} \sum_{k=1}^n y_k \right) - 2 \sum_{j=1, j \neq i}^n \beta_{ij} R_{ij}(\mathbf{x}) (y_i - y_j).
\]

Define the \( n \times n \) diagonal matrix
\[
A = \text{diag}(\alpha_1^1, \alpha_1^2, \ldots, \alpha_n^1, \alpha_n^2).
\]

Then our system (5) becomes the gradient system
\[
\dot{\mathbf{x}} = -A (\nabla L(\mathbf{x})), \quad \mathbf{x}_0 := \mathbf{x}(t_0), \quad t_0 \geq 0,
\]
the \( i \)th term of which is given by (6).

We now establish that \( L(\mathbf{x}) \) is indeed a Lyapunov function for system (7), and that it provides a stable equilibrium point for the system.

**Theorem 3** The function \( L(\mathbf{x}), \mathbf{x} \in D_1(L), \) is a Lyapunov function for system (7), a stable equilibrium point of which is
\[
\mathbf{x}^* = (\mathbf{x}_1^*, \ldots, \mathbf{x}_n^*) := \left( \frac{1}{n} \sum_{k=1}^n x_k, \frac{1}{n} \sum_{k=1}^n y_k, \ldots, \frac{1}{n} \sum_{k=1}^n x_k, \frac{1}{n} \sum_{k=1}^n y_k \right) \in D_1(L).
\]

**Proof.** By the Chain Rule
\[
\dot{L}_{(7)}(\mathbf{x}) = \nabla L(\mathbf{x}) \cdot \dot{\mathbf{x}} = \nabla L(\mathbf{x}) \cdot [-A (\nabla L(\mathbf{x})].
\]
Since \( A \) is an \( n \times n \) diagonal matrix with real-valued entries, \( \alpha_i^s > 0, \ i = 1, \ldots, n \) and \( s = 1, 2, \) if \( \lambda := \max\{\alpha_i^s; i = 1, \ldots, n, s = 1, 2\} \), then
\[
\dot{L}_{(7)}(\mathbf{x}) \leq -\lambda |\nabla L(\mathbf{x})|^2 \leq 0,
\]
showing that \( L(\mathbf{x}), \) with \( \mathbf{x} \in D_1(L), \) is a Lyapunov function for system (7). In particular \( \dot{L}(\mathbf{x}) = 0 \) if and only \( \nabla L(\mathbf{x}) = 0 \). Since \( \dot{L}_{(7)}(\mathbf{x}) = 0 \) at \( \mathbf{x} = \mathbf{x}^* \), it follows easily that \( \mathbf{x}^* \) is a stable equilibrium point of system (7) and is in \( D_1(L) \).
As discussed in Gazi and Passino (2004a), swarming in nature normally occurs in a distributed fashion; there is no leader and each individual decides independently its direction of motion. Our model captures this since system (6) gives the equations of motion of each individual, and does not depend on an external variable (such as a command from a leader or another agent), but only on the position of the individual itself and its observation of the positions (or relative positions) of the other individuals. Moreover, the individuals do not have to know the global Lyapunov function \( L(x) \). Instead, they only know the local or their internal Lyapunov function \( L_i(x) \).

5.2 Insight into the form of the Lyapunov Function and Cohesiveness

Let us now discuss the idea behind the construction of our Lyapunov function

\[
L(x) = \sum_{i=1}^{n} L_i(x) = \sum_{i=1}^{n} \left( \gamma_i R_i(x) + \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij} R_i(x)}{R_{ij}(x)} \right).
\]

At large distances between the \( i \)th and the \( j \)th individuals, the ratio,

\[
\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{\beta_{ij} R_i(x)}{R_{ij}(x)},
\]

is negligible, and the attraction term \( \sum_{i=1}^{n} \gamma_i R_i(x) \) dominates. Thus, the long-range attraction requirement in a swarm model is met, and \( \sum_{i=1}^{n} \gamma_i R_i \) acts as the attraction function. Indeed, since \( \sum_{i=1}^{n} \gamma_i R_i(x) \) is allowed to be zero at \( x = x^* \) where \( L \) is also zero, and \( dL/dt \leq 0 \) for all \( t \geq 0 \) along every solution of (7), each individual is attracted to the centroid, and therefore the swarm system (7) maintains centering and hence cohesiveness at all times. Indeed, Theorem 3 proves the cohesiveness of the swarm since stability means that for each \( \epsilon > 0 \) and \( t_0 \geq 0 \), there is a \( \delta = \delta(t_0, \epsilon) > 0 \) such that \( |x(t) - x^*| < \delta \) implies \( |x(t) - x^*| < \epsilon \) for all \( t \geq t_0 \), with \( x, x^* \in D_1(x) \); this boundedness of solution for all time \( t \geq t_0 \) implies that distances between individuals are bounded from above at all times.

Note that the parameter \( \gamma_i > 0 \) can be considered as a measurement of the strength of attraction between an individual \( i \) and the swarm centroid, and hence between each other. The smaller the parameter is, the weaker the attraction between the members is; hence, \( \gamma_i \) can be considered a coupling parameter.

Consider the situation where any two individuals \( i \) and \( j \) approach each other. In this case, \( R_{ij} \) decreases and the ratio increases, with \( \beta_{ij} > 0 \) acting as a cohesion parameter that is a measurement of the strength of interaction between the individuals. Now, because, with respect to time \( t \geq 0 \), we have that \( dL/dt \leq 0 \) along a trajectory of system (7), and \( L \) is a positive definite function, \( L \) cannot increase in \( t \geq t_0 \geq 0 \). Hence, for every initial condition \( x(t_0) \in D_1(x) \), the ratio cannot be unbounded in \( t \). However, at the initial time \( t_0 \geq 0 \), large values of \( L(x(t_0)) \) – and hence large controls efforts – could be required for collision-avoidance and cohesion. This is so because the ratio (8) is large for small arguments. Fuelling with this large value of \( L \) which can only decrease in \( t \), any change in the value of the ratio (8) could only correspond to either an increase or decrease in \( |dL/dt| \). Analogously, \( |dL/dt| \) is
the rate of dissipation of energy from the system in absolute value. Thus, if two individuals approach each other, the rate of energy dissipation from system (7), in absolute value, gets larger. This increased dissipation of energy along a trajectory of system (7) could only be directed towards a stable equilibrium point, such as $x^*$, where $R_{ij} \neq 0$, away from any potential collision between individuals $i$ and $j$. In other words, we cannot have a situation where $R_{ij} = 0$ along any system trajectory that starts in $D_1(x)$. Hence, the ratio (8) acts as an obstacle-avoidance function and system trajectories originating at $x(t_0) \in D_1(x)$ remain in $D_1(x)$ for all time $t \geq t_0 \geq 0$. In turn, this means that the individuals in a swarm cannot collapse onto themselves.

If the two parameters $\gamma_i$ and $\beta_{ij}$ are the same for all individuals, then we have an isotropic and a reciprocal swarm model. If they differs between at least two individual, then the model is anisotropic and non-reciprocal.

Finally, we note that we have used two other parameters, $\alpha_1^i > 0$ and $\alpha_2^i > 0$, $i = 1, \ldots, n$ in system (7). Because the parameters are a measure of the instantaneous velocities, we name them convergence parameters. The larger the convergence parameters, the quicker the movements of the individuals towards and about the centroid.

5.3 Constant Arrangement About the Centroid

Theorem 3 shows that the swarm members can converge to a constant arrangement about the swarm centroid. Indeed, if

$$S := \{ x \in \mathbb{R}^{2n} : \frac{\partial L}{\partial x_i} = \frac{\partial L}{\partial y_i} = 0, i = 1, \ldots, n \},$$

which is the set of all equilibrium points of system (7), and

$$E := \{ x \in D_1(x) : L(x) = 0 \}, \quad \dot{L} = \dot{L}(5) = \dot{L}(7)$$

then it follows easily that $S = E$ since

$$\dot{L}(x) = -\sum_{i=1}^{n} \alpha_1^i \left( \frac{\partial L}{\partial x_i} \right)^2 + \alpha_2^i \left( \frac{\partial L}{\partial y_i} \right)^2 = 0,$$

if and only if

$$\frac{\partial L}{\partial x_i} = \frac{\partial L}{\partial y_i} = 0, \quad i = 1, \ldots, n.$$

Hence, by LaSalle’s Invariance Principle (Theorem 2), the equilibrium points in $S$, which include $x^*$, are attractive.

5.4 Size and Density of the Swarm

Given that a member $i$ of the swarm resides in a disk defined in (3), with radius $r_i$, we can follow the argument by Gazi and Passino (2004a) to estimate the size and density of the swarm in a stable arrangement, but without using their assumption that the swarm members had to be squeezed cohesively as closely as possible in an area (a disk) of radius $r$, since Theorem 3 already provides this cohesiveness. Indeed, since Theorem 3 establishes the stability of system (7) in $D_1(x)$, there is no collision between members in $D_1(x)$. Accordingly, between two members $i$ and $j$,

$$\|x_i(t) - x_j(t)\| > (r_i + r_j), \quad x_i = (x_i, y_i),$$

for all time $t \geq t_0 \geq 0$. Now, the safety areas are disjoint, so the total area occupied by the swarm is $\pi \sum_{i=1}^{n} r_i^2$. By Theorem 3, for each $\epsilon > 0$ and $t_0 \geq 0$, there is a $\delta = \delta(t_0, \epsilon) > 0$ such
that \( \| \mathbf{x}(t_0) - \mathbf{x}^* \| < \delta \) implies \( \| \mathbf{x}(t) - \mathbf{x}^* \| < \epsilon \) for all \( t \geq t_0 \), with \( \mathbf{x}, \mathbf{x}^* \in D_1(\mathbf{x}) \). In such a stable arrangement, where all the solutions of (7) are bounded above by \( \epsilon > 0 \), we can therefore find a disk of radius, say, \( r = r(\epsilon) \), around \( \mathbf{x}^* \) such that

\[
\pi r^2(\epsilon) \geq \pi \sum_{i=1}^{n} r_i^2.
\]

From this we get

\[
r_{\min} := \sqrt{\sum_{i=1}^{n} r_i^2},
\]

a lower bound on the radius of the smallest circle which can enclose all the individuals. It is clear that the swarm size will will scale with the size of the individual.

If we define the density of the swarm as the number of individuals per unit area, and let it be \( \rho \), then it is simple to see that \( \rho \) is upper bounded, with

\[
\rho \leq \frac{n}{\pi \sum_{i=1}^{n} r_i^2}.
\]

Hence, the swarm cannot become arbitrarily dense.

6 Computer Simulations

Extensive computer simulations show that for a sufficiently large number of individuals the proposed model (7) generates four types of swarming-like behaviors. They are (1) the cruise formation (linear or nonlinear) reminiscent of a cruising and leaderless school of fish, a moving herd of cattle or elephants with a leader (leader-following), (2) random walks similar to the swarming behavior of fruit flies *Drosophila melanogaster*, (3) constant arrangements requiring individuals to aggregate and stop, as in fruiting body formation by bacteria, and (4) circular motions reminiscent of the behavior of a school of fish when threatened by a predator.

Table 1 summarizes the emergent behaviors as we modify the three parameters.

6.1 Examples of Type A and Type B Arrangements

6.1.1 Straight Line Formation

Our first diagram shows an example of Type A arrangement. In Figure 1 (a), randomly-positioned 30 individuals, each with bin 10, at the initial time of \( t = 0 \) are shown. As time evolves, they cluster around the centroid and cruise along a straight line as a well-spaced cohesive group as shown in Figure 1 (b). The path of the centroid is shown thick.
Table 1: Parameters produce different types of emergent behaviors for sufficiently large populations.

