*Progress of code development*

# *2nd-order Einstein-Boltzmann solver for CMB anisotropy*

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# Introduction : Inflation paradigm

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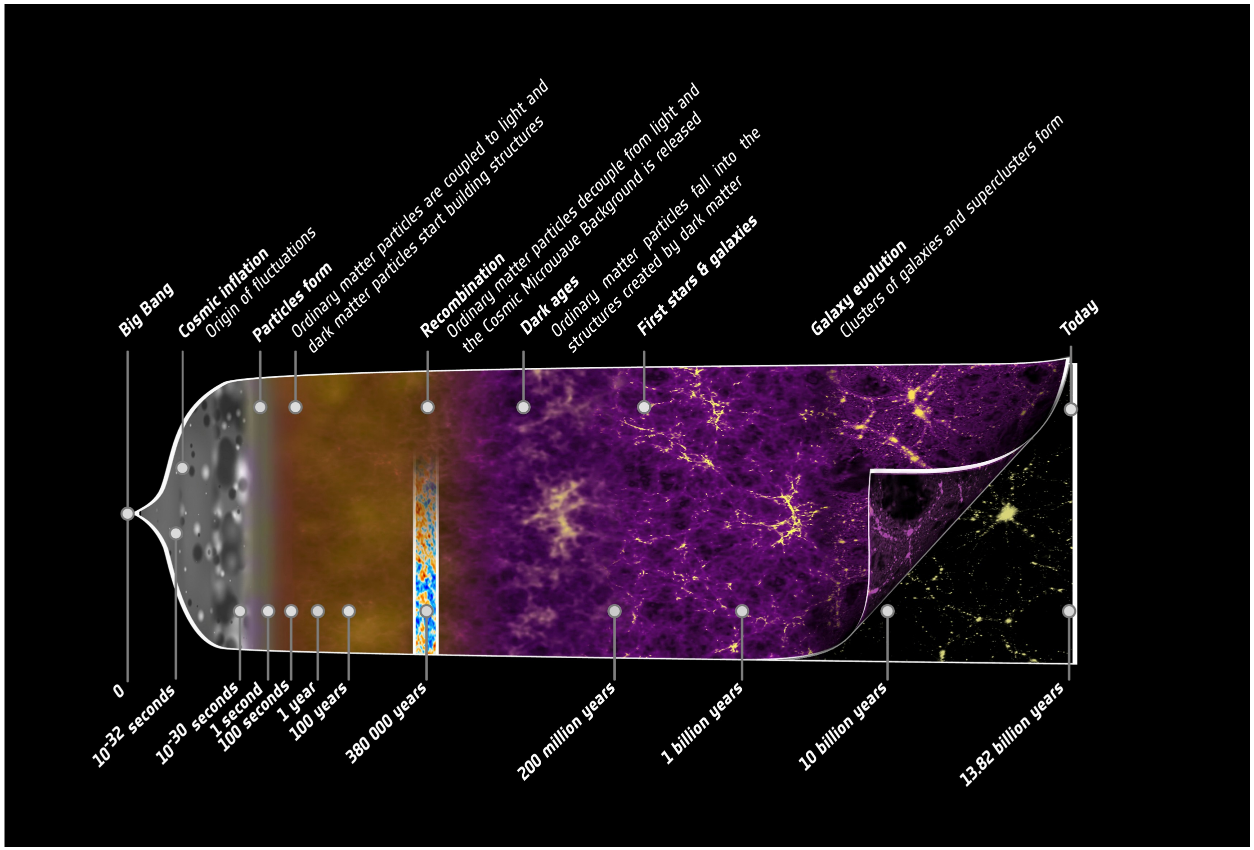
*A natural (?) way to give a seed of observed large-scale structure.*



