Numerical Methods for Geodynamo Simulation

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Part 2

2007.08 Les Houches Summer School “Dynamos”

Geodynamo Simulations
in a Sphere or a Spherical Shell

Outline

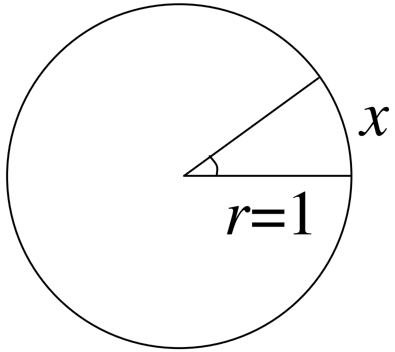
1. Various numerical methods used in the spherical geodynamo simulation
2. How to make a large scale simulation code
   - A sample Fortran90 code for kinematic dynamo in a box geometry.
   - Basic and useful Fortran90 features.
   - A sample data analysis (visualization) program.

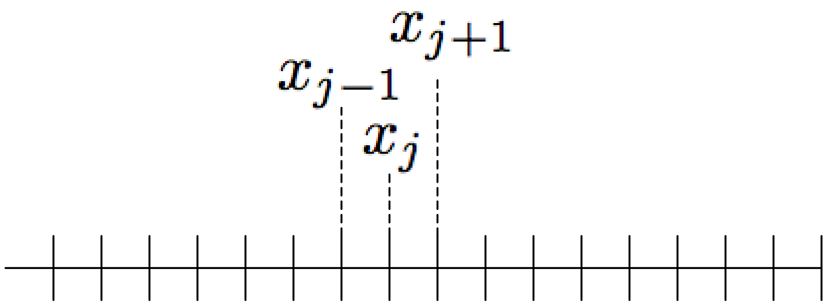
Methods used in geodynamo simulations

- Spectral-based methods
  - Spherical harmonics expansion
  - Double Fourier expansion
  - Beltrami function expansion
- Other methods
  - Finite difference method
  - Finite volume method
  - Finite element method
  - Cartesian grid method

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2}$$

(1) FDM: Finite Difference Method



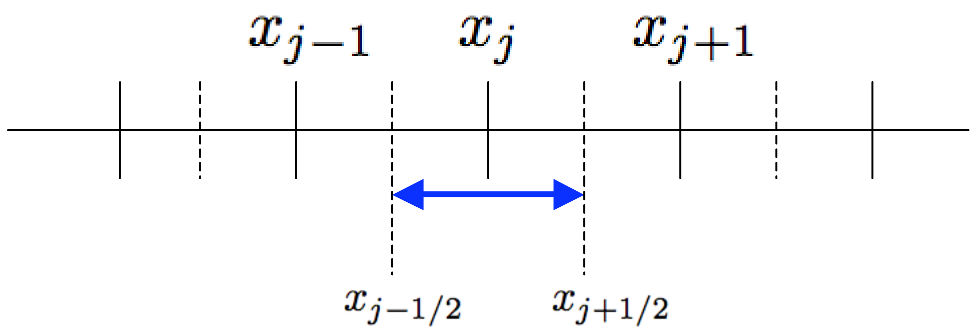


$$\frac{d\psi_j}{dt} = \frac{\psi_{j+1} - 2\psi_j + \psi_{j-1}}{(\Delta x)^2}$$

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2}$$

(2) FVM: Finite Volume Method

$$\frac{\partial \psi}{\partial t} = \frac{\partial F}{\partial x}, \qquad F = \frac{\partial \psi}{\partial x}$$



$$\int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial \psi}{\partial t} dx = \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial F}{\partial x} dx,$$

cell average

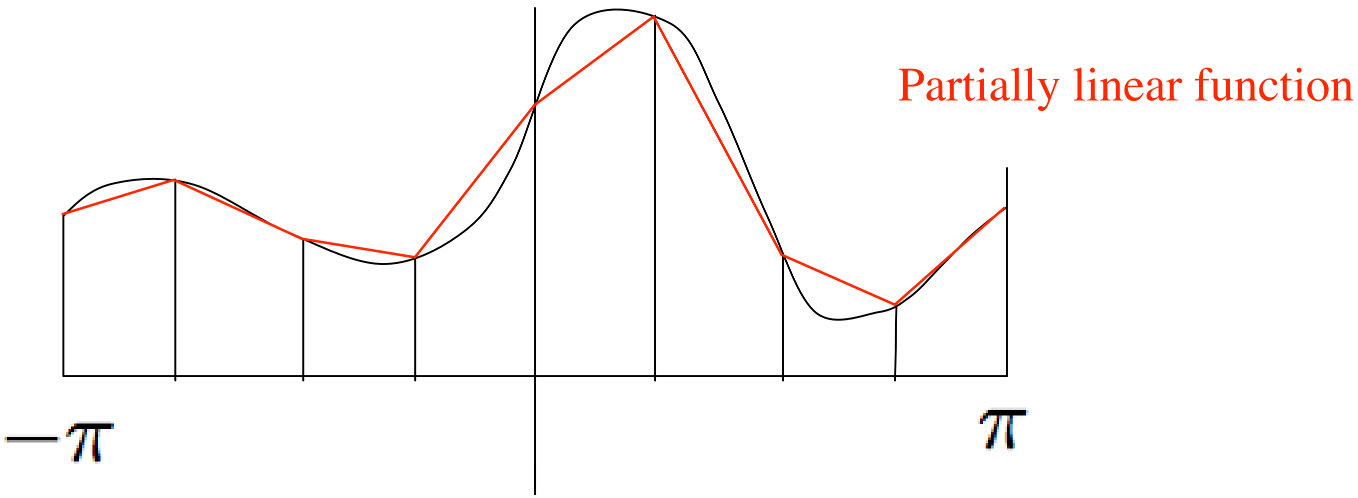
$\left(\bar{f} \equiv \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} f dx \right)$

$$\frac{d\bar{\psi}_j}{dt} = \left[F(x_{j+1/2}) - F(x_{j-1/2}) \right] / \Delta x$$