<table>
<thead>
<tr>
<th>Type</th>
<th>Convergence parameter, $\alpha^s_i &gt; 0$, $s = 1, 2; i \in N$</th>
<th>Coupling parameter, $\gamma_i &gt; 0$, $i \in N$</th>
<th>Cohesion parameter, $\beta_{ij} &gt; 0$, $i, j \in N, i \neq j$</th>
<th>Some emergent arrangement about centroid</th>
</tr>
</thead>
</table>
| A    | same $\alpha^s_i$ for all $s, i,$ or random $\alpha^s_i$ | same $\gamma_i$ for all $i$ | same $\beta_{ij}$ for all $i, j$ | - coherent compact cluster cruising in a straight line;  
|      |                                                 |                  |                                 | - constant arrangement.                       |
| B    | same $\alpha^s_i$ for all $s, i$                | random $\gamma_i$ | same $\beta_{ij}$ for all $i, j$ | - coherent compact cluster cruising in a nonlinear fashion, with leader(s) possibly emerging;  
|      |                                                 |                  |                                 | - circular motion.                            |
| C    | same $\alpha^s_i$ for all $s, i,$ or, random $\alpha^s_i$ | same $\gamma_i$ for all $i$ | random $\beta_{ij}$ | - Lévy-like random walk;                      |
| D    | random $\alpha^s_i$                             | random $\gamma_i$ | same $\beta_{ij}$ for all $i, j$ | - same as in B                               |
| F    | same $\alpha^s_i$ for all $s, i$               | random $\gamma_i$ | random $\beta_{ij}$ | - same as in C                               |
| G    | random $\alpha^s_i$                             | random $\gamma_i$ | random $\beta_{ij}$ | - any of the above                           |

Figure 1: Cruise. Example of Type A arrangement, where $\alpha^s_i = 1$, $\gamma_i = 2$ and $\beta_{ij} = 50$. The path of the centroid is shown thick. The swarm is cruising non-stop as a well-spaced cohesive group in a stable formation.

6.1.2 Constant Arrangement about Centroid

In 2006, Sozinova and colleagues developed a three-dimensional model of myxobacterial fruiting-body formation, in which myxobacterial cells, when sensing starvation, change their
movement pattern from outward spreading to inward concentration and form aggregates nucleated by a stationary traffic jam or nonsymmetric initial aggregates [Sozinova et al (2006)].

In our second example (Figure 2), the members \((n = 70, \text{bin 20})\) converge to a constant arrangement about the centroid, reminiscent of the shape of the base level of such bacterial swarm formation.

Figure 2: Balanced forces. Example of constant arrangement about centroid (Type A), where \(\alpha_i = 0.1, \gamma_i = 0.2\) and \(\beta_{ij} = 50\). Part (a) and (b) show the initial and final positions of the individuals, respectively. The centroid remains stationary.

6.1.3 Leader-following Behavior

The example shown in Figure 3, with \(n = 30\) individuals and bin 10, shows a nonlinear path taken by the swarm, with an emergent leader.

6.2 Examples of Type C Arrangement

6.2.1 Random Walks

In 2006, Majkut modelled the flight paths of fruit flies \textit{Drosophila melanogaster}, which utilize scent to locate food sources in their vicinity [Majkut (2006)]. Fruit fly flight is characterized by a series of straight segments interrupted by rapid changes in horizontal heading known as saccades. Majkut used Lévy flights to model the foraging behavior of fruit flies. Levy flights are a class of continuous time random walks, which are often found in biological behavior and are prevalent in foraging.

Our diagram in Figure 4 shows a Lévy-like random walk, with \(n = 30\) and bin 10.

6.2.2 Circular Motion from Random Walks

Our second example of Type C arrangement shows the formation of a circular motion out from a random walk (Figure 5).
Figure 3: Leader of the pack. Example of Type B arrangement, where $\alpha_s = 1$, $\gamma_i$ is randomized between 0.01 and 1, and $\beta_{ij} = 100$. The path of the centroid is shown thick. The swarm is cohesive throughout. Part (b), without paths drawn, clearly shows a leader and those that trail behind.

Figure 4: Random walks. Example of Type C arrangement, where $\alpha_s^5 = 5$, $\gamma_i = 1$ and $\beta_{ij}$ is randomized between 200 and 500. The path of the centroid is shown thick in (a). The swarm is cohesive throughout. In (b), the path of an individual is magnified, showing a saccade-like flight path.

7 Conclusion

Recent work on swarm modelling, especially by Edelstein-Keshet (2001), Mogilner and Edelstein-Keshet (1999), Mogilner et al (2003), and Gazi and Passino (2003, 2004b,a) shows
that an element of the swarming phenomenon is a long-range attraction and a short-range repulsion between individuals in the swarm. The Lagrangian approach is a means to do this. This study also supports this heuristic argument with a novel technique to construct a Lagrangian model. Utilizing the Lyapunov method, we create a gradient system that is stable, implying the congregation of individuals about their centroid to form cohesive and well-spaced swarms.

Our model shares the disadvantage of other Lagrangian models which require every individual to know (or sense) the (relative) position of all the other individuals. Obviously such models are not scalable. Nonetheless, it has several characteristics that make it stand out from other swarm models: (1) It is a distributed system that not only captures the basic feature of aggregation, cohesion and stability of a swarm, but also exhibits more complex dynamics such as random walks and self-organized oscillatory motions via the use of only three parameters: the convergence, coupling and cohesion parameters; (2) It is general enough to be either an isotropic and a reciprocal swarm model, or anisotropic and non-reciprocal swarm model by manipulating the coupling and cohesion parameters appropriately; (3) Finally, the results may be applicable to distributed robotic systems, or considered for the control of heterogeneous robotic swarms by creating, for instance, a kinematic model of an individual robot in a swarm and constructing its instantaneous velocity along the method expounded in this article. The recent work by the authors and colleagues in Sharma et al (2009, 2010), which include fixed obstacles, is in this direction, and will be further developed in light of the new results in this paper.

References


Tunnel Passing Maneuvers of a Team of Car-like Robots in Formation

Bibhya SHARMA¹, Jito VANUALAILAI¹, Shin-Ichi NAKAGIRI² and Shonal SINGH¹

¹. University of the South Pacific, FIJI
². Kobe University, JAPAN

Abstract

The research essays the design of a motion planner that will simultaneously manage collision and obstacle avoidances of a team of nonholonomic car-like robots fixed in prescribed formation and ensure desirable tunnel passing maneuvers. This decentralized planner, derived from the Lyapunov-based control scheme works within a leader-follower framework to generate either split/rejoin or expansion/contraction of the formation, as feasible solutions to the tunnel passing problem. In either scenario, the prescribed formation will be re-established after the tunnel has been passed. Moreover, avoidance of the walls of a tunnel will be accomplished via the minimum distance technique. The results can be viewed as a significant contribution to the intelligent vehicle systems discipline.

Keywords: Lyapunov-based control scheme; split/rejoin; expansion/contraction; formation control; tunnel passing.

Subject Classification 34D20; 37J60; 68T40; 70E60; 93C85; 93D05.

1 Introduction

The concept of formation control has in recent years garnered monumental attention from researcher all over, for both theoretical research and real-world applications. Formation control is basically to control the posture (position and orientation) of a team of agents while normally maintaining constant their relative locations and allowing them to travel to their desired destinations [1–5]. Formation control in difficult and constrained environments has been favored because of the wide spectrum of formation stiffness possible (eg. split/rejoin, low degree, rigid) and its relevancy to different real-life applications. The applications include surveillance; transportation; reconnaissance; save and rescue; pursuit-evasion; and exploration in either fully known or partially known environments, environments that may be very harsh, or hazardous, or even inaccessible to humans.
The literature harbors leader-follower, virtual structure, nearest neighbors, social potentials and the behavior based approaches to address the problem of formation control. Although each has its share of advantages and disadvantages, the leader-follower approach seems to be favored because of its simplicity and scalability [2, 6–8]. Furthermore, the approach has the ability to contain a wide range of formations with richer specifications and complexities. Generally, when the posture of the leader-robot is known, desired postures of the follower-robots can be achieved by appropriate control laws. However, the leader-follower approach is widely known for its poor disturbance rejection properties [2] and the dependence placed upon a single agent which can be crucial in atrocious and adverse conditions.

The main strength of this research lies in the emphasis placed upon the tunnel passing maneuvers, a practical situation commonly seen on our roads and highways. We take this tunnel passing problem to a higher level by considering formation control of a team of nonholonomic car-like robots through a tunnel. From the authors’ viewpoint, although there are a number of techniques that can be successfully deployed to generate feasible algorithms of the tunnel passing problem, namely; scaling; prioritizing; caging [9]; split/rejoin [4, 16–18]; and contraction/expansion of the teams, only the latter two can possibly maintain the prescribed formations, at least, before and after tunnel passing. The two strategies are elegant, simple to implement and yet meet the expectations of the researcher. For the very first time we deploy the two techniques to solve this interesting but complicated problem within the overarching framework of a new control scheme.

Operations within the control scheme are guided by the principles of the Direct Method of Lyapunov, hence the control scheme is appropriately classified as a Lyapunov-based control scheme (LBCS), an artificial potential field method [19]. The reader is referred to [19] for a detailed account of the LbCS.

This paper is organized as follows: in Section 2 the car-like robot model is defined; in Section 3 the tunnel problem is designed and the two strategies are described; in Section 4 the attractive and repulsive potential field functions are designed and explained; in Section 5 the acceleration controllers are designed and stability of the robot system carried out; in Section 6 computer simulations of interesting scenarios are carried out; and Section 7 concludes the paper and outlines future work in the area.

2 Car-like Robot Model

Definition 1 The $i$th front-wheel steered car-like mobile robot is a disk with radius $r_{V_i}$ and is positioned at center $(x_i, y_i)$. Precisely, the $i$th car-like robot is the set

$$A_i = \{(z_1, z_2) \in \mathbb{R}^2 : (z_1 - x_i)^2 + (z_2 - y_i)^2 \leq r_{V_i}\},$$

where $A_1$ and $A_i$ for $i = 2, \ldots, n$ are the lead robot and the follower-robots, respectively, of a team of nonholonomic car-like robots.
With reference to Fig. 1, \([x_i, y_i]^T\) denotes the CoM of \(A_i\), \(\phi_i\) gives its steering wheel’s angle with respect to the longitudinal axis, \(l_1\) is the distance between the center of the rear and front axles, while \(l_2\) is the length of each axle.

The configuration of \(A_i\) is given by \(\mathbf{q}_i = [x_i, y_i, \theta_i]^T \in \mathbb{R}^3\), where \(\mathbf{d}_i = [x_i, y_i]^T \in \mathbb{R}^2\) is its position and \(\theta_i \in \mathbb{R}\) is its angle with respect to the \(z_1\)-axis.

If we let \(m_i\) be the mass of the robot, \(F_i\) the force along the axis of the robot, \(\Gamma_i\) the torque about a vertical axis at \([x_i, y_i]^T\) and \(I_i\) the moment of inertia of the robot, then the dynamic model of \(A_i\) with respect to its CoM is

\[
\begin{align*}
\dot{x}_i &= v_i \cos \theta_i - \frac{l_1}{2} \omega_i \sin \theta_i, \\
\dot{y}_i &= v_i \sin \theta_i + \frac{l_1}{2} \omega_i \cos \theta_i, \\
\dot{\theta}_i &= \frac{v_i}{l_1} \tan \theta_i = \omega_i, \\
\dot{v}_i &= F_i/m_i = \sigma_i, \\
\dot{\omega}_i &= \Gamma_i/I_i = \eta_i,
\end{align*}
\]

where \(v_i\) and \(\omega_i\) are, respectively, the instantaneous translational and rotational velocities, while \(\sigma_i\) and \(\eta_i\) are the instantaneous translational and rotational accelerations of \(A_i\). In addition, we assume no slippage (i.e. \(\dot{x}_i \sin \theta_i - \dot{y}_i \cos \theta_i = 0\)) and pure rolling (i.e. \(\dot{x}_i \cos \theta_i + \dot{y}_i \sin \theta_i = v_i\)) of the wheels. These generate non-integrable constraints of the system, constraints that are passionately denoted as the nonholonomic constraints. We note that these constraints are already reflected in system (1). The dynamic constraints tagged to the system will be treated in a later section.