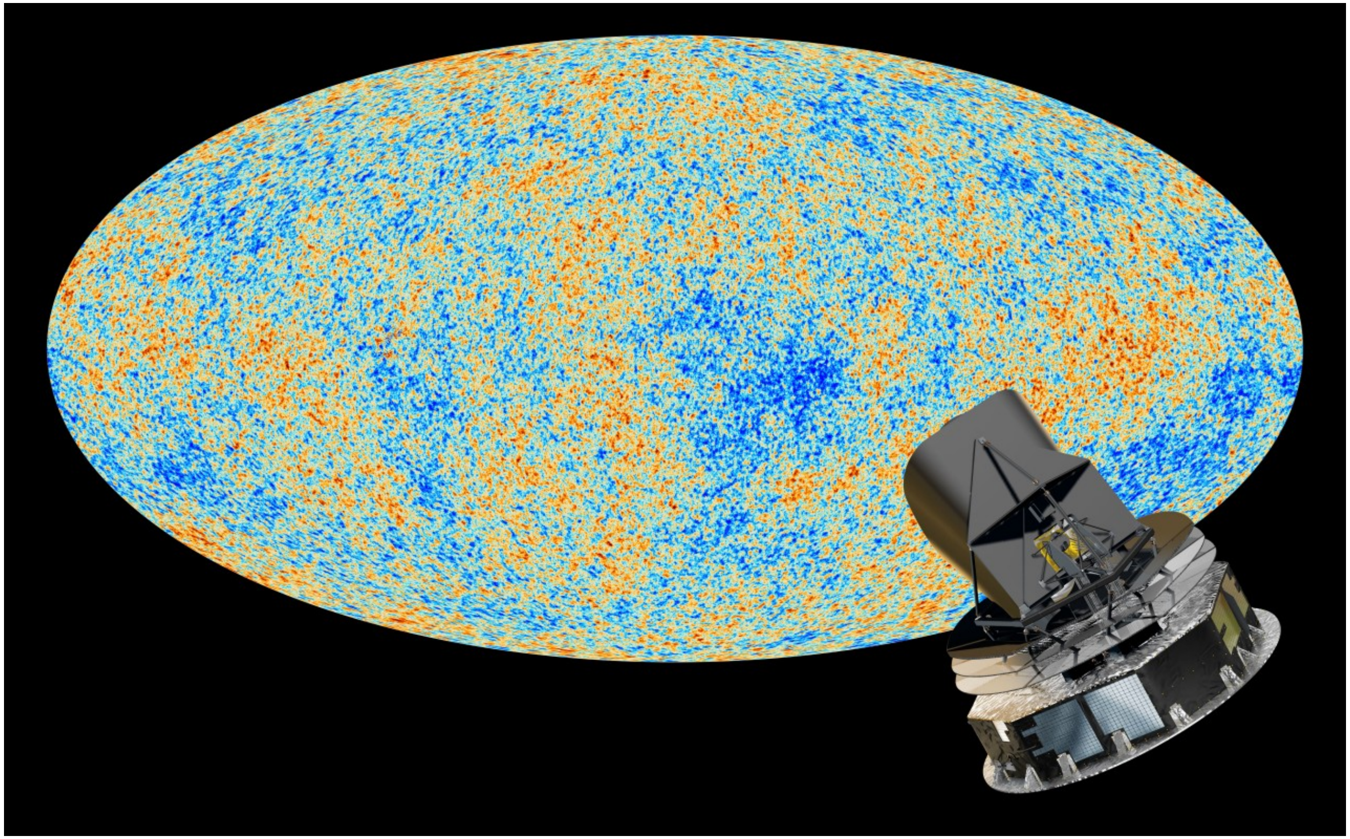

*http://www.sciops.esa.int*

Quantum fluctuations of inflationary space-time

= inhomogeneity of gravitational field in the Universe

Inhomogeneity of dark matter and baryons and photons

Cosmic Microwave Background (CMB)



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Temperature fluctuations of $\mathcal{O}(10)\mu\mathrm{K}$

embedded into the Planck distribution with $2.726\mathrm{K}$

$\longrightarrow$ A probe for the extraordinarily deep Universe.