$$\boxed{\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2}}$$

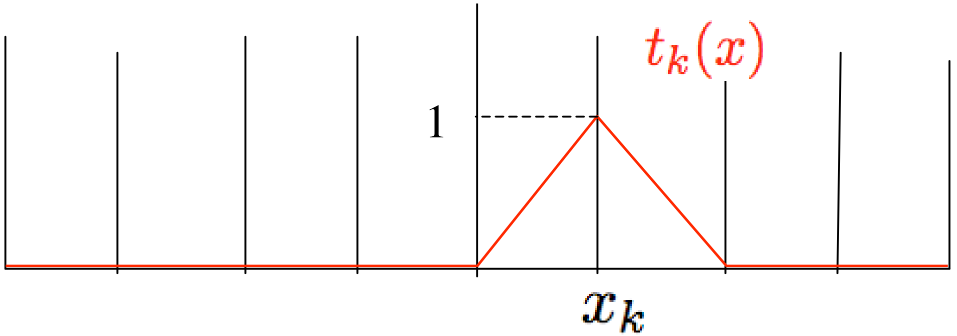
(3) FEM: Finite Element Method

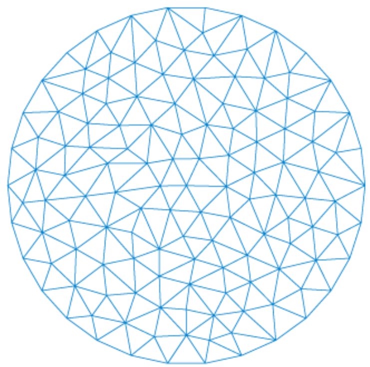
$\forall a(x)$, (periodic)
$$\int_{-\pi}^{\pi} a(x) \frac{\partial \psi}{\partial t} dx = -\int_{-\pi}^{\pi} a'(x) \frac{\partial \psi}{\partial x} dx$$



$$\psi(x,t) = \sum_k \psi_k(t)\, t_k(x)$$

$$a(x) = t_j(x), \quad (j = 1, 2, 3, \ldots)$$





$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2}$$

(4) Spectral Method

Expansion by a set of orthonormal basis.

$$\psi(x,t) = \sum_k \psi_k(t) a_k(x)$$

$$(a_j, a_k) \equiv \int_{-\pi}^{\pi} a_j^\dagger(x)\, a_k(x) dx = \delta_{jk}$$

Example: Fourier functions.

$$a_k(x) = \frac{\exp(ikx)}{\sqrt{2\pi}}$$

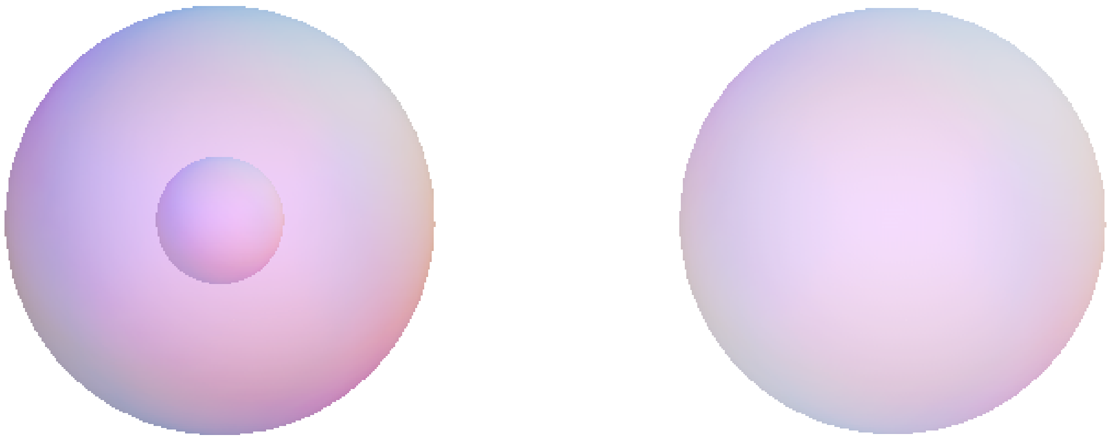
$$\frac{d\psi_m(t)}{dt} = -m^2\, \psi_m(t)$$

Geodynamo simulation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \quad \nabla \cdot \mathbf{v} = 0, \quad \nabla \cdot \mathbf{B} = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla p' + 2\mathbf{v} \times \mathbf{\Omega} + \nu \nabla^2 \mathbf{v} + \mathbf{j}' \times \mathbf{B} + \mathbf{F}$$

To solve this kind of PDE system in the spherical geometry.

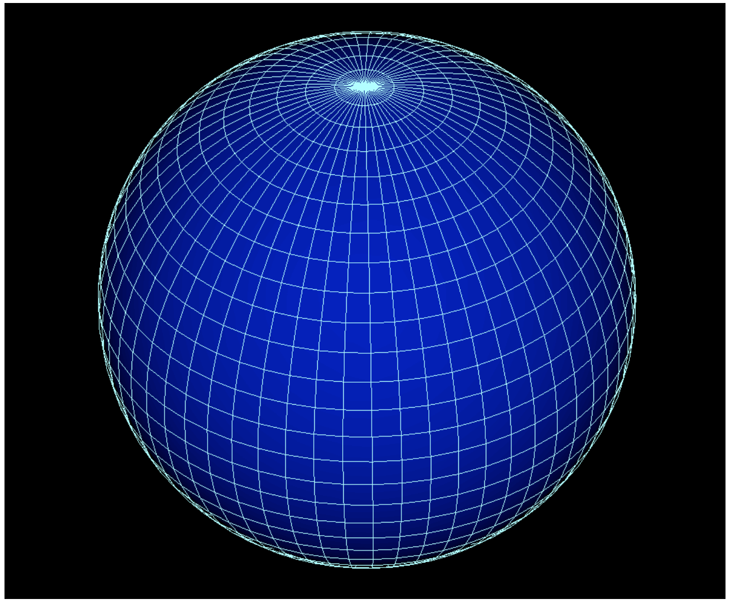


Geodynamo simulation by full FDM

Lat-lon (latitude-longitude) grid

Coordinate singularity?

===> No problem.