The state of robot \(A_i\) is captured in \(\mathbf{x}_i = [x_i, y_i, \theta_i, v_i, \omega_i]^T \in \mathbb{R}^5\) and its acceleration controls in \(\mathbf{u}_i = [\sigma_i, \eta_i]^T \in \mathbb{R}^2\). We collect the states of all the \(n\) robots in the vector \(\mathbf{x} = [\mathbf{x}_1^T, \ldots, \mathbf{x}_n^T]^T \in \mathbb{R}^{5 \times n}\) and the acceleration controls in \(\mathbf{u} = [\mathbf{u}_1^T, \ldots, \mathbf{u}_n^T]^T \in \mathbb{R}^{2 \times n}\).

Next, given the clearance parameters \(\epsilon_1\) and \(\epsilon_2\), we enclose each \(A_i\) in a protective circular region centered at \(\mathbf{d}_i\) with radius \(r_v = \sqrt{(l_1 + 2\epsilon_1)^2 + (l_2 + 2\epsilon_2)^2}/2\) to maximize the free space and ensure an easier construction of the potential field functions [4, 19].

3 Devising the Tunnel Passing Problem

**Definition 2** Tunnel passing is a geometric problem of generating collision-free maneuvers of agents from arbitrary initial positions through a 2D-tunnel of given geometry.
In this research, we drive a team of robots through a 2D-tunnel of given geometry. This tunnel passing problem can be divided into a number of sub-tasks: guiding the team in formation to the front of the tunnel; driving the team through the tunnel; and finally re-establishing the original formation of the team. We assume that the robots will be able to measure the distances from the tunnel walls using sensors.

Let us treat the top tunnel wall in the $z_1 z_2$-plane as a line segment with initial coordinates $(q_{11}, r_{11})$ and final coordinates $(q_{12}, r_{12})$, while the bottom tunnel wall has initial and final coordinates as $(q_{21}, r_{21})$ and $(q_{22}, r_{22})$, respectively. Hence, we have region $t_f = \{0 < z_1 < q_{11}, r_{21} < z_2 < r_{11}\}$ and region $t_b = \{q_{22} < z_1, r_{22} < z_2 < r_{12}\}$. Therefore

**Definition 3** A point $z \in \mathbb{R}^2$ is behind the tunnel if $z \in t_b$ and is in front of the tunnel if $z \in t_f$. The size of the tunnel entrance is denoted by $h_t$ and it is a measure with reference to the $z_2$-axis.

**Definition 4** $h_f$ is the spread of the prescribed formation and it is a measure of the maximum inter-robot distance in relation to the $z_2$-axis.

**Assumption 1** The team will be required to be fixed in a prescribed formation, at least, before and after the tunnel passing maneuver.

**Remark 1** The assumption legislates a change in the formation to facilitate the passing maneuvers through the tunnel. This will be required when the size of the tunnel entrance will not allow the prescribed formation of the team to pass through, per se.

In this research we will deploy contraction/expansion and split/rejoin of a team to provide feasible solutions to the tunnel passing problem. We will discuss each strategy in detail now.

### 3.1 Strategy I: Split/Rejoin of the Team

**Definition 5** Split/rejoin strategy is where multiple agents fixed in a specific formation split to steer past the encountering obstacle(s) and then rejoin to establish the prescribed formation.

This research evokes the split/rejoin maneuver of a team of nonholonomic car-like mobile robots fixed in a formation in order to pass a tunnel of an arbitrary configuration. As illustrated in Figure 2(a), when $h_f + \epsilon > h_t$, the spread of the formation is greater than the size of the tunnel entrance, within a safety of $\epsilon$, hence requiring split/rejoin. The split/rejoin maneuver, in the context of the tunnel passing problem, can be broken down into the following sub-tasks:

**Sub-task 1:** Drive the team into the prescribed formation;

**Sub-task 2:** Maintain the prescribed formation;

**Sub-task 3:** Steer the formation to the front entrance of the tunnel;
**Sub-task 4:** Activate the split maneuver of the team;

**Sub-task 5:** Drive the team members through the tunnel;

**Sub-task 6:** Activate the rejoin maneuver of the team to attain the prescribed formation within a maximum distance $d_0 \in t_b$. Note $d_0$ is measured relative to the leader position.

![Diagram showing split/rejoin and contraction/expansion strategies](image)

*Fig. 2:* Schemes for the tunnel passing problem with $A_1$ as the leader robot.

### 3.2 Strategy II: Contraction/Expansion of the Team

**Definition 6** Contraction/expansion strategy is where the prescribed formation of the multiple agents is allowed to resize in order to steer past the encountering obstacle(s) and then return to the original size of the formation. Nonetheless, the prescribed shape is preserved throughout the journey.

As illustrated in Figure 2(b), within a distance $d_1$, there has to be a successful contraction of the formation $h_f \implies h_{f*}$ such that $h_{f*} + \epsilon < h_t$. It means that the spread of the formation is less than the size of the tunnel entrance, within a safety of $\epsilon$. We will develop an algorithm which again involves six major sub-tasks:

**Sub-task 1:** Drive the team into the prescribed formation;

**Sub-task 2:** Maintain the prescribed formation;

**Sub-task 3:** Steer the formation to a distance of $d_1 \in t_f$. Note $d_1$ is measured relative to the leader position;

**Sub-task 4:** Activate the contraction maneuver of the team to reduce the size of the formation. Rate of contraction of formation will be relative to the $h_f$ and $h_t$ measures;

**Sub-task 5:** Drive the team fixed in the reduced size through the tunnel;

**Sub-task 6:** Activate the expansion maneuver of the team to attain the original size of the prescribed formation within a maximum distance $d_2 \in t_b$. 


3.3 Control Objective

The overall control objective of this paper is to design decentralized acceleration controllers, \( \sigma_i \) and \( \eta_i \), for each \( A_i \) in system (1), within the framework of LbCS, to navigate safely through the tunnel either from a split/rejoin or a contraction/expansion maneuver of a team in formation, within a finite period of time.

4 Artificial Potential Field Functions

In this section, we will construct attractive and repulsive potential field functions required to tackle each sub-task tagged to the split/rejoin and the contraction/expansion strategies. For simplicity we make the following assumption:

**Assumption 2** The dimensions, the maximum speed \( v_{\text{max}} \) and the maximum steering angle \( \phi_{\text{max}} \) of the \( n \) car-like robots are kept the same.

We now look into the various aspects of the two strategies and carefully consider the associated potential field functions.

4.1 Drive the team into the prescribed formation

There are basically two phases of Sub-task 1: (1) initiate movement of the team members, and (2) establish the prescribed formation. We note that irrespective of the differences contained in the two strategies, the mathematical treatment of the two parts are the same for the two strategies. We shall consider these two parts separately.

4.1.1 Drive

To initiate movement we propose to have a target for each member of the team. Therefore, for \( A_i \), we define a target \( T_i = \{ (x_i, y_i) \in \mathbb{R}^2 : (x_i - t_{i1})^2 + (y_i - t_{i2})^2 \leq r_{ti}^2 \} \) with center \((t_{i1}, t_{i2})\) and radius \( r_{ti} \). For each \( A_i \) to be attracted to \( T_i \) and its center finally positioned at \((t_{i1}, t_{i2})\) we utilize an attractive potential field function \( U_{\text{att}} : \mathbb{R}^4 \rightarrow \mathbb{R}^+ \) with

\[
U_{\text{att}}(x) = \sum_{i=1}^{n} H_{N_i}(x)
\]

where

\[
H_{N_i}(x) = \frac{1}{2} \ln(H_i + 1)
\]

and the corresponding target attractive function is of the form

\[
H_i(x) = (x_i - t_{i1})^2 + (y_i - t_{i2})^2 + v_i^2 + \omega_i^2, \quad \text{for } i = 1, \ldots, n.
\]
While the function is the measure of the distance between $A_i$ and the target $T_i$, it can also be treated as a measure of convergence. This invariably substantiates the first phase of Sub-task 1.

### 4.1.2 Establish Prescribed Formation

To realize the second phase of Sub-task 1 we adopt the leader-follower scheme described by Sharma et al. in [4]. The scheme was designed to navigate a flock in a constrained environment. This elegant yet simple scheme will be instrumental in establishing the prescribed formation of the team, for either strategy. In the scheme, see Figure 3, the follower robots follow the lead robot via mobile ghost targets. The $i$th mobile ghost target is positioned relative to the position of the lead robot.

This is a user-defined Euclidean measure of $a_i$ units right or left and $b_i$ units up or down, while the center of the ghost target is given by $(t_{i1}, t_{i2}) = (x_1 - a_i, y_1 - b_i)$, for $i = 2$ to $n$.

As the lead robot $A_1$ moves towards its target $T_1$, the mobile ghost targets will move relative to the position of the lead robot.

In turn each follower-robot of the team moves towards a designated mobile ghost target at every iteration $t > 0$, hence establishing the prescribed formation [4]. A specific and prescribed formation can be established with appropriate values of the Euclidean measures $a_i$ and $b_i$.

Figure 4 shows the potential valleys created by the attractive forces in a continuous potential field in relation to the moving ghost targets. The ultimate goal is for each robot to move to its designated valley (mobile ghost target) via steepest descent of the potential gradient.

### 4.2 Maintain the Prescribed Formation

To realize Sub-task 2, we design specific modules that govern the prescribed formation of the team. We note that these modules will greatly differ for the two strategies since we will want to activate two significantly different maneuvers to engender tunnel passing.

While for Strategy I the leader-follower scheme and the attractive potentials governed by equation (2) are sufficient to maintain the prescribed formation, the following bounds are enacted specifically for Strategy II to maintain a continued cohesion of the robots.
Fig. 4: Total potentials and the corresponding contour plot generated using the target attractive function governed by equation (2). The ghost targets are fixed at (60, 24) and (27, 30).

4.2.1 Maximum Inter-robot Bound

The maximum distance between any two robots of the team needs to be bounded so that a robot cannot drift off the prescribed formation. We thus desire a bound \( \| \mathbf{d}_i - \mathbf{d}_j \|^2 < M_{ij}^2 \) where \( M_{ij} \) is the maximum Euclidian distance between \( A_i \) and \( A_j \) on \( \mathbb{R}^2 \).

The only way this bound could be treated within the LbCS framework is to develop an artificial obstacle for it. We can choose \( AO_{ij} = \{ \mathbf{x} \in \mathbb{R}^2 : (x_i - x_j)^2 + (y_i - y_j)^2 \geq M_{ij}^2 \} \). To ensure that each robot of the team operates within these bounds, we need only avoid the corresponding artificial obstacles by appropriate repulsive potential field functions. Hence, we introduce tuning parameter \( \zeta_{ij} > 0 \), and adopt potential fields defined by \( U_{rep_1} : \mathbb{R}^2 \rightarrow \mathbb{R}^+ \) with

\[
U_{rep_1}(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\zeta_{ij}}{R_{ij}(\mathbf{x})},
\]

where the obstacle avoidance function is of the form

\[
R_{ij}(\mathbf{x}) = \frac{1}{2} \left[ M_{ij}^2 - \{(x_i - x_j)^2 + (y_i - y_j)^2\} \right], \quad \text{for } i, j \in \{1, 2, \ldots, n\}, \ j \neq i.
\]

4.2.2 Minimum Inter-robot bound

The minimum inter-robot bounds prevent a robot from getting very close to (or colliding with) another robot. We desire the bound \( \| \mathbf{d}_i - \mathbf{d}_j \|^2 > N_{ij}^2 \) where \( N_{ij} \) is the minimum Euclidian distance given as \( (r_V + r_V')^2 = 4r_V^2 \) on \( \mathbb{R}^2 \). We can choose to have an artificial obstacle \( AO_{2ij} = \{ \mathbf{x} \in \mathbb{R}^2 : (x_i - x_j)^2 + (y_i - y_j)^2 \leq N_{ij}^2 \} \}. \) For avoidance we introduce tuning parameter \( \xi_{ij} > 0 \), and adopt repulsive potential fields defined by \( U_{rep_2} : \mathbb{R}^2 \rightarrow \mathbb{R}^+ \) with
with

\[ U_{\text{rep}}(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\xi_{ij}}{MO_{ij}(x)} \]  

(7)

where the obstacle avoidance function is of the form

\[ MO_{ij}(x) = \frac{1}{2} \left[ (x_i - x_j)^2 + (y_i - y_j)^2 - 4r^2 \right], \text{ for } i, j \in \{1, 2, \ldots, n\}, j \neq i. \]  

(8)

### 4.3 Drive Team to Front Entrance

In either strategy, to realize Sub-task 3 we simply ensure that the team is driven towards the tunnel. It is easy to accept that we basically require \( T_1 \in t_b \). As the lead robot approaches the tunnel entrance, the follower-robots follow their mobile ghost targets and hence also approach the front entrance.

