Euler Angles in Four Dimension

Akira Kageyama

Department of Computational Science, Kobe University, Kobe, Japan

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Abstract

Rotations in four dimensional (4-D) space has six degrees of freedom. An example set of the six Euler angles in 4-D rotations is derived.

1 Rotations in Four Dimensional Space

It was shown in the 19th century by Cole that a rotation in four dimensional (4-D) space is reduced to a combination of two rotations whose fixed planes are absolutely perpendicular each other [1]. Here we derive the result in a concise way—Cole's paper is 20 pages long.

Because a 4-D rotation R is a unitary transformation, it is represented by a diagonal matrix $R = \mathrm{diag}(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$, under (normalized) complex eigenvectors $\boldsymbol{u}_i$. Here σ_i are complex eigenvalues: $R\boldsymbol{u}_i = \sigma_i \boldsymbol{u}_i$, with $|\sigma_i| = 1$ and $\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle = \delta_{ij}$. The angle brackets denote the inner product of complex vectors. Complex conjugate of this equation means that $\boldsymbol{u}_i^\dagger$ is an eigenvector of R with eigenvalue $\sigma_i^\dagger$, where the dagger denotes a complex conjugate. When σ_i is real, it is ± 1. We exclude negative determinant cases such as $R = \mathrm{diag}(\sigma_0, \sigma_0^\dagger, 1, -1)$. Then we can assume, without loss of generality, that R is represented as $R = \mathrm{diag}(\sigma_0, \sigma_0^\dagger, \sigma_2, \sigma_2^\dagger)$ on the basis vectors $\{\boldsymbol{u}_0, \boldsymbol{u}_0^\dagger, \boldsymbol{u}_2, \boldsymbol{u}_2^\dagger\}$. Now we define real vectors

$$\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\} = \left\{ \frac{\boldsymbol{u}_0 + \boldsymbol{u}_0^\dagger}{\sqrt{2}}, \frac{\boldsymbol{u}_0 - \boldsymbol{u}_0^\dagger}{i\sqrt{2}}, \frac{\boldsymbol{u}_2 + \boldsymbol{u}_2^\dagger}{\sqrt{2}}, \frac{\boldsymbol{u}_2 - \boldsymbol{u}_2^\dagger}{i\sqrt{2}} \right\}. \tag{1}$$

The factor $1/\sqrt{2}$ is introduced to satisfy the orthonormal relations: $\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \delta_{ij}$. Inversely,

$$\left\{\boldsymbol{u}_0, \boldsymbol{u}_0^\dagger, \boldsymbol{u}_2, \boldsymbol{u}_2^\dagger\right\} = \left\{ \frac{\boldsymbol{e}_0 + i\boldsymbol{e}_1}{\sqrt{2}}, \frac{\boldsymbol{e}_0 - i\boldsymbol{e}_1}{\sqrt{2}}, \frac{\boldsymbol{e}_2 + i\boldsymbol{e}_3}{\sqrt{2}}, \frac{\boldsymbol{e}_2 - i\boldsymbol{e}_3}{\sqrt{2}} \right\}. \tag{2}$$

Denoting $\sigma_0 = \mathrm{e}^{i\alpha}$, and using eqs. (1) and (2), The real vector $\boldsymbol{e}_0$ is transformed by R as

$$R\boldsymbol{e}_0 = \frac{\sigma_0 \boldsymbol{u}_0 + \sigma_0^\dagger \boldsymbol{u}_0^\dagger}{\sqrt{2}} = \frac{\sigma_0(\boldsymbol{e}_0 + i\boldsymbol{e}_1) + \sigma_0^\dagger(\boldsymbol{e}_0 - i\boldsymbol{e}_1)}{2} = \cos\alpha\, \boldsymbol{e}_0 - \sin\alpha\, \boldsymbol{e}_1 \tag{3}$$

Similarly,

$$R\boldsymbol{e}_1 = \sin\alpha\, \boldsymbol{e}_0 + \cos\alpha\, \boldsymbol{e}_1 \tag{4}$$

Also, denoting $\sigma_2 = \mathrm{e}^{i\beta}$, we get $R\boldsymbol{e}_2 = \cos\beta\, \boldsymbol{e}_2 - \sin\beta\, \boldsymbol{e}_3$,and $R\boldsymbol{e}_3 = \sin\beta\, \boldsymbol{e}_2 + \cos\beta\, \boldsymbol{e}_3$.Therefore, on the basis vectors $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, the rotation matrix R is represented as

$$R_{01,23}(\alpha, \beta) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{pmatrix}. \tag{5}$$

This is called double rotation. The subscripts of $R_{01,23}$ stand for rotations in the $\boldsymbol{e}_0$–$\boldsymbol{e}_1$ plane and in the $\boldsymbol{e}_2$–$\boldsymbol{e}_3$ plane.

Note that a double rotation is a commutable product of two rotations

$$R_{01}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6}$$

and

$$R_{23}(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{pmatrix}. \tag{7}$$

These are called simple rotations. The subscripts of R_{01} indicate that it is a rotation in the $\boldsymbol{e}_0$–$\boldsymbol{e}_1$ plane. The $\boldsymbol{e}_2$–$\boldsymbol{e}_3$ plane is called the fixed plane for this rotation. The double rotations are a combination of two simple rotations around two fixed planes that are absolutely perpendicular each other [2]. The simple rotations are special case when one of the two angles in the double rotation is zero.

2 Euler Angles in 4-D

Any 4-D rotation can be represented by a series of simple rotations of six in maximum, under a fixed coordinate system. It is proved as follows: Suppose that unit vectors $\{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z, \boldsymbol{e}_w\}$ along each axis of a four dimensional x–y–z–w space are rotated to $\{\boldsymbol{e}'_x, \boldsymbol{e}'_y, \boldsymbol{e}'_z, \boldsymbol{e}'_w\}$ by a rotation R. We can always construct simple rotations R_{wy}, R_{yx}, and R_{wz} in such a way that their product $R_{wy}R_{yx}R_{wz}$ reverts $\boldsymbol{e}'_w$ to the original direction $\boldsymbol{e}_w$. The other three vectors $\{\boldsymbol{e}'_x, \boldsymbol{e}'_y, \boldsymbol{e}'_z\}$ are then in the three dimensional x–y–z space and they can be reverted to the original directions $\{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}$ by the standard (extrinsic) Euler angles R_{yx}, R_{yz}, and R_{xz}. Therefore, $R_{yx}R_{yz}R_{xz}R_{wy}R_{yx}R_{wz}R = I$, where I is the identity matrix. In other words, R is represented by six simple rotations: $R = R_{zw}R_{xy}R_{yw}R_{zx}R_{zy}R_{xy}$, since $R_{ij}^{-1} = R_{ji}$. Note that R can also be represented by two double rotations and two simple rotations: $R = R_{zw,xy}R_{yw,zx}R_{zy}R_{xy}$.

References

[1] F.N. Cole, American Journal of Mathematics pp. 191–210 (1890)

[2] H.P. Manning, *Geometry of four dimensions* (MACMILLAN, New York, 1914)