# Introduction : Non-Gaussianity

## Focus on the statistical property of fluctuations...

Pure de Sitter inflation provides Gaussian fluctuations,

$$\langle\zeta(\mathbf{k})\zeta(\mathbf{k}')\rangle = (2\pi)^3 P_\zeta(k)\delta^{(3)}(\mathbf{k}+\mathbf{k}')$$

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle = 0$$

Non-gaussianity (NG) indicates the deviation from de Sitter space-time
(slow-roll inflation, multi-field inflation, etc.)

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle = (2\pi)^3 B_\zeta(k_1,k_2,k_3)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)$$

... parameterised by $f_{\rm NL}$

→ Observed NG has a possibility to screen enormous kinds of inflation models

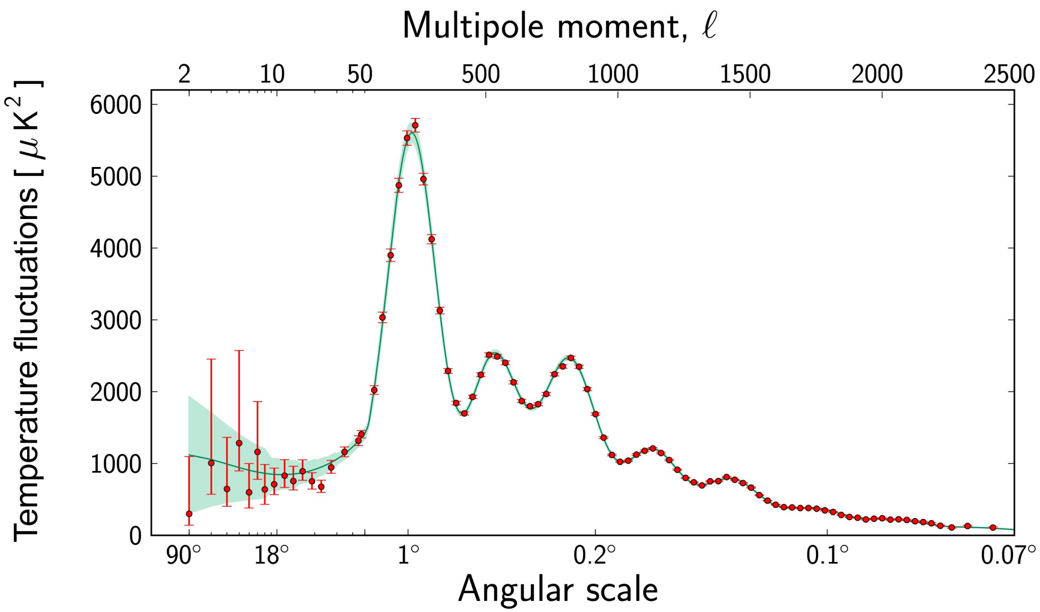
But, non-linear evolution of fluctuations can also generate NG...

→ To specify the total amount of intrinsic $f_{\rm NL}$ is an important task.

# Introduction : current status



Linear Boltzmann solvers (CAMB, CMBfast, CLASS, etc. ) have been available, giving familiar angular power spectrum of temperature fluctuations.

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3 Boltzmann solvers for 2nd-order perturbations are available, but the resultant $f_{\rm NL}$ is not converged...

Do it ourselves ! (At least, my code may be the first domestically-produced CMB code in Japan...?)

# Introduction : current status of my code

| *1st-order perturbations* | *2nd-order perturbations* |
|---|---|
| Implementing basic equations | |
| History of electron density | |
| line-of-sight integral +angular power spectrum | Implementing basic equations 1 |
| Qualitative check | |
| Quantitative check with CAMB ← | NOW |
| NOW → | 2nd-order line-of-sight formula |
| | R.Saito, Naruko, Hiramatsu, Sasaki, JCAP10(2014)051 [arXiv:1409.2464] |
| NOW → | Bispectrum estimator |
| | Implementing basic equations 2 |
| | Speed-up + Optimisation |