### 4.4 Drive Team through the Tunnel

Sub-task 5 of either strategy addresses the ultimate goal of this research. This is to attain the tunnel passing maneuver and to make sure that all the team members are able to safely steer through the tunnel.

Since \( T_1 \in t_b \) the target attractive functions constructed in Section 4.3 are sufficient to drive the team members through the tunnel. However, to garner an overall success, we need to address an important issue inherently tagged to Sub-task 5: obstacle and collision avoidances. We will now address the various avoidances deemed important to this sub-task.

#### 4.4.1 Fixed Obstacles: Tunnel Walls

We assume that the walls of the tunnel are fixed obstacles that need to be avoided in either strategy while passing through. This also ensures containment of the motion within the walls of the tunnel.

**Definition 7** The \( k \)th tunnel wall is collapsed into a line segment in the \( z_1z_2 \)-plane with initial coordinates \((q_{k1}, r_{k1})\) and final coordinates \((q_{k2}, r_{k2})\). The parametric representation of the \( k \)th tunnel wall can be given as \( x_k = q_{k1} + \lambda_k(q_{k2} - q_{k1}) \) and \( y_k = r_{k1} + \lambda_k(r_{k2} - r_{k1}) \) where \( \lambda_k : \mathbb{R}^2 \to [0, 1] \).

We will adopt the minimum distance technique (MDT) from [19] to facilitate the avoidance of these line segments. We calculate the minimum Euclidian distance from the center of \( A_i \) to the \( k \)th line segment and avoid the resultant point of the line segment. From geometry, coordinates of this point can be given as \( x_{ik} = q_{k1} + \lambda_{ik}(q_{k2} - q_{k1}) \) and \( y_{ik} = r_{k1} + \lambda_{ik}(r_{k2} - r_{k1}) \) where \( \lambda_{ik} = (x_i - q_{k1})c_k + (y_i - r_{k1})d_k \).
\[ c_k = \frac{(q_{k_2} - q_{k_1})}{(q_{k_2} - q_{k_1})^2 + (r_{k_2} - r_{k_1})^2}, \quad d_k = \frac{(r_{k_2} - r_{k_1})}{(q_{k_2} - q_{k_1})^2 + (r_{k_2} - r_{k_1})^2} \]

and the saturation function is given by

\[
\lambda_{ik}(x_i, y_i) = \begin{cases} 
0 , & \text{if } \lambda_{ik} < 0 \\
\lambda_{ik}, & \text{if } 0 \leq \lambda_{ik} \leq 1 \\
1 , & \text{if } \lambda_{ik} > 1 
\end{cases}
\]

We note that \( \lambda_{ik}(x_i, y_i) \) is a nonnegative scalar such that it is restricted to the interval \([0, 1]\). Hence, there is always an avoidance of the \( k \)th tunnel wall at every iteration \( t \geq 0 \). For example, \( \lambda_{ik} = 0 \) would mean that \( A_i \in t_f \) and the point closest to it will be \((q_{k_1}, r_{k_1})\) which the robot has to avoid. Now, for each robot to avoid the closest point on each of the \( k \)th tunnel wall we introduce tuning parameter \( \alpha_{ik} > 0 \), for \( i = 1 \) to \( n \) and \( k = 1, 2 \), and consider repulsive potential fields defined by

\[
U_{rep_3}(x) = \sum_{i=1}^{n} \sum_{k=1}^{2} \alpha_{ik} W_{ik}(x),
\]

where the associated obstacle avoidance function is of the form

\[
W_{ik}(x) = \frac{1}{2} \left\{ \left| x_i - (q_{k_1} + \lambda_{ik}(q_{k_2} - q_{k_1})) \right|^2 + \left| y_i - (r_{k_1} + \lambda_{ik}(r_{k_2} - r_{k_1})) \right|^2 - r_v^2 \right\},
\]

### 4.4.2 Moving Obstacles: Car-like Mobile Robots

While the minimum inter-robot bounds designed in Section 4.2.2 govern the formation of the team in Strategy II, they will also prevent inter-robot collisions.

We now need to address the collision avoidance issue for Strategy I. This is because once the team members are split there is a possibility that they can collide amongst each other, that is, each robot itself becomes a moving obstacle for all the other robots. For this we will simply deploy the repulsive potential field function given by equation (7) in Section 4.2.2 which will also prevent all possible inter-robot collisions.

### 4.5 Split/Rejoin and Contraction/Expansion Maneuver

Sub-task 4 and Sub-task 6 involve activating the split/rejoin and contraction/expansion maneuvers of the team for Strategy I and Strategy II, respectively. While the lack of strong constraints help attain split/rejoin maneuvers in Strategy I, we will require the following updating rule to engender contraction/expansion of the formation in Strategy II:

\[
a_i(t + 1) = \begin{cases} 
- \rho_1 (a_i(0) - a_i^*) + a_i(t), & \text{if } \lambda_{1k}(x_1, y_1) < 0, \\
\phantom{- \rho_1 (a_i(0) - a_i^*) + a_i(t)}, & \lambda_{1k}(x_1, y_1) = 1 \\
- \rho_2 (a_i^* - a_i(0)) + a_i(t), & \text{if } \lambda_{1k}(x_1, y_1) > 1,
\end{cases}
\]
and
\[ b_i(t + 1) = \begin{cases} 
- \rho_1 (b_i(0) - b^*_i) + b_i(t), & \text{if } \lambda_{ik}(x_i, y_i) < 0, \\
b^*_i, & \text{if } 0 \leq \lambda_{ik}(x_i, y_i) \leq 1, \\
- \rho_2 (b^*_i - b_i(0)) + b_i(t), & \text{if } \lambda_{ik}(x_i, y_i) > 1,
\end{cases} \]

where the distance measures, \( a_i \) and \( b_i \), will be iteratively updated from the attractive potential field function governed by equation (4) and from the saturation function \( \lambda_{ik}(x_i, y_i) \) defined in Subsection 4.4. When \( \lambda_{ik}(x_i, y_i) < 0 \), \( A_1 \in t_f \) which implies that the contraction \( h_f = h_f^* \) continues such that \( h_f^* + \epsilon < h_t \). However, if \( A_1 \in t_b \) then we continue to expand the formation, until the original size is re-established. This also establishes that the feedback gains \( \rho_1, \rho_2 \in \mathbb{R}^+ \) are directly dependent on distances \( d_1 \) and \( d_2 \). Furthermore, the critical measures to allow tunnel passing \( a^*_i > 2 \times r_V \), and \( b^*_i > 2 \times r_V \) need to be observed to avoid saturations.

4.6 Other Requirements

4.6.1 Auxiliary Function

To ensure that the total potentials vanish when the team converges to the final target configuration we design an auxiliary function defined by \( U_{aux} : \mathbb{R}^3 \rightarrow \mathbb{R}^+ \) with
\[ U_{aux} = \sum_{i=1}^{n} G_i(x) \quad (11) \]
where
\[ G_i(x) = \frac{1}{2} \left[ (x_i - t_{i1})^2 + (y_i - t_{i2})^2 + (\theta_i - t_{i3})^2 \right], \quad (12) \]
for \( i = 1 \) to \( n \), where \( t_{i3} \) is the desired orientation of \( A_i \).

4.6.2 Artificial Obstacles: Dynamics Constraints

In practice, the translational speed and the steering angle of the car-like robots are limited. If \( v_{max} > 0 \) and \( 0 < \phi_{max} < \frac{\pi}{2} \) then the constraints imposed on the translational and the rotational velocities are, respectively, \( |v_i| < v_{max} \) and \( |\omega_i| < \frac{v_{max}}{|\rho_{min}|} \) where \( \rho_{min} = \frac{l_1}{\tan \phi_{max}} \). Again, the only way these dynamic constraints could be treated within the LbCS framework is to develop an \textit{artificial obstacle} for each. Hence we have:
\[
AO_{3i1} = \{ v_i \in \mathbb{R} : v_i \leq -v_{max} \text{ or } v_i \geq v_{max} \}, \\
AO_{3i2} = \{ \omega_i \in \mathbb{R} : \omega_i \leq -v_{max}/|\rho_{min}| \text{ or } \omega_i \geq v_{max}/|\rho_{min}| \}.
\]
To avoid these artificial obstacles we introduce tuning parameter \( \beta_{im} > 0 \), for \( i = 1 \) to \( n \) and \( m = 1, 2 \), and use the repulsive potential fields defined by \( U_{rep} : \mathbb{R}^2 \rightarrow \mathbb{R}^+ \) with
\[ U_{rep} = \sum_{i=1}^{n} \sum_{m=1}^{2} \frac{\beta_{im}}{U_{im}(x)} \quad (13) \]
where the associated avoidance functions are of the form
\[
U_{i1}(x) = \frac{1}{2}(v_{\text{max}} - v_i)(v_{\text{max}} + v_i) \\
U_{i2}(x) = \frac{1}{2} \left( \frac{v_{\text{max}}}{\rho_{\text{min}}} - \omega_i \right) \left( \frac{v_{\text{max}}}{\rho_{\text{min}}} + \omega_i \right)
\]

5 Controller Design and Stability Issues

The total attractive and repulsive APFs for system (1) are defined by \( U_{\text{att}}(x) \) and \( U_{\text{rep}}(x) = \sum_{j=1}^{4} U_{\text{rep},j} \), respectively. The resulting total force for system (1) is \( F(x) : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) with
\[
F(x) = - (\nabla U_{\text{att}}(x) + U_{\text{rep}}(x) \times \nabla U_{\text{aux}}(x) + U_{\text{aux}}(x) \times \nabla U_{\text{rep}}(x))
\]

The collision-free trajectories are harvested following the notion of steepest descend. We begin with the following theorem:

**Theorem 1** Consider a team of car-like mobile robots, the motion of which is governed by ODEs described by system (1). The objective is to, amongst considering other integrated subtasks, establish and control a prescribed formation, facilitate tunnel passing maneuvers of the robots within a constrained environment and attain the target configuration in its original formation. Utilizing the potential field functions the following continuous time-invariant acceleration control laws can be generated for \( A_i \) that per se guarantees stability, in the sense of Lyapunov, of system (1) as well:

\[
\sigma_i = - \left[ \delta_{i1} v_i + f_{1i} \cos \theta_i + f_{2i} \sin \theta_i \right] / f_{4i}, \\
\eta_i = - \left[ \delta_{i2} \omega_i + \frac{l_i}{2} (f_{2i} \cos \theta_i - f_{1i} \sin \theta_i) + f_{3i} \right] / f_{5i},
\]

for \( i = 1 \) to \( n \) where \( \delta_{i1}, \delta_{i2} > 0 \) are constants commonly known as convergence parameters.

**Remark 2** The generalized controls are applicable to both strategies. Strategy that does not require a particular repulsive potential function will have a zero value of the corresponding control parameter.

