Derivation of the dissipation function*

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Take a small parcel δV of a compressible viscous fluid* with the viscosity μ. The equation of motion reads $\rho \frac{\partial v_i}{\partial t} = \cdots + \partial_j T_{ij}$, where T_{ij} is the stress tensor proportional to the viscosity:

$$T_{ij} = \mu\,(\partial_i v_j + \partial_j v_i) - (2\mu/3)\,\partial_k v_k \tag{1}$$

The energy integral (dot product of v_i) of the equation of motion leads to

$$(\partial/\partial t)(\rho v^2/2) = \cdots + v_i\,(\partial_j T_{ij}).$$

where $v_i\,(\partial_j T_{ij})\,\delta V$ is the <u>gain</u> of the flow energy due to viscosity, or $\delta E := -v_i\,(\partial_j T_{ij})\,\delta V$ is the <u>loss</u> of kinetic energy due to the viscosity. The lost energy δE is converted into two parts: (i) the work δW done by the fluid parcel to other part of the fluid, and (ii) the internal thermal heating δH:

$$\delta E = \delta W + \delta H.$$

The work δW is the negative of the work done by other part of the fluid to the parcel, which is calculated by the surface integral of the dot product of the viscous force F_i and velocity v_i:

$$\delta W = -\int_{\partial\delta V} v_i\,F_i\,dS = -\int_{\partial\delta V} v_i\,T_{ij}\,n_i\,dS = -\partial_j\,(v_i\,T_{ij})\;\delta V.$$

Therefore, the thermal heating is given by

$$\delta H = -\delta W + \delta E = [\,\partial_j\,(v_i\,T_{ij}) - v_i\,(\partial_j T_{ij})\,]\,\delta V = (\partial_i v_j)\,T_{ij}\,\delta V$$

The dissipation function is defined as $\Phi = \delta H/\delta V = (\partial_i v_j)\,T_{ij}$. Substituting eq. (1), we get†

$$\Phi = (\mu/2)(\partial_i v_j + \partial_j v_i)(\partial_i v_j + \partial_j v_i) - (2\mu/3)\,(\partial_k v_k)^2$$

In the cartesian coordinates, it is simply,

$$\begin{aligned}\Phi &= 2\,\mu\left[(\partial_1 v_1)^2 + (\partial_2 v_2)^2 + (\partial_3 v_3)^2\right] \\ &\quad + \mu\left[(\partial_1 v_2 + \partial_2 v_1)^2 + (\partial_2 v_3 + \partial_3 v_2)^2 + (\partial_3 v_1 + \partial_1 v_3)^2\right] - (2\mu/3)\,(\partial_k v_k)^2\end{aligned}$$

*A note for B4 students in our lab.

*Here we suppose the bulk viscosity $\zeta := \lambda + (2/3)\,\mu = 0$.

†Note that $(\partial_i v_j)\,T_{ij} = (\partial_j v_i)\,T_{ij} = (1/2)(\partial_i v_j + \partial_j v_i)\,T_{ij}$.