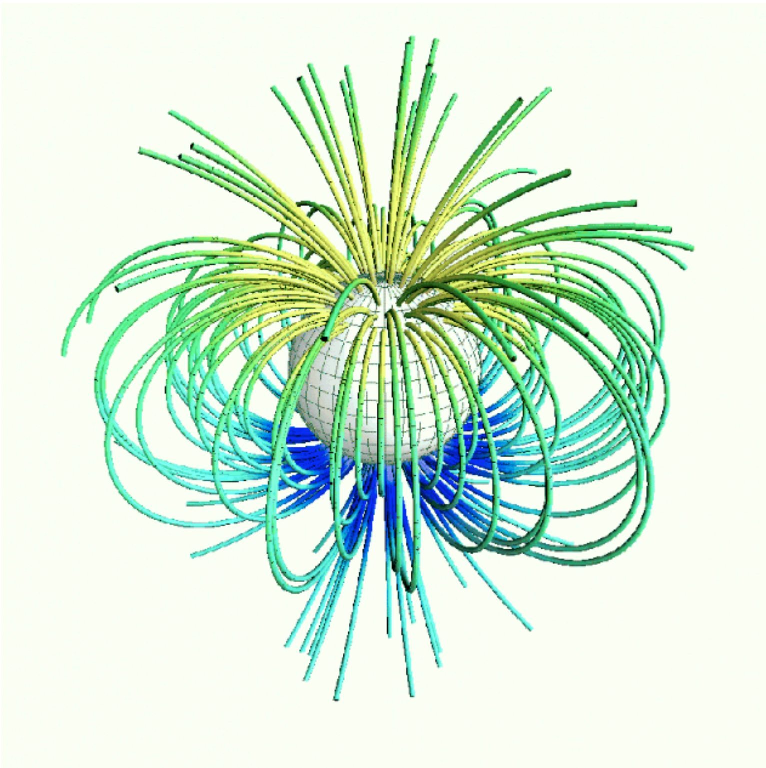


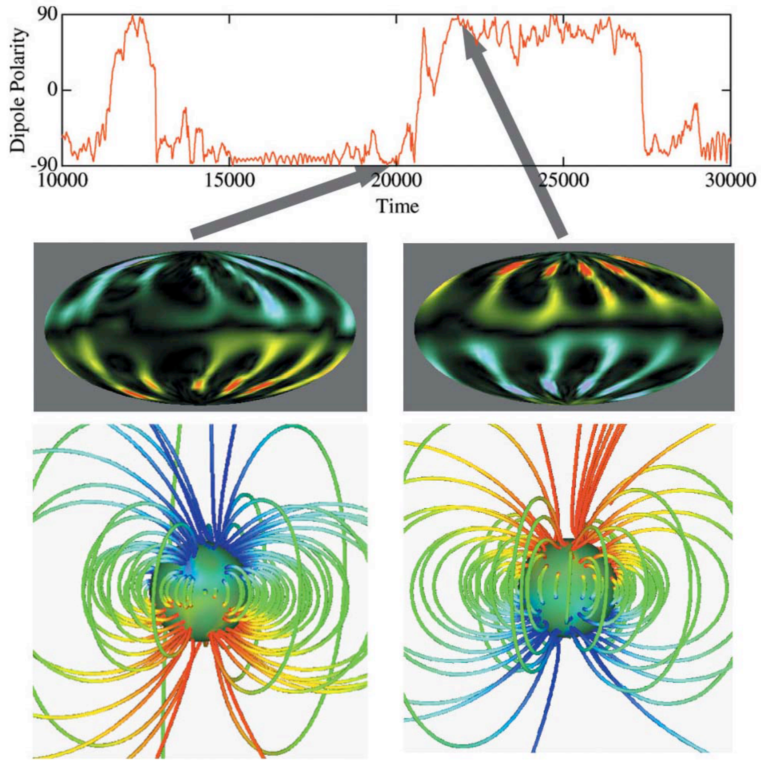
l'Hôpital's rule

$$\lim_{\theta \to 0} \frac{1}{r \sin\theta} \frac{\partial f}{\partial \theta} = \frac{1}{r} \frac{\partial^2 f}{\partial \theta^2} \bigg|_{\theta=0}$$

Geodynamo simulation by full FDM

Lat-lon (latitude-longitude) grid

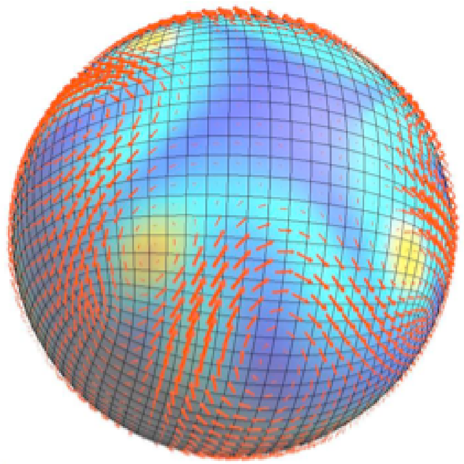




Kageyama et al., 1997, 1999

Geodynamo simulation by FVM

H. Harder and U. Hansen: on a cubic projected grid



P. Hejda and M. Reshetnyak: on the latitude-longitude grid

Control Volume Method for the Dynamo Problem ...