**Proof:**

We propose a Lyapunov function candidate for system (1):
\[
L(x) = \sum_{i=1}^{n} \left\{ H_{N_i}(x) + G_i(x) \left[ \sum_{j=1}^{n} \left( \frac{\xi_{ij}}{R_{ij}(x)} + \frac{\xi_{ij}}{MO_{ij}(x)} \right) \right] + \sum_{k=1}^{2} \alpha_{ik} W_{ik}(x) + \sum_{m=1}^{2} \beta_{im} U_{im}(x) \right\}
\]

(19)
Assumption 3 The point \( x^* = (t_{11}, t_{12}, t_{13}, 0, \ldots, t_{n1}, t_{n2}, t_{n3}, 0, 0) \in \mathbb{R}^{5 \times n} \in D(L) \) is an equilibrium point of system (1).

Then one can easily verify that \( L \) is continuous and positive on the domain \( \mathbb{D} \), \( L(x^*) = 0 \), \( x^* \in \mathbb{D} \) and \( L(x) > 0 \ \forall \ x \in \mathbb{D}, x \neq x^* \). Now let us consider the first derivatives of the Cartesian quantities of our Lyapunov function candidate \( L(x) \). Along a particular trajectory of system (1), we have, upon collecting terms with \( v_i \) and \( \omega_i \) separately

\[
L_{(1)}(x) = \sum_{i=1}^{n} \left[ (f_{i1} \cos \theta_i + f_{i2} \sin \theta_i + f_{i4} \sigma_i) v_i - \left( \frac{1}{2} f_{i1} \sin \theta_i - f_{i2} \cos \theta_i - f_{i3} - f_{i5} \eta_i \right) \omega_i \right],
\]

where functions \( f_{i1} \) to \( f_{i5} \) are defined as (on suppressing \( x \)):

\[
f_{11} = \left( \frac{1}{H_1 + 1} + \sum_{k=1}^{2} \frac{\alpha_{1k}}{W_{1k}} + \sum_{m=1}^{2} \beta_{1m} \right) \left( \frac{\zeta_{ij}}{R_{ij}} + \frac{\xi_{ij}}{MO_{ij}} \right) (x_1 - t_{11}) - \sum_{i=2}^{n} \left( \frac{1}{H_i + 1} + \sum_{k=1}^{2} \frac{\alpha_{ik}}{W_{ik}} + \sum_{m=1}^{2} \beta_{im} \right) \left( \frac{\zeta_{ij}}{R_{ij}} + \frac{\xi_{ij}}{MO_{ij}} \right) (x_i - t_{i1}) + G_1 \sum_{j=1, j \neq i}^{n} \left( \frac{\zeta_{1j}}{R_{1j}^2} - \frac{\xi_{1j}}{MO_{1j}^2} \right) (x_1 - x_j) + \sum_{j=1, j \neq i}^{n} G_j \left( \frac{\zeta_{1j}}{MO_{j1}^2} - \frac{\zeta_{j1}}{R_{j1}^2} \right) (x_j - x_1) - G_1 \sum_{k=1}^{2} \frac{\alpha_{1k}}{W_{1k}^2} c_k (r_{k2} - r_{k1})(x_1 - (q_{k1} + \lambda_{1k}(q_{k2} - q_{k1}))(1 - c_k(q_{k2} - q_{k1})) + G_1 \sum_{k=1}^{2} \frac{\alpha_{1k}}{W_{1k}^2} c_k (r_{k2} - r_{k1})(y_1 - (r_{k1} + \lambda_{1k}(r_{k2} - r_{k1}))\right).
\]

\[
f_{21} = \left( \frac{1}{H_1 + 1} + \sum_{k=1}^{2} \frac{\alpha_{1k}}{W_{1k}} + \sum_{m=1}^{2} \beta_{1m} \right) \left( \frac{\zeta_{ij}}{R_{ij}} + \frac{\xi_{ij}}{MO_{ij}} \right) (y_1 - t_{12}) - \sum_{i=2}^{n} \left( \frac{1}{H_i + 1} + \sum_{k=1}^{2} \frac{\alpha_{ik}}{W_{ik}} + \sum_{m=1}^{2} \beta_{im} \right) \left( \frac{\zeta_{ij}}{R_{ij}} + \frac{\xi_{ij}}{MO_{ij}} \right) (y_i - t_{i2}) + G_1 \sum_{j=1, j \neq i}^{n} \left( \frac{\zeta_{1j}}{R_{1j}^2} - \frac{\xi_{1j}}{MO_{1j}^2} \right) (y_1 - y_j) + \sum_{j=1, j \neq i}^{n} G_j \left( \frac{\zeta_{1j}}{MO_{j1}^2} - \frac{\zeta_{j1}}{R_{j1}^2} \right) (y_j - y_1) - G_1 \sum_{k=1}^{2} \frac{\alpha_{1k}}{W_{1k}^2} (y_1 - (r_{k1} + \lambda_{1k}(r_{k2} - r_{k1}))(1 - d_k(r_{k2} - r_{k1})) + G_1 \sum_{k=1}^{2} \frac{\alpha_{1k}}{W_{1k}^2} d_k(q_{k2} - q_{k1})(x_1 - (q_{k1} + \lambda_{1k}(q_{k2} - q_{k1}))).
\]
For $i = 2$ to $n$

\[ f_{1i} = \left( \frac{1}{H_i + 1} + \sum_{k=1}^{2} \frac{\alpha_{ik}}{W_{ik}} + \sum_{m=1}^{2} \frac{\beta_{im}}{U_{im}} + \sum_{j=1,j\neq i}^{n} \left( \frac{\xi_{ij}}{R_{ij}} + \frac{\xi_{ij}}{MO_{ij}} \right) \right) (x_i - t_{1i}) \]

\[ + G_i \sum_{j=1,j\neq i}^{n} \left( \frac{\xi_{ij}}{R_{ij}^2} - \frac{\xi_{ij}}{MO_{ij}^2} \right) (x_i - x_j) + \sum_{j=1,j\neq i}^{n} G_j \left( \frac{\xi_{ij}}{MO_{ij}^2} - \frac{\xi_{ij}}{R_{ij}^2} \right) (x_j - x_i) \]

\[ - G_i \sum_{k=1}^{2} \frac{\alpha_{ik}}{W_{ik}^2} c_k (r_{k2} - r_{k1})(y_i - (r_{k1} + \lambda_{ik}(r_{k2} - r_{k1})) - G_i \sum_{k=1}^{2} \frac{\alpha_{ik}}{W_{ik}^2} d_k (q_{k2} - q_{k1})(x_i - (q_{k1} + \lambda_{ik}(q_{k2} - q_{k1}))). \]

For $i = 1$ to $n$

\[ f_{3i} = \left( \sum_{k=1}^{2} \frac{\alpha_{ik}}{W_{ik}} + \sum_{m=1}^{2} \frac{\beta_{im}}{U_{im}} + \sum_{j=1,j\neq i}^{n} \left( \frac{\xi_{ij}}{R_{ij}} + \frac{\xi_{ij}}{MO_{ij}} \right) \right) (\theta_i - t_{3i}), \]

\[ f_{4i} = \frac{1}{H_i + 1} + G_i \frac{\beta_{i1}}{U_{i1}^2}, \quad f_{5i} = \frac{1}{H_i + 1} + G_i \frac{\beta_{i2}}{U_{i2}^2}. \]

Substituting the controllers given in (17) - (18) and the governing ODEs for system (1) we obtain a semi-negative definite function

\[ \dot{L}_{(1)}(x) = -\sum_{i=1}^{n} \left( \delta_{1i} v_i^2 + \delta_{2i} \omega_i^2 \right) \leq 0. \]

We have thus provided a working proof of the fact that \( \frac{d}{dt}[L(x)] \leq 0 \ \forall \ x \in \mathbb{D}. \)

Finally, it can easily be verified that the first partial’s of $L_{(1)}(x)$ is $C^1$ which satisfies the final property of a Lyapunov function. Hence $L(x)$ is a feasible Lyapunov function for system (1) and $x^*$ is a stable equilibrium point in the sense of Lyapunov. In our case, this practical limitation is well within the framework of the Lyapunov-based control scheme and there is no contradiction with Brockett’s Theorem.
6 Computer Simulations

In this section we illustrate the effectiveness of the Lyapunov-based control scheme vis-a-vis the continuous time-invariant control laws, by simulating two interesting scenarios. We verify numerically the stability and the convergence results obtained from the control scheme. We present tunnel passing maneuver of a 3-robot team whereby each robot starts from an arbitrary position as depicted in Figure 2(a) and Figure 2(b). The teams get into the prescribed formations and translate to the front entrance of the tunnel. Split/rejoin and contraction/expansion maneuvers are carried out to facilitate tunnel passing. The original formations are established within a specified period of time after passing the tunnel.

6.1 Scenario 1: Split/Rejoin

Assuming the units have been appropriately taken care of, initial conditions of the 3-robot team, obstacles and target configurations, limitations on velocities, and values of different parameters are given in Table 1. We witness the split/rejoin maneuver of the team in order to facilitate tunnel passing (see Figure 5). Note the attraction functions from the leader-follower scheme ensures the members return to the prescribed formation within a predefined distance. Initially the follower-robots travel backwards to coalesce into the prescribed formation.

Figures 6 to 7 show the acceleration components of the individual robots of the team. One can clearly notice the convergence of the variables at the final state implying the effectiveness of the nonlinear controllers.

<table>
<thead>
<tr>
<th>Table 1: Numerical values of initial states, constraints and parameters for a simulation of Scenario 1.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Initial Conditions</strong></td>
</tr>
<tr>
<td>Positions</td>
</tr>
<tr>
<td>$(x_1, y_1) = (5, 10), (x_2, y_2) = (5, 15), (x_3, y_3) = (5, 5)$.</td>
</tr>
<tr>
<td>Velocities</td>
</tr>
<tr>
<td>$\theta_i = 0, v_i = 0.5, \omega_i = 0$, for $i = 1$ to $3$.</td>
</tr>
<tr>
<td><strong>Constraints and Parameters</strong></td>
</tr>
<tr>
<td>Dimension of robots</td>
</tr>
<tr>
<td>$l_1 = 1.6, l_2 = 1.2$.</td>
</tr>
<tr>
<td>Target for leader, centre</td>
</tr>
<tr>
<td>$(t_{11}, t_{12}) = (50, 10)$, and radius $r t_1 = 0.3$.</td>
</tr>
<tr>
<td>Final orientations</td>
</tr>
<tr>
<td>$t_{i3} = 0$, for $i = 1$ to $3$.</td>
</tr>
<tr>
<td>Position of ghost targets</td>
</tr>
<tr>
<td>$(a_2, b_2) = (5, -5), (a_3, b_3) = (-5, -5)$</td>
</tr>
<tr>
<td>Max. translational speed</td>
</tr>
<tr>
<td>$v_{max} = 5$.</td>
</tr>
<tr>
<td>Min. turning radius</td>
</tr>
<tr>
<td>$\rho_{min} = 0.14$.</td>
</tr>
<tr>
<td>Clearance parameter</td>
</tr>
<tr>
<td>$\epsilon_1 = 0.1, \epsilon_2 = 0.05$.</td>
</tr>
<tr>
<td>Coordinates for tunnel boundaries</td>
</tr>
<tr>
<td>$(q_{11}, r_{11}) = (20, 13), (q_{12}, r_{12}) = (30, 13)$,</td>
</tr>
<tr>
<td>$(q_{21}, r_{21}) = (20, 7), (q_{22}, r_{22}) = (30, 7)$.</td>
</tr>
<tr>
<td><strong>Control and Convergence Parameters</strong></td>
</tr>
<tr>
<td>Obstacle avoidance</td>
</tr>
<tr>
<td>$\alpha_{ik} = 0.001$, for $i=1$ to $3$, $k = 1$ to $2$.</td>
</tr>
<tr>
<td>Dynamics constraints</td>
</tr>
<tr>
<td>$\beta_{ij} = 0.01$ for $i, j = 1$ to $3$, $i \neq j$.</td>
</tr>
<tr>
<td>Convergence</td>
</tr>
<tr>
<td>$\delta_{i1} = 8, \delta_{i2} = 2, \delta_{21} = 1, \delta_{22} = 2, \delta_{31} = 1, \delta_{32} = 2$.</td>
</tr>
</tbody>
</table>
6.2 Scenario 2: Expansion/Contraction

Assuming the units have been appropriately taken care of, initial conditions of the 3-robot team, obstacles and target configurations, limitation on velocities, and values of different parameters are given in Table 2, however, only those that are different from Scenario 1.

The control laws were implemented to generate feasible contraction/expansion maneuvers of the team to facilitate tunnel passing (see Figure 8). Note that while the attraction functions in the leader-follower scheme make sure that the team members return to the prescribed formation, the inter-robot bounds guarantee and maintain the shape of the formation although the size of the formation is continually changed. We also see that initially the follower-robots travel backwards to coalesce into the prescribed formation.

Figures 9 and 10 show the acceleration components of the team members. Once again we
Table 2: Numerical values of initial states, constraints and parameters for a simulation of Scenario 2.

<table>
<thead>
<tr>
<th>Constraints and Parameters</th>
<th>Maximum distance $M_{ij}$ for $i = 1, 2, 3, j = 1, 2, 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{12} = 8.4$, $M_{13} = 8.4$, $M_{21} = 8.4$, $M_{23} = 10.3$, $M_{31} = 8.4$, $M_{32} = 10.3$.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Minimum distance $N_{ij}$ for $i, j = 1, 2, 3, i \neq j$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{ij} = 4.9$ for $i, j = 1$ to $3, i \neq j$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Control and Convergence Parameters</th>
<th>Dynamics constraints $\beta_{ij} = 3$ for $i, j = 1$ to $3, i \neq j$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max. inter-robot bound $\zeta_{ij} = 0.001$ for $i, j = 1$ to $3, i \neq j$.</td>
<td></td>
</tr>
<tr>
<td>Min. inter-robot bound $\xi_{ij} = 0.1$ for $i, j = 1$ to $3, i \neq j$.</td>
<td></td>
</tr>
<tr>
<td>Convergence $\delta_{11} = 10, \delta_{12} = 2, \delta_{21} = 1, \delta_{22} = 2, \delta_{31} = 1, \delta_{32} = 2$.</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 8: The evolution of the team trajectories to facilitate the contraction/expansion maneuver.

clearly notice the convergence of the variables at the final state implying the effectiveness of the controllers. The velocity components share similar convergence trends.

7 Conclusions and Future Research

In this paper, the Lyapunov based control scheme provides a decentralized planning architecture which stands poised to tackle the tunnel passing problem in more than one possible way with its time invariant nonlinear controllers. The controllers enable a team of nonholonomic robots fixed in a prescribed formation to obtain collision-free tunnel passing maneuvers by deploying either split/rejoin or contraction/expansion of the formation. Inter alia, subtasks such as satisfying the nonholonomic and kinodynamic constraints associated with the system are also appropriately encompassed with the framework of the Lyapunov-based control scheme.

Although computationally intensive, the control scheme can invariably be extended to three-dimensional cases as well. All-in-all, the paper highlights a fairly broad conception that reflects at least some of the autonomy of swarms in nature. The Lyapunov function extracted
from the control scheme also guaranteed stability of the system. Future work includes fine tuning the trajectories by parameter optimization, introducing curvature to the geometry of the tunnels and introducing non-leader strategies to the tunnel passing problem.

References


Continuous dependence among isotropic-anisotropic total variation flows associated with phase transitions

Shirakawa, Ken
Department of Applied Mathematics, Graduate School of System Informatics, Kobe University
1-1 Rokkodai, Nada, Kobe, 657-8501, Japan
E-mail: kenboich@kobe-u.ac.jp

1 Introduction.

Let \( \Omega \subset \mathbb{R}^2 \) be a two-dimensional bounded domain with a Lipschitz boundary \( \Gamma := \partial \Omega \). Let \( \kappa > 0 \) be a fixed constant, which is settled to be sufficiently small for the diameter of \( \Omega \). Also, let \( \theta \in L^2_{\text{loc}}(\mathbb{R}^+; L^2(\Omega)) \) be a given function. For any two-dimensional convex set \( C \subset \mathbb{R}^2 \), including the origin, let \( g_C \) be the gauge function of \( C \).

In this paper, we take an origin-symmetric compact and convex set \( W \subset \mathbb{R}^2 \), to consider the following evolution inclusion, denoted by \((E; \theta)_W\):

\[
(E; \theta)_W\ u'(t) + \kappa \partial V_W(u(t)) \ni u(t) + \theta(t) \text{ in } L^2(\Omega).
\]

Here, “′” denotes the time-derivative of functions, and \( \partial V_W \) denotes the \( L^2 \)-subdifferential of a proper l.s.c. and convex function \( V_W \) on \( L^2(\Omega) \), which is defined as:

\[
z \in L^2(\Omega) \mapsto \text{Var}_W(z) + \int_\Omega I_{[-1,1]}(z) \, dx;
\]

by using the generalized total variation:

\[
z \in L^2(\Omega) \mapsto \text{Var}_W(z) := \sup \left\{ \int_\Omega z \text{div} \varphi \, dx \middle| \varphi \in C^1_c(\Omega; \mathbb{R}^2), g_W(\varphi) \leq 1 \text{ on } \Omega \right\};
\]

and the indicator function \( I_{[-1,1]} \) on the closed interval \([-1,1]\).

Evolution inclusion \((E)_W\) (for each \( W \)) is derived as a \( L^2 \)-gradient flow of the following functional:

\[
u \in L^2(\Omega) \mapsto \mathcal{F}_W(u; \theta) := \kappa \text{Var}_W(u) + \int_\Omega \left\{ I_{[-1,1]}(u) - \frac{1}{2} u^2 - \theta u \right\} \, dx;
\]

which is a generalized version of free-energy for a certain solid-liquid phase transition, proposed by Visintin [19, Chapter VI]. Hence, mathematically, \((E)_W\) can be regarded as a kind of (generalized) total variation flow, and physically, it can be regarded as a kinetic equation of solid-liquid phase field dynamics, such as Allen-Cahn equation. Here, the given function \( \theta = \theta(t, x) \) is the relative temperature, assuming the critical temperature.
to be 0, and \( u = u(t, x) \) is the order parameter that indicates the physical state (phase) of material in the following way:

\[
\begin{cases}
\bullet u(t, x) = 1 \text{ (resp. } u(t, x) = -1) & \text{if the phase is liquid (resp. solid),} \\
\bullet -1 < u(t, x) < 1 & \text{if it is on the free boundary between solid and liquid states (the so-called interface), or the phase is intermediate.}
\end{cases}
\]

The compact convex set \( W \), as in \((E; \theta)_W\), is called “Wulff shape”, and its shape is supposed to correspond to the structural unit of crystal. Then, due to the assumed origin-symmetricity of \( W \), the gauge function \( g_W \) turns out to form a norm in \( \mathbb{R}^2 \), such that:

\[
W = \{ \ y \in \mathbb{R}^2 \mid g_W(y) \leq 1 \ \};
\]

and the polar \( g_W^\circ \) turns out to be its dual norm.

Besides, the indicator function \( I_{[-1,1]} \), as in (1), is built in to constrain the range of the parameter \( u \) onto the supposed one \([-1, 1]\). However, this indicator function also makes the integrand:

\[
\omega \in \mathbb{R} \mapsto I_{[-1,1]}(\omega) - \frac{1}{2} \omega^2 - \vartheta \omega \ (\vartheta \in \mathbb{R});
\]

be the so-called double-well type function that characterizes the phase bi-stability in the observing solid-liquid phase transition.

As a mathematical model, the evolution inclusion \((E; \theta)_W\) is described in a simplified form, and hence it is not so difficult to check the basic properties, such as the well-posedness and the large-time behavior. In fact, if we assume that:

\[
(*) \ \theta - \theta_* \in L^2(0, +\infty; L^2(\Omega)) \text{ and } \theta(t) \to \theta_* \text{ in } L^2(\Omega) \text{ as } t \to +\infty, \text{ for some } \theta_* \in \mathbb{R};
\]

then it will be seen from [10, Theorem 9.1] that the following inclusion, denoted by \((E_\infty; \theta_*)_W\):

\[
(E_\infty; \theta_*)_W \quad \kappa \partial V_W(w) \ni w + \theta_* \text{ in } L^2(\Omega);
\]

will correspond to the governing equation of the large-time behavior for \((E; \theta)_W\). Therefore, under situations similar to \((*)\), the inclusion \((E_\infty; \theta_*)_W\) can be said as the steady-state problem for \((E; \theta)_W\), and each solution \( w \) of \((E_\infty; \theta_*)_W\) can be said as the steady-state solution of \((E; \theta)_W\).

In this paper, we will consider the situation that the constant degree of the equilibrium temperature \( \theta_* \) is close to the critical one, by assuming \(-1 < \theta_* < 1\), and we will focus on special steady-state solutions, to see some geometric association between represented interfaces and Wulff shapes. Then, the Wulff shape \( W \subset \mathbb{R}^2 \) will be supposed to belong to one of the following two cases.

(Case 0) (Isotropic case) the case when \( W = \mathbb{D}^2 := \text{conv}(S^1) \).

(Case 1) (Anisotropic case of crystalline type) the case when:

\[
W \in \mathcal{P} := \left\{ P \subset \mathbb{R}^2 \left| P \text{ is origin-symmetric compact convex polygon, such that } \partial P \text{ is circumscribed to } S^1 \right. \right\}.
\]
On that basis, the following three items will be settled as the discussion points in this paper:

(a) to see the geometric structure of the interfacial patterns in steady-state (steady-state patterns), in (Case 0) and (Case 1);
(b) to study the stability for the steady-state patterns, in (Case 0) and (Case 1);
(c) to see some continuous dependence of stable steady-state patterns with respect to Wulff shapes, when (Case 0) is regarded as a limiting situation of (Case 1).

Notation. Here are listed the notations that are used throughout this paper.

For an abstract Banach space $X$, we denote by $|·|_X$ the norm of $X$. In particular, when $X$ has the Hilbert structure, we denote by $(·, ·)_X$ the inner product in $X$. In particular, when $X = \mathbb{R}^2$, we simply denote by:

$$|\xi| := \sqrt{\xi_1^2 + \xi_2^2} \quad \text{and} \quad \xi \cdot \eta := \xi_1 \eta_1 + \xi_2 \eta_2$$

for all $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in \mathbb{R}^2$; the Euclidean norm of $\xi \in \mathbb{R}^2$, and the usual scalar product of $\xi, \eta \in \mathbb{R}^2$, respectively.

Besides, the Hausdorff distance between two subsets $A, B \subset X$ is denoted by $\operatorname{dist}_X(·, ·)$, namely:

$$\operatorname{dist}_X(A, B) := \max \left\{ \sup_{x \in A} \inf_{y \in B} |x - y|_X, \sup_{y \in B} \inf_{x \in A} |y - x|_X \right\}.$$ 

For any proper l.s.c. and convex function $\Phi$ defined on an abstract Hilbert space $H$, we denote by $D(\Phi)$ the effective domain of $\Phi$, and denote by $\partial \Phi$ the subdifferential of $\Phi$. The subdifferential $\partial \Phi$ of each proper l.s.c. and convex function $\Phi$ is known as a maximal monotone graph in the product space $H \times H$, that corresponds to weak derivative (gradient) of $\Phi$. More precisely, for each $v_0 \in H$, the value $\partial \Phi(v_0)$ of the subdifferential at $v_0$ is defined as a set of all elements (derivatives) $v^*_0 \in H$ which satisfy the following variational inequality:

$$(v^*_0, v - v_0)_H \leq \Phi(v) - \Phi(v_0) \quad \text{for any } v \in D(\Phi).$$

Then, we put $D(\partial \Phi) := \{ z \in H \mid \partial \Phi(z) \neq \emptyset \}$, and call this set the domain of $\partial \Phi$. Furthermore, we often denote by $[v^*_0, v_0] \in \partial \Phi$ in $H \times H$, to say that $v^*_0 \in \partial \Phi(v_0)$ in $H$ for $v_0 \in D(\partial \Phi)$, by identifying the operator $\partial \Phi$ as its graph in $H \times H$.