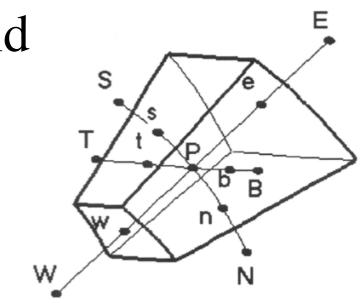
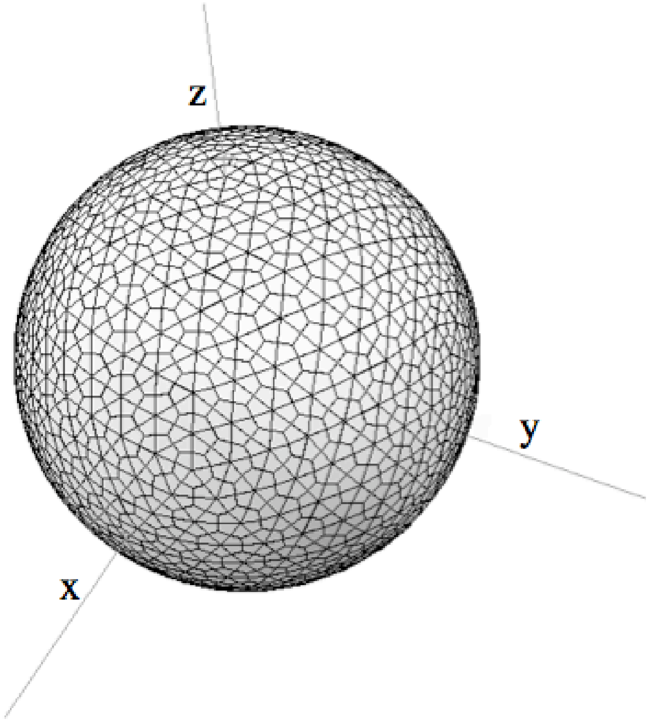
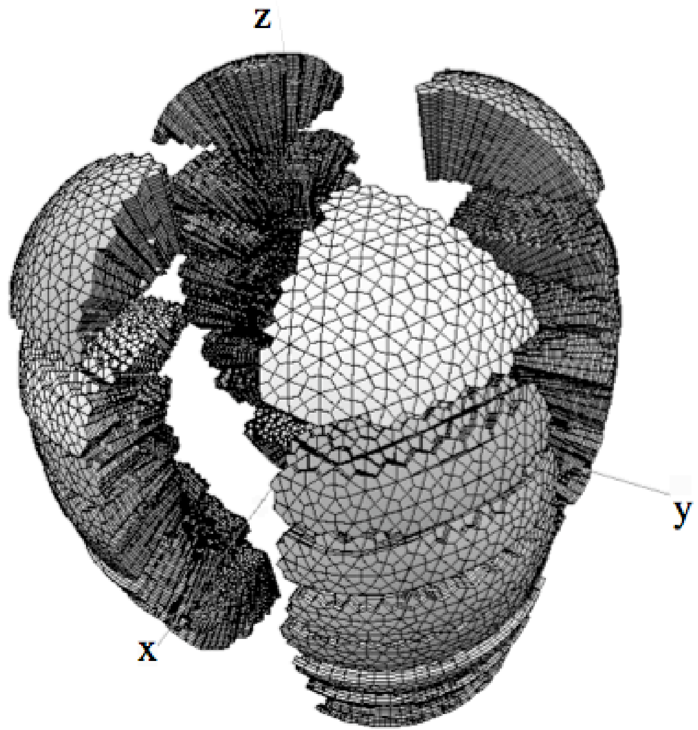


Fig. 1. Control volume in the spherical grid.

Geodynamo simulation by FEM





Matsui and Okuda

Geodynamo simulation by FEM

K.H. Chan et al. / Physics of the Earth and Planetary Interiors 157 (2006) 124–138

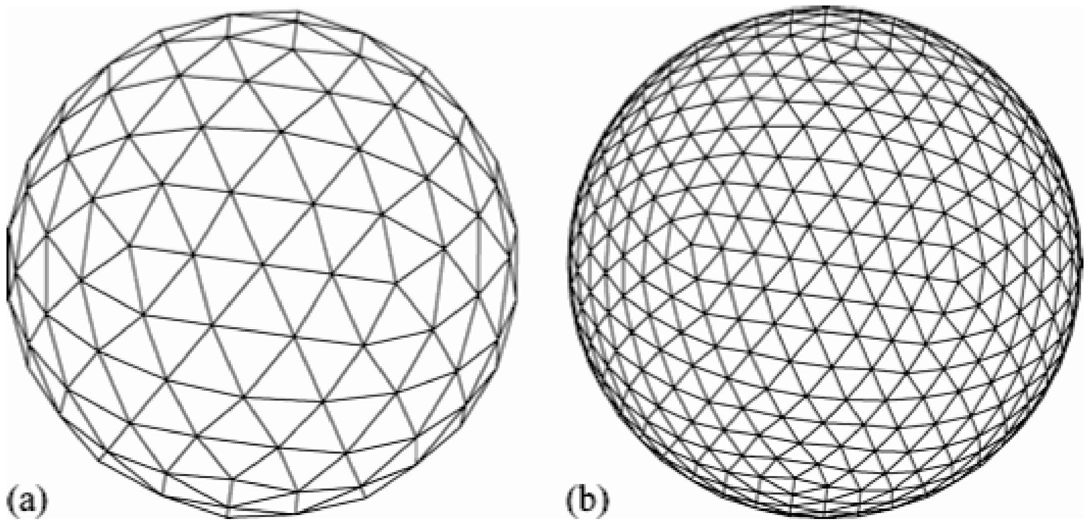
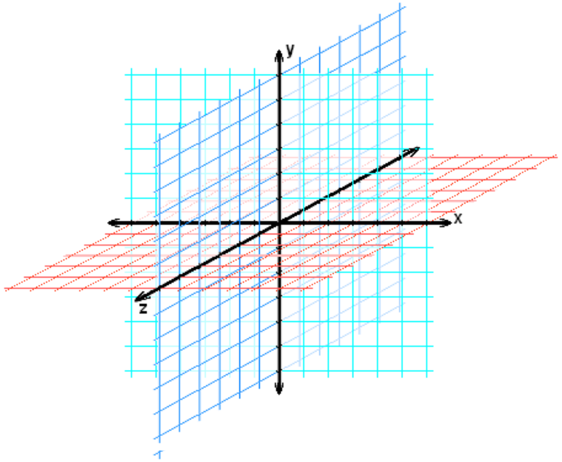


Fig. 1. Triangulation of the inner boundary: (a) 320 triangles and (b) 1280 triangles.

K.H. Chan, Ligang Li, and Xinhao Liao, 2006,

Gedynamo simulation by FDM on Cartesian grid



D.G. McMillan, G.R. Sarson / Physics of the Earth and Planetary Interiors 153 (2005) 124–135 133

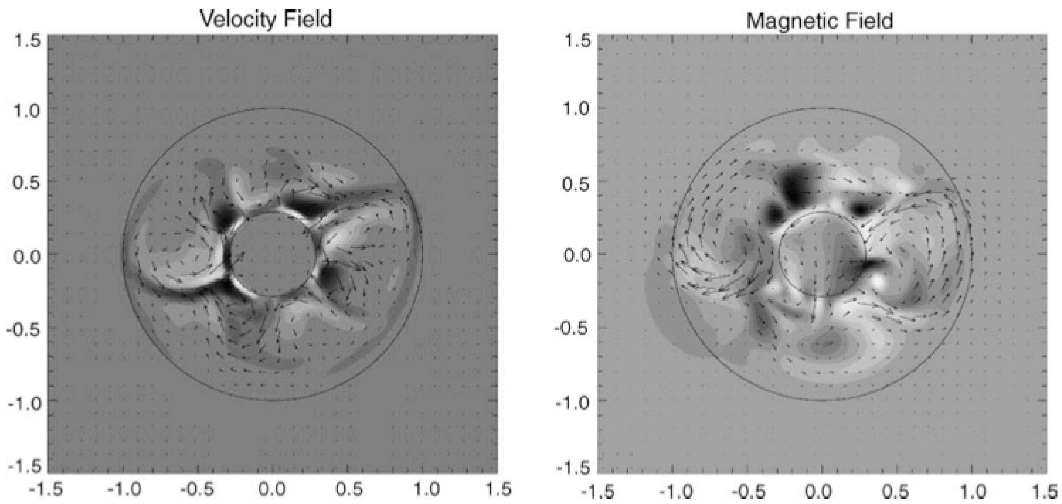
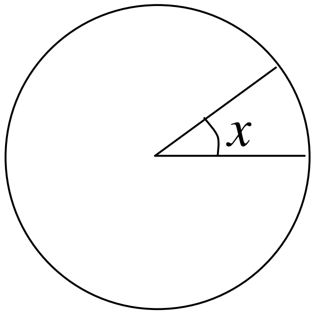


Fig. 9. Equatorial velocity and magnetic field structure. From a snapshot near the end of run 2F, the greyscale shows the vertical components of vorticity ω_z (left) and magnetic field B_z (right). The arrows give the horizontal components of velocity (left) and magnetic field (right). With the smallest magnetic Prandtl number ($Pm = 20.2$), this run retains the least disordered flow after saturation of the magnetic energy and still displays remnants of columnar structures.

D.G. McMillan and G.R. Sarson, 2005

Geodynamo simulation by spectral method

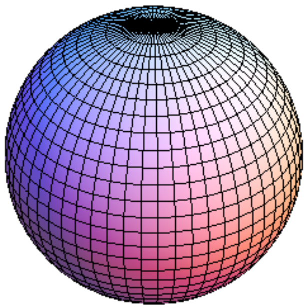
2-D sphere (circle)



$$\psi(x,t) = \sum_k \psi_k(t) a_k(x)$$

$$(a_j, a_k) \equiv \int_{-\pi}^{\pi} a_j^{\dagger}(x)\, a_k(x) dx = \delta_{jk}$$

3-D sphere



$$\psi(\theta,\phi,t) = \sum_k \psi_k(t)\, a_k(\theta,\phi)$$

$$(a_j, a_k) \equiv \int_S a_j^{\dagger}(\theta,\phi)\, a_k(\theta,\phi)\, d\cos\theta d\phi$$

Spherical harmonics

$$Y_{\ell,m}(\theta,\phi) = C_{\ell,m}\, P_\ell^m(\cos\theta)\, \exp(im\phi)$$

Normalized spherical harmonics

$$\int_0^\pi d\cos\theta \int_{-\pi}^\pi d\phi\; Y_{\ell,m}(\theta,\phi)\; Y^\dagger_{\ell,m}(\theta,\phi) = \delta_{\ell,m}$$

$$\psi(\theta,\phi) = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \psi_{\ell,m}\, Y_{\ell,m}(\theta,\phi)$$

Expand physical variables by the spherical harmonics and time-integrate them in the spectral space.

The method used in most (>90%?) geodynamo simulations.

Pseudo-spectral method for nonlinear terms

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla p' + 2\mathbf{v}\times\mathbf{\Omega} + \nu\nabla^2\mathbf{v} + \mathbf{j}'\times\mathbf{B} + \mathbf{F}$$

Spectral space	Real space
$T_{\ell,m},\ P_{\ell,m}$	$\mathbf{j}(\theta,\phi),\ \ \mathbf{B}(\theta,\phi)$
$\hat{T}_{\ell,m},\ \hat{P}_{\ell,m}$	$\mathbf{f}(\theta,\phi) = \mathbf{j}(\theta,\phi)\times\mathbf{B}(\theta,\phi)$

Spherical harmonics transform

Spherical harmonics transform = Fourier transform + Legendre transform

$$Y_{\ell,m}(\theta,\phi) = C_{\ell,m}\, P_\ell^m(\cos\theta)\ \exp(im\phi)$$

Legendre transform

$$Y_{\ell,m}(\theta,\phi) = C_{\ell,m}\, P_\ell^m(\cos\theta)\ \exp(im\phi)$$

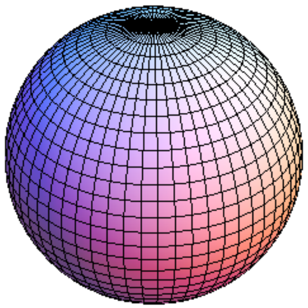
$$\ell \leq L, \quad m \leq M$$

For Fourier transform, “fast” algorithm exists: FFT

- $O(M \log M)$

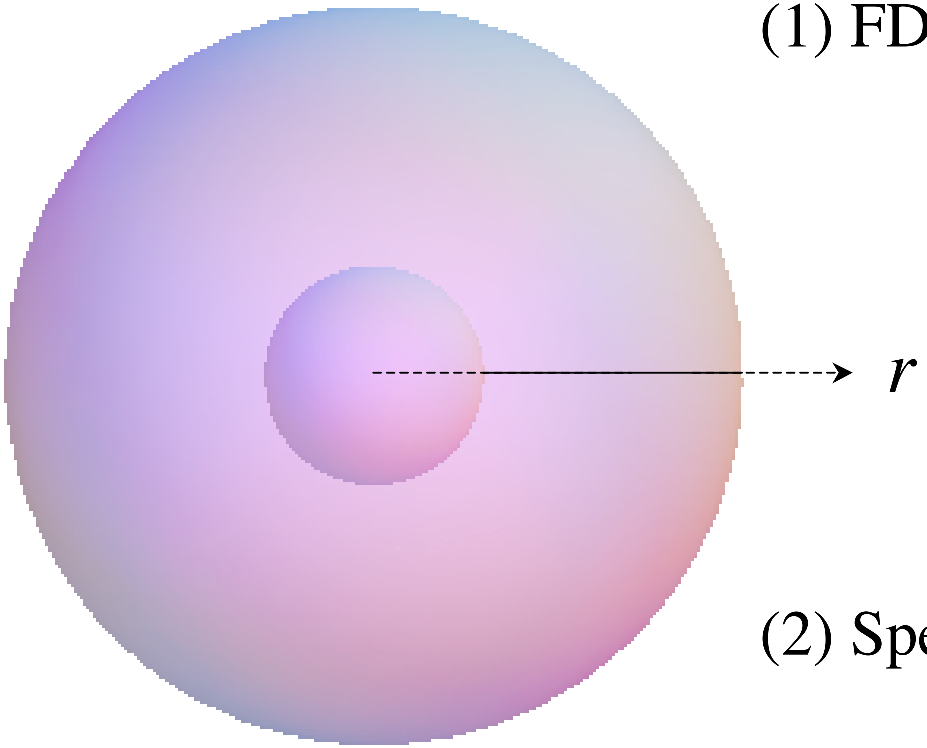
But for Legendre transform, there is no fast (practical) algorithm:

- $O(L^2)$

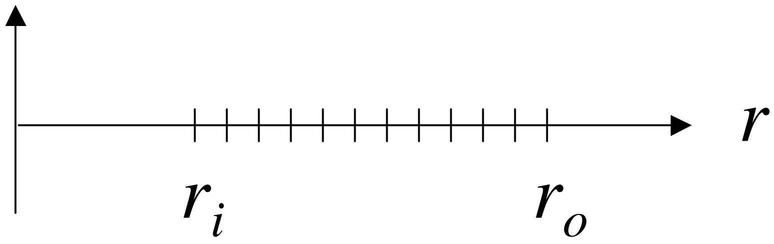


- Data size
  - $O(LM)$
- Computation for MHD eqs.
  - $O(LM)$
- Computations for transformation
  - $O(L^2 M \log M) \gg O(LM)$

Numerical methods for spherical shell geometry



(1) FDM (only) in the radial direction



(2) Spectral method in radial direction

Chebyshev polynomials

$$\psi(r) = \sum_n \psi_n T_n(x), \quad x \equiv \frac{r - r_i}{r_o - r_i}$$

Chebyshev polynomials

Definition:

$$T_n(x) = \cos(n \arccos(x))$$

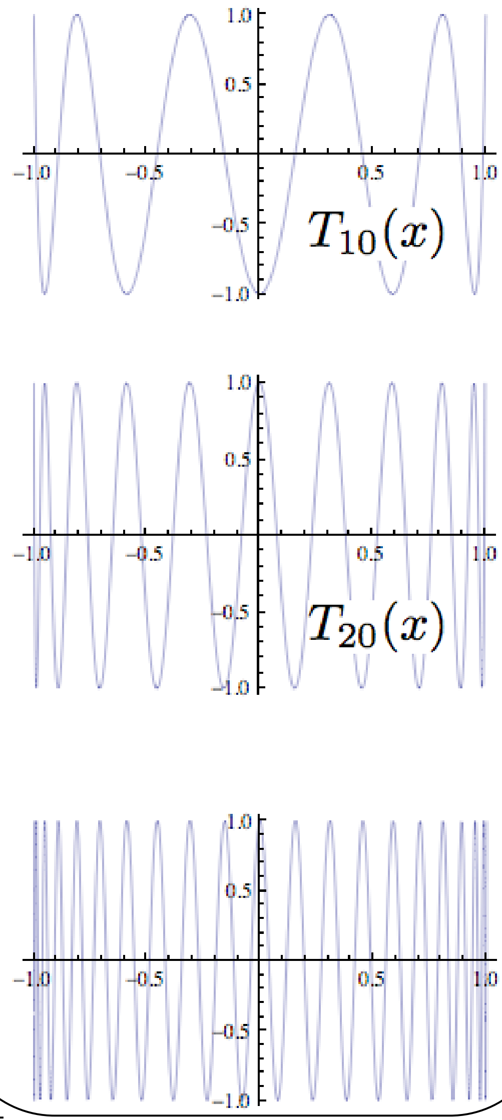
Orthogonal relation:

$$\int_{-1}^{1} \frac{T_m(x)\, T_n(x)}{\sqrt{1-x^2}} dx = 0, \quad (m \neq n)$$

Fast Fourier Cosine Transform:

$$x_k = -\cos\left(\frac{k\pi}{N}\right), \quad k = 0, 1, \ldots, N$$

Naturally high resolution on the outer and inner spherical boundaries:



Geodynamo simulation by the fully spectral method

Glatzmaier and many others

Geodynamo simulation by Fourier + FDM

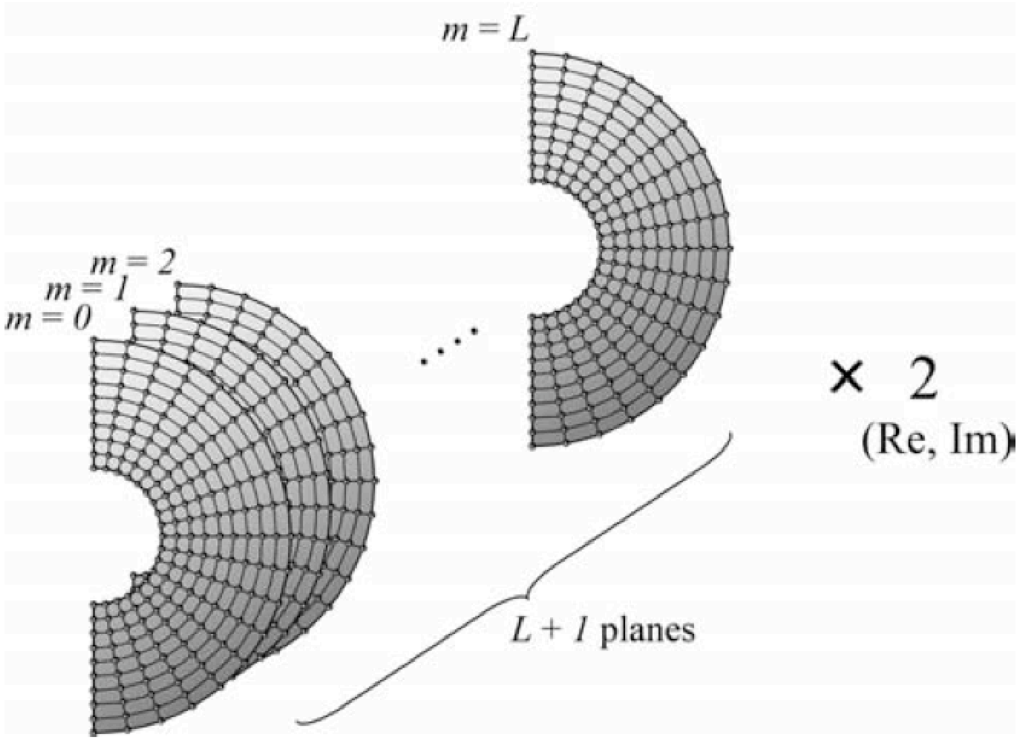
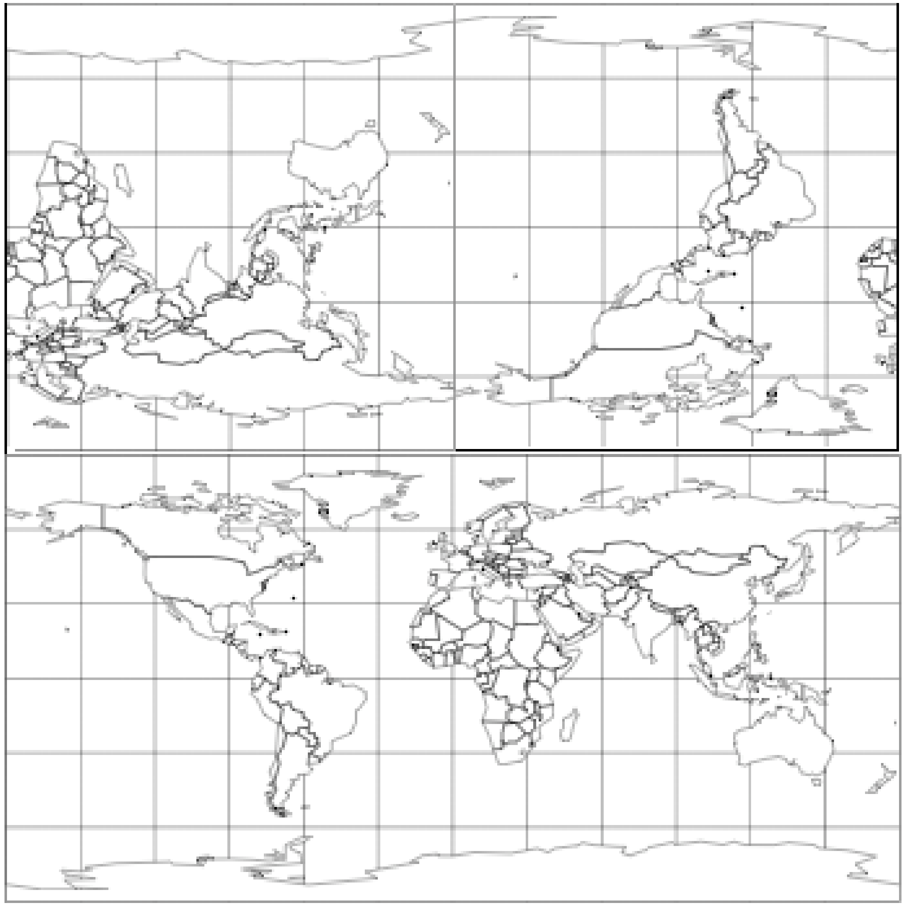


Fig. 1 Grid system in spectral space. Since Fourier coefficients are imaginary numbers, we solve a differntial equation on $2(L+1)$ meridional planes.

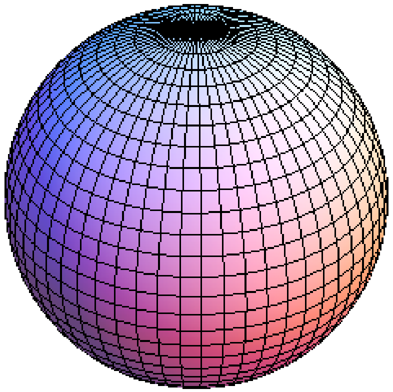
Oishi, Sakuraba, and Hamano, 2007
Hejda and Reshetnyak, 2000

Double Fourier transform method

2π



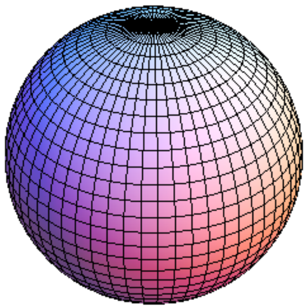
2π



Nishikawa and Kusano

Spectral Method for a full sphere (or a ball)

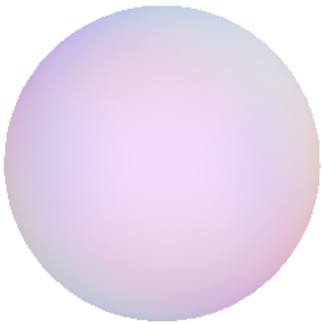
Sphere (surface)



$$\psi(\theta,\phi,t) = \sum_k \psi_k(t)\, a_k(\theta,\phi)$$

$$(a_j, a_k) \equiv \int_S a_j^\dagger(\theta,\phi)\, a_k(\theta,\phi)\, d\cos\theta d\phi$$

Ball



$$\psi(r,\theta,\phi,t) = \sum_k \psi_k(t)\, a_k(r,\theta,\phi)$$

$$(a_j, a_k) \equiv \int_V a_j^\dagger(\theta,\phi)\, a_k(\theta,\phi)\, r^2\, dr\, d\cos\theta\, d\phi$$

Beltrami field

Beltrami field = Eigen-vector of the curl operator:

$$\nabla \times \mathbf{B}_i = \lambda_i \mathbf{B}_i$$

$$\lambda_i = \text{eigen value (real)}$$

- As a fluid flow, a Beltrami field

$$\nabla \times \mathbf{u}_i = \lambda_i \mathbf{u}_i$$

gives a solution of the stationary Euler equation:

$$(\mathbf{u}_i \cdot \nabla)\mathbf{u}_i = -\nabla p$$

- For MHD, a Beltrami field is a force-free magnetic field:

$$\mathbf{j}_i \times \mathbf{B}_i = 0$$

Beltrami fields

Beltrami fields in a simply connected domain *V*, like a ball, form a set of complete orthogonal system.