For each dimension $d \in \mathbb{N}$, we denote by $\mathcal{L}^d$ the $d$-dimensional Lebesgue measure, and we use this measure unless otherwise specified. Also, we denote by $\mathcal{H}^d$ the Hausdorff measure in each observing dimension $d \in \mathbb{N}$.

Additionally, in this paper, $\Omega \subset \mathbb{R}^2$ is fixed as a bounded domain with a Lipschitz boundary $\Gamma$, and the product space $(0, +\infty) \times \Omega$ of the time interval $(0, +\infty)$ and the spatial domain $\Omega$ is simply denoted by $Q$. Besides, $\theta_s$ is fixed as a constant degree of the equilibrium temperature, such that $-1 < \theta_s < 1$. For any open subset $D \subset \Omega$, the external part $\Omega \setminus \overline{D}$ in $\Omega$ is denoted by $D^{\text{ex}}$. 
For any compact and convex set $C \subset \mathbb{R}^2$ including the origin, the *gauge function* of $C$, more precisely:

$$g_C(\xi) := \inf \left\{ \lambda \geq 0 \mid \xi \in \lambda C \right\} \text{ for any } \xi \in \mathbb{R}^2.$$ 

As is easily checked, if the convex set $C$ is origin-symmetric, then $g_C$ forms a norm in $\mathbb{R}^2$.

In view of this, for any $r > 0$, any $x \in \mathbb{R}^2$ and any origin-symmetric compact convex set $W \subset \mathbb{R}^2$, we denote by $B_W(x; r)$ the interior of the bounded set $(x + rW) \cap \Omega$, namely:

$$B_W(x; r) := \{ y \in \Omega \mid g_W(y - x) < r \}.$$ 

## 2 Main Theorems

In this paper, four theorems are presented as the main conclusions, and they are respectively stated as follows.

The first and the second theorems are respectively concerned with the item (a) in (Case 0) and (Case 1), as in introduction.

**Main Theorem 1.** (Structural observation in (Case 0), cf. [14, 18]) Let us denote by $\mathcal{X}_0$ the solution class of the steady-state problem $(E_\infty; \theta_*)_{\mathbb{D}^2}$ in (Case 0). Then, the class $S(\mathbb{D}^2)$, defined below, is a subclass of $\mathcal{X}_0$.

$$S(\mathbb{D}^2) := \left\{ w_D := \chi_D - \chi_{D^{\text{ex}}} \mid D \subset \Omega \text{ is a domain satisfying the margined conditions, labeled as (A1)$_0$-(A3)$_0$ (see Fig. 1 to get the general idea).} \right\}.$$ 

(A1)$_0$ $\Gamma_D := \partial D \cap \Omega$ is a Jordan curve.

(A2)$_0$ $\exists r > 2\kappa, \text{ s.t.}$

$$D = \bigcup_{x \in D, B_{\mathbb{D}^2}(x; r) \subset D} B_{\mathbb{D}^2}(x; r) \text{ and } D^{\text{ex}} = \bigcup_{x \in D^{\text{ex}}, B_{\mathbb{D}^2}(x; r) \subset D^{\text{ex}}} B_{\mathbb{D}^2}(x; r). \quad (2)$$

(A3)$_0$ The tubular domain $\left\{ x \in \Omega \mid \inf_{y \in \Gamma_D} |y - x| < r \right\}$ is $C^2$-diffeomorphic with the rectangle $[0, 1] \times (-1, 1)$.

![Fig. 1](image1.png)

![Fig. 2](image2.png)
Main Theorem 2. (Structural observation in (Case 1), cf. [15, 16, 17]) Let us fix any 
$P \in \mathcal{P}$, and let us denote by $X_P$ the solution class of the steady-state problem $(E_{\infty}; \theta_*)_P$ 
in (Case 1) when $W = P$. Then, the class $S(P)$, defined blow, is a subclass of $X_P$.

$$S(P) := \left\{ w_D := \chi_D - \chi_{D^{\infty}} \middle| D \subseteq \Omega \text{ is a domain satisfying the marginated conditions, labeled as (A1)$_P$-(A3)$_P$ (see Fig. 2 to get the general idea).} \right\}.$$  

(A1)$_P$ $\Gamma_D := \partial D \cap \Omega$ is a polygonal (piecewise linear) Jordan curve, such that its any 
edge (the part of segment) is parallel to one of those of the polygon $P$.

(A2)$_P$ $\exists r > 2\kappa$, s.t.

$$D = \bigcup_{x \in D, B_P(x;r) \subseteq D} B_P(x;r) \quad \text{and} \quad D^{ex} = \bigcup_{x \in D^{ex}, B_P(x;r) \subseteq D^{ex}} B_P(x;r).$$

(A3)$_P$ $0 < \forall \rho < r$, two compact sets:

$$\left\{ x \in D \middle| \inf_{y \in \Gamma_D} g_P(y-x) = \rho \right\} \quad \text{and} \quad \left\{ x \in D^{ex} \middle| \inf_{y \in \Gamma_D} g_P(y-x) = \rho \right\};$$

are both Jordan curves, included in $\Omega$.

The next third main theorem is concerned with the stability analysis, mentioned in 
the item (b) in introduction.

Main Theorem 3. (Stability analysis, cf. [14, 15, 16, 17]) Let us fix any Wulff shape 
$W$ belonging to either of (Case 0)-(Case 1), and let us take any $w_D \in S(W)$ with the 
characteristic domain $D = w_D^{-1}(1) \subset \Omega$. Here, let us take two positive constants $\varepsilon_*$ and 
$\delta_*$, such that:

$$0 < \frac{2\kappa}{1 - |\theta_*| - 3\varepsilon_*} < r_* - \delta_*;$$

and let us set:

$$\Gamma_D(\rho)_W := \left\{ x \in \Omega \middle| \inf_{y \in \Gamma_D} g_W(y-x) \leq \delta \right\}, \quad \text{for any } 0 < \delta < \delta_*.$$ 

Then, the steady-state solution $w_D \in S(W)$ shows the stability in the following sense.

(s)$_*$ For any $0 < \varepsilon < \varepsilon_*$ and any $0 < \delta < \delta_*$, there exists a finite time $t_W = t_W(\varepsilon, \delta)$, 
depending on $W$, $\varepsilon$ and $\delta$, such that $u(t) = w_D$ a.e. in $\Omega \setminus \Gamma_D(\delta)$, for any $t \geq t_W$, 
any function $\theta \in L^2_{loc}([0, +\infty); L^2(\Omega))$ and any solution $u$ of $(E; \theta)_W$, satisfying:

$$|\theta - \theta_*|_{L^\infty(Q)} \leq \varepsilon_* \quad \text{and} \quad |u(0) - w_D|_{L^\infty(\Omega \setminus \Gamma_D(\delta))} \leq \varepsilon.$$ 

Remark. (Supplements toward the continuous dependence) In Main Theorem 3, let us 
note that the constants $\varepsilon_*$ and $\delta_*$ are taken, independently of the Wulff shape $W \in \mathcal{P} \cup \{D^2\}$. Furthermore, in the item (s)$_*$, setting $t_W = \log 2$ is actually one of possible 
choice, and hence the convergence time $t_W$ can be taken, independently of $W \in \mathcal{P} \cup \{D^2\}$, 
too. Such uniform situation will be an important key in the observation of the continuous 
dependence, as in the item (c) in introduction, and the results on this theme is stated in 
the following final Main Theorem 4.
Main Theorem 4. (Continuous dependence from (Case 1) to (Case 0)) Let us define:

\[
\omega \cdot S(\mathcal{P}) := \left\{ w_\infty \in BV(\Omega) \left| \exists \{P_n\} \subset \mathcal{P}, \exists \{w_n\} \mid w_n \in S_{P_n}, n \in \mathbb{N} \right., \text{s.t.} \right. \begin{align*}
&\text{dist}_{\mathbb{R}^2}(\partial P_n, S^1) \to 0 \\
&w_n \to w_\infty \text{ in } L^2(\Omega) \left. \right\} \text{ as } n \to +\infty \right\}.
\]

where \( \text{dist}_{\mathbb{R}^2}(\cdot, \cdot) \) is the Hausdorff distance between subsets in \( \mathbb{R}^2 \). Then, the following three statements hold.

(I) (Upper-bound) \( \omega \cdot S(\mathcal{P}) \subset X_0 \), and furthermore:

\[
\omega \cdot S(\mathcal{P}) \subset S^* := \left\{ w_D := \chi_D - \chi_D^\text{ex} \left| D \subset \Omega \text{ is a domain which satisfies the condition } (A1)_0 \text{, and satisfies (2), as in } \right. \right. (A2)_0, \text{ for some } r \geq 2\kappa. \right. \right\}.
\]

(II) (Lower-bound)

\[
\omega \cdot S(\mathcal{P}) \supset S_\ast := \left\{ w_D := \chi_D - \chi_D^\text{ex} \left| D \subset \Omega \text{ is a domain, which satisfies just two conditions } (A1)_0 \text{ and } (A2)_0 \right. \right. \right\}.
\]

Namely, \( S_\ast \) is also a subclass of \( X_0 \), which is strictly wider than \( S(\mathbb{D}^2) \), as in Main Theorem 1.

(III) (Stability) For any \( w_D \in S_\ast \), the steady-state solution \( w_D \) shows the stability, just mentioned in (s)\(_*\) of Main Theorem 3. To conclude, the third condition \( (A3)_0 \), as in Main Theorem 1, is eventually unnecessary for the stability analysis in the isotropic case.

The above Main Theorems will be proved, with helps from a lot of mathematical theories, obtained by Amar-Bellettini [1], Ambrosio-Fusco-Pallara [2], Andreu-Caselles-Mazón [3], Attouch [4], Bellettini-Caselles-Chambolle-Novaga [5], Caselles-Chambolle-Moll-Novaga [6], Evans-Gariepy [7], Giusti [9], Giga-Giga [8], Kenmochi [10], Moll [11], Mosco [12], e.t.c..

References


Non-homogeneous semilinear elliptic equations involving critical Sobolev exponent

Yūki Naito\textsuperscript{a} and Tokushi Sato\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, Ehime University, Matsuyama 790-8577, Japan
\textsuperscript{b} Mathematical Institute, Tohoku University, Sendai 980-8578, Japan

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial \Omega$ with $N \geq 3$. We consider the existence of multiple positive solutions of the following semilinear elliptic equations

\begin{equation}
(1.1)_\lambda \begin{cases}
-\Delta u + \kappa u = u^p + \lambda f(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

where $\lambda > 0$ is a parameter, $\kappa \in \mathbb{R}$ is a constant, $p = (N + 2)/(N - 2)$ is the critical Sobolev exponent, and $f(x)$ is a non-homogeneous perturbation satisfying $f \in H^{-1}(\Omega)$ and $f \geq 0$, $f \not\equiv 0$ in $\Omega$. Let $\kappa_1$ be the first eigenvalue of $-\Delta$ with zero Dirichlet condition on $\Omega$. Since $(1.1)_\lambda$ has no positive solution if $\kappa \leq -\kappa_1$ (see Remark 1 below), we will consider the case $\kappa > -\kappa_1$.

Let us recall the results for the homogeneous semilinear elliptic problem involving critical Sobolev exponent

\begin{equation}
(1.2) \begin{cases}
-\Delta u + \kappa u = u^p & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation}

It is well known that $(1.2)$ admits no nontrivial solution for each $\kappa \geq 0$ via Pohozaev identity, provided that $\Omega$ is star-shaped. On the other hand, when $\kappa < 0$, Brezis and Nirenberg [2] showed the following remarkable phenomenon:

(i) if $N \geq 4$, then for every $\kappa \in (-\kappa_1, 0)$, there exists a positive solution;

(ii) if $N = 3$ and $\Omega$ is a ball, then there exists a positive solution if and only if $\kappa \in (-\kappa_1, -\kappa_1/4)$.