- Boundary condition on the surface of *V*,

$$\mathbf{B}_i \cdot \mathbf{n} = 0$$

- Orthogonal relation:

$$\int_V \mathbf{B}_i \cdot \mathbf{B}_j \, dV = 0 \quad (\text{for} \ \ \lambda_i \neq \lambda_j)$$

Orthogonality of Beltrami fields

$$\begin{cases} \nabla \times \mathbf{B}_1 = \lambda_1 \mathbf{B}_1 \\ \nabla \times \mathbf{B}_2 = \lambda_2 \mathbf{B}_2 \end{cases}$$

$$\nabla \times \mathbf{B}_i \cdot \mathbf{n} = 0$$

$$\mathbf{B}_i = \nabla_{\parallel} \phi_i \ \text{(on the surface)}$$

$$(\lambda_1 - \lambda_2) \int_V \mathbf{B}_1 \cdot \mathbf{B}_2 \, dV = \int_V (\mathbf{B}_2 \cdot \nabla \times \mathbf{B}_1 - \mathbf{B}_1 \cdot \nabla \times \mathbf{B}_2) \, dV$$

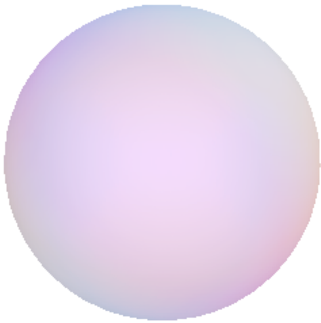
$$= \int_V \nabla \cdot (\mathbf{B}_1 \times \mathbf{B}_2) \, dV$$

$$= \int_{\partial V} (\mathbf{B}_1 \times \mathbf{B}_2) \cdot \mathbf{n} \, dS$$

$$= \int_{\partial V} (\nabla_{\parallel} \phi_1 \times \mathbf{B}_2) \cdot \mathbf{n} \, dS$$

$$= \int_{\partial V} \nabla \times (\phi_1 \mathbf{B}_2) \cdot \mathbf{n} \, dS = 0$$

Beltrami fields in a ball



- A solution of the Helmholtz equation

$$(\nabla^2 + \lambda^2)\,\psi = 0$$

gives a Beltrami field in the form

$$\mathbf{J} = \lambda \nabla \times (\psi \mathbf{r}) + \nabla \times \nabla \times (\psi \mathbf{r})$$

$$\nabla \times \mathbf{J} = \lambda \mathbf{J}$$

$$\psi_{q\ell m} = c_{q\ell}\, j_\ell(|\lambda_{q\ell}|\, r)\, Y_{\ell m}(\theta, \phi)$$

spherical Bessel functions

P. D. Mininni and D. C. Montgomery, Phys. Fluids **18**, 11602 (2006)

MHD dynamo simulation in a ball by the spectral method based on Beltrami expansion

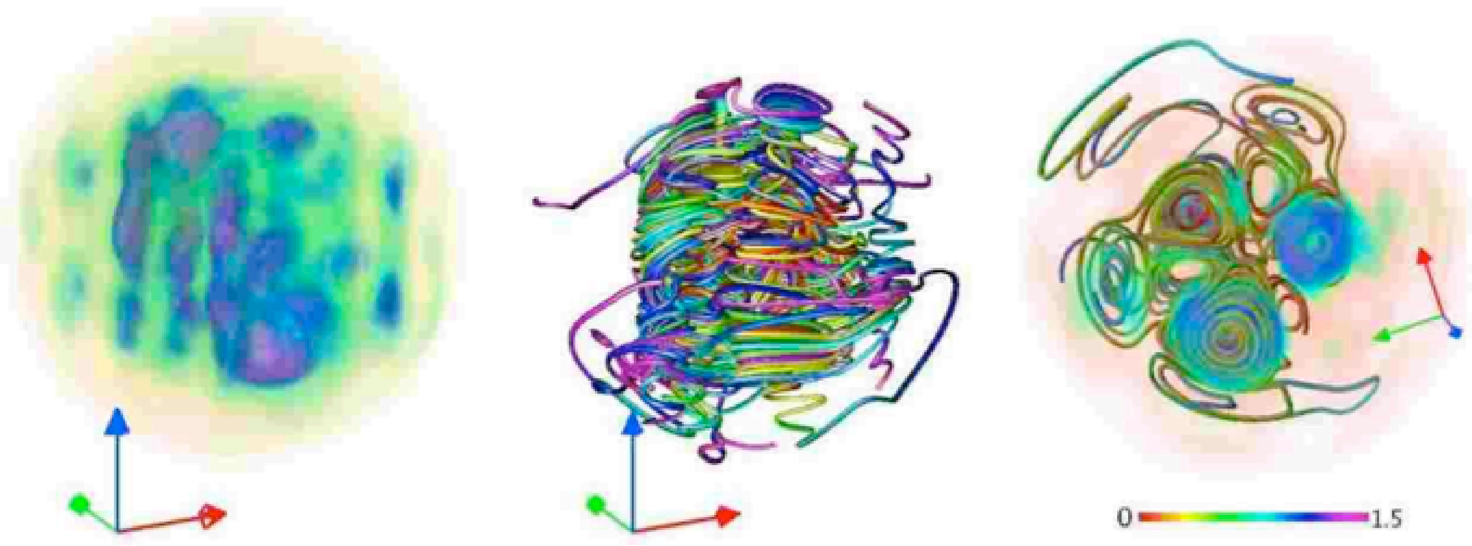


Figure 15. Kinetic energy density (left), velocity field lines (middle), and view from top of the kinetic energy density superposed with velocity field lines (right) in run D10, before magnetic energy is introduced. Note the columnar structures in the velocity field aligned along $\boldsymbol{\Omega}$ (in the z direction).

P. D. Mininni, D. C. Montgomery, and L. Turner, arXiv:physics/070208, (2007)

Outline

1. Various numerical methods used in the spherical geodynamo simulation
2. How to make a large scale simulation code
   - A sample Fortran90 code for kinematic dynamo in a box geometry.
   - Basic and useful Fortran90 features.
   - A sample data analysis (visualization) program.

Source codes

In source_codes.tar.gz,

- src/Kindanb/src: kinematic dynamo code
- src/Kindanb/analizer/: visualization code