The results in [2] show that the space dimension plays a fundamental role when one seeks solutions of $(1.2)$. Similar phenomena have been shown for more general problems.
involving the $p$-Laplacian [7, 6, 8], higher order elliptic problems [15, 16, 11], elliptic problems with Hardy potential [12], and quasilinear elliptic equations [1].

For the non-homogeneous problem, let us first consider the problem (1.1)$_\lambda$ with $\kappa = 0$;

$$
\begin{align*}
-\Delta u &= u^p + \lambda f(x) \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

(1.3)

Tarantello [18] investigated suitable minimization and minimax principles of mountain pass-type, and showed that (1.3) has at least two positive solution for $\lambda > 0$ sufficiently small. The existence of two nontrivial solutions was shown by Cao and Zhou [3] for more general problem

$$
\begin{align*}
-\Delta u &= u^p + g(x, u) + \lambda f(x) \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

where $g(x, u)$ is a suitable lower-order perturbation of $u^p$. The achievement, existence of at least two nontrivial solutions for $\lambda > 0$ sufficiently small, has been extended to the problems involving the $p$-Laplacian by Chabrowski [4] and Zhou [19], and to more general problems by Squassina [17].

In this paper we investigate the multiplicity of positive solutions to the problem (1.1)$_\lambda$ with $\kappa > -\kappa_1$, and show that the problem exhibits the phenomenon depending on the space dimension. Precisely, we find that, when $\kappa > 0$, the situation is drastically different between the cases $N = 3, 4, 5$ and $N \geq 6$.

We show first the existence of the extremal value of $\lambda$ for the existence of solutions to (1.1)$_\lambda$. We call a positive minimal solution $u_\lambda$ of (1.1)$_\lambda$, if $u_\lambda$ satisfies $u_\lambda \leq u$ in $\Omega$ for any positive solution $u$ of (1.1)$_\lambda$.

**Theorem 1.** Let $\kappa > -\kappa_1$. Then there exists $\bar{\lambda} \in (0, \infty)$ such that

(i) if $0 < \lambda < \bar{\lambda}$ then the problem (1.1)$_\lambda$ has a positive minimal solution $u_\lambda \in H^1_0(\Omega)$. Furthermore, if $0 < \lambda < \hat{\lambda} < \bar{\lambda}$ then $u_\lambda < u_{\hat{\lambda}}$ a.e. in $\Omega$, and $u_\lambda \rightharpoonup 0$ in $H^1_0(\Omega)$ as $\lambda \rightarrow 0$;

(ii) if $\lambda > \bar{\lambda}$ then the problem (1.1)$_\lambda$ has no positive solution $u \in H^1_0(\Omega)$.

**Remark 1.** There is no positive solution of (1.1)$_\lambda$ with $\kappa \leq -\kappa_1$. In fact, assume to the contrary that there exists a positive solution $u$ of (1.1)$_\lambda$ with $\kappa \leq -\kappa_1$. Let $\varphi_1$ be the eigenfunction corresponding to the first eigenvalue $\kappa_1$ with $\varphi_1 > 0$ on $\Omega$. Then we have

$$
0 = \int_\Omega (\nabla u \cdot \nabla \varphi_1 - \kappa_1 u \varphi_1) dx \geq \int_\Omega (\nabla u \cdot \nabla \varphi_1 + \kappa u \varphi_1) dx = \int_\Omega (u^p \varphi_1 + \lambda f \varphi_1) dx > 0.
$$
This is a contradiction.

Let us consider the existence of solutions of \((1.1)_\lambda\) at the extremal value \(\lambda = \overline{\lambda}\), so-called extremal solutions. For the semilinear elliptic equation \(-\Delta u = \lambda F(u)\) in \(\Omega\) with Dirichlet condition \(u = 0\) on \(\partial\Omega\), the existence of extremal solutions has been well studied, see e.g., [9, 13, 5, 14], and it is well known that the existence and nonexistence of the extremal solutions depend strongly on the dimension, domain, and nonlinearity. For non-homogeneous problem \((1.1)_\lambda\), we obtain the following.

**Theorem 2.** If \(\lambda = \overline{\lambda}\) then the problem \((1.1)_\lambda\) has a unique positive solution in \(H^1_0(\Omega)\).

Let us consider the existence and nonexistence of second positive solutions to \((1.1)_\lambda\) for \(0 < \lambda < \overline{\lambda}\).

**Theorem 3.** Assume that either (i) or (ii) holds.

(i) \(\kappa \in (-\kappa_1, 0]\) and \(N \geq 3\);  
(ii) \(\kappa > 0\) and \(N = 3, 4, 5\).

If \(0 < \lambda < \overline{\lambda}\) then \((1.1)_\lambda\) has a positive solution \(\overline{u}_\lambda \in H^1_0(\Omega)\) satisfying \(\overline{u}_\lambda > u_{\lambda}\).

**Theorem 4.** Assume that \(\kappa > 0\) and \(N \geq 6\).

(i) There exists \(\lambda^* = \lambda^*(\kappa) \in (0, \overline{\lambda})\) such that if \(\lambda^* \leq \lambda < \overline{\lambda}\) then the problem \((1.1)_\lambda\) has a positive solution \(\overline{u}_\lambda \in H^1_0(\Omega)\) satisfying \(\overline{u}_\lambda > u_{\lambda}\).

(ii) Let \(\Omega = \{x \in \mathbb{R}^N : |x| < R\}\) with some \(R > 0\), and let \(f = f(|x|)\) be radially symmetric about the origin. Assume that \(f \in C^\alpha([0, R])\) with some \(0 < \alpha < 1\), and \(f(r)\) is nonincreasing in \(r \in (0, R)\). Then there exists \(\lambda_* \in (0, \lambda^*)\) such that \((1.1)_\lambda\) has a unique positive solution \(\overline{u}_\lambda\) for \(\lambda \in (0, \lambda_*]\).

**Remark 2.** (i) In Theorem 3, the existence of non-minimal solution was shown by Tarantello [18] in the case \(\kappa = 0\). However, in the case \(\kappa > 0\), the result for the existence of non-minimal solutions depends on the space dimension by Theorem 4 (ii), and seems to be new.

(ii) In Theorem 4, it is an open question whether the problem \((1.1)_\lambda\) has a non-minimal solution for \(\lambda_* < \lambda < \lambda^*\).

(iii) When one of the conditions in Theorem 4 (ii) does not hold, for example, when \(f\) is not radially symmetric, we do not know the uniqueness of the positive solution of \((1.1)_\lambda\) for \(\lambda > 0\) sufficiently small.
In the proof of Theorem 1, we will employ the bifurcation results and the comparison argument for solutions of \((1.1)\lambda\) to obtain the minimal solutions. We will prove Theorem 2 by establishing a priori bound for the solutions of \((1.1)\lambda\) at \(\lambda = \bar{\lambda}\).

In order to find a second positive solution of \((1.1)\lambda\), we introduce the problem

\[
\begin{align*}
-\Delta v + \kappa v &= (v + \underline{u}_\lambda)^p - \underline{u}_\lambda^p \quad \text{in } \Omega, \quad v \in H^1_0(\Omega),
\end{align*}
\]

where \(\underline{u}_\lambda\) is the minimal positive solution of \((1.1)\lambda\) for \(\lambda \in (0, \bar{\lambda})\) obtained in Theorem 1. In fact, assume that \((1.4)\) has a positive solution \(v\), and put \(\underline{u}_\lambda v + \underline{u}_\lambda^p\). Then \(\underline{u}_\lambda \in H^1_0(\Omega)\) and solves \((1.1)\lambda\) and satisfies \(\underline{u}_\lambda > \underline{u}_\lambda\) in \(\Omega\). In the proof of Theorem 3, we will show the existence of solutions of \((1.4)\) by using a variational method. To this end we define the corresponding variational functional of \((1.4)\) by

\[
I_\kappa(v) = \frac{1}{2} \int_\Omega (|\nabla v|^2 + \kappa v^2) \, dx - \int_{\mathbb{R}^N} G(v, \underline{u}_\lambda) \, dx
\]

for \(v \in H^1_0(\Omega)\), where

\[
G(t, s) = \frac{1}{p+1} (t_+ + s)^{p+1} - \frac{1}{p+1} s^{p+1} - s^p t_+.
\]

It is easy to see that \(I_\kappa : H^1_0(\Omega) \to \mathbb{R}\) is \(C^1\) and the critical point \(v_0 \in H^1_0(\Omega)\) satisfies

\[
\int_\Omega (\nabla v_0 \cdot \nabla \psi + \kappa v_0 \psi + g(v_0, \underline{u}_\lambda)\psi) \, dx = 0
\]

for any \(\psi \in H^1_0(\Omega)\), where

\[
g(t, s) = (t_+ + s)^p - s^p.
\]

Denote by \(S\) the best Sobolev constant of the embedding \(H^1_0(\Omega) \subset L^{p+1}(\Omega)\), which is given by

\[
S = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx}{\left( \int_\Omega |u|^{p+1} \, dx \right)^{2/(p+1)}}.
\]

We will obtain Theorem 3 as a consequence of the following two propositions.

**Proposition 1.** Let \(\lambda \in (0, \lambda^*)\). Assume that there exists \(v_0 \in H^1_0(\Omega)\) with \(v_0 \geq 0\), \(v_0 \not\equiv 0\) such that

\[
\sup_{t > 0} I_\kappa(tv_0) < \frac{1}{N} S^{N/2}.
\]

Then there exists a positive solution \(v \in H^1_0(\Omega)\) of \((1.4)\).

**Proposition 2.** Assume that either (i) or (ii) holds.

(i) \(\kappa \in (-\kappa_1, 0]\) and \(N \geq 3\); (ii) \(\kappa > 0\) and \(N = 3, 4, 5\).
Then there exists a positive function \( v_0 \in H^1_0(\Omega) \) such that (1.5) holds.

In the proof of Proposition 1, we will derive some estimates to establish inequalities relating certain minimizing sequences. In order to prove Proposition 2, for \( \varepsilon > 0 \), we will set
\[
u_\varepsilon(x) = \frac{\phi(x)}{\varepsilon + |x|^2(N-2)/2},
\]
where \( \phi \in C^\infty_0(\mathbb{R}^N), 0 \leq \phi \leq 1 \), is a cut off function, and will show that (1.5) holds with \( v_0 = u_\varepsilon \) for sufficiently small \( \varepsilon > 0 \).

In the proof of Theorem 4 (ii), we will verify the nonexistence of positive solutions of (1.4) in the radial case by the Pohozaev type argument for the associated ODE. In fact, by [10], the solution \( v \) of (1.4) must be radially symmetric, and \( v = v(r), r = |x| \), satisfies the problem of the following ordinary differential equation
\[
\begin{cases}
(r^{N-1}v_r)_r - \kappa r^{N-1}v + r^{N-1}g(v, u_\lambda) = 0, & 0 < r < R, \\
v_r(0) = v(R) = 0.
\end{cases}
\]
(1.6)

For the solution \( v \) to (1.6), we will obtain the following Pohozaev type identity:
\[
\int_0^R r^{N-1} \left[ \frac{2N}{N-2} G(u, u_\lambda) - g(u, u_\lambda)u \right] dr + \frac{2}{N-2} \int_0^R r^N G_s(u, u_\lambda)u_\lambda dr \\
+ \frac{2\kappa}{N-2} \int_0^\infty r^{N-1}u^2 dr = \frac{1}{N-2} R^N v_r(R)^2.
\]

In the proofs of Theorems 2, 3 and 4, the results concerning the eigenvalue problems to the linearized equations around the minimal solutions
\[
-\Delta \phi + \phi = \mu p(u_\lambda)^{p-1} \phi \quad \text{in} \Omega, \quad \phi \in H^1_0(\Omega).
\]
play a crucial role.

References