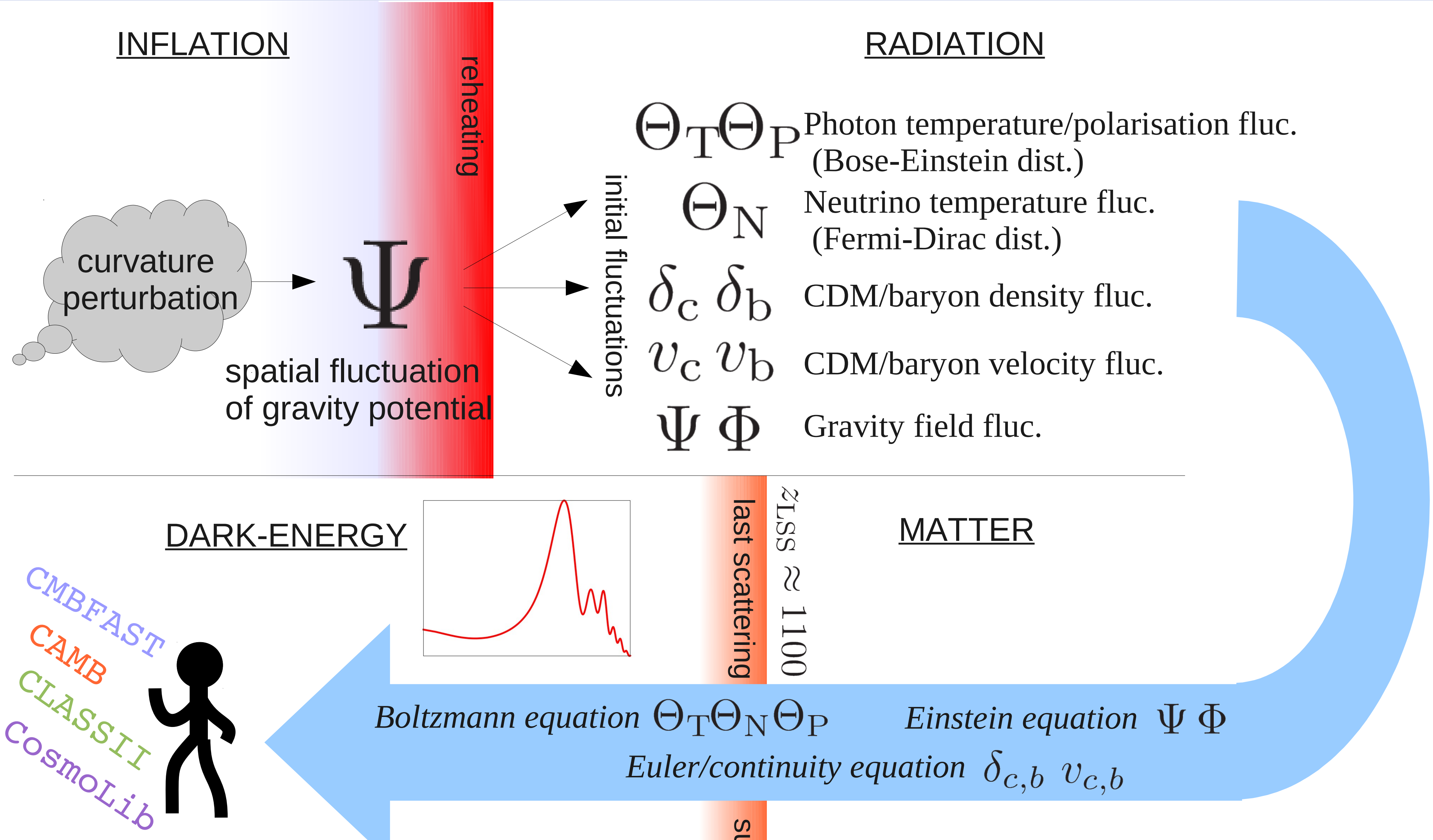
1st-order 95%

2nd-order 35%

# 1st-order

```
// ------------------------------------------------------------------//
//  1st-order
// ------------------------------------------------------------------//

void eval1st( Param * par, TParam * tpar, double * ans, double * field, SourceArray &src, int ik,
uble sigmastar, double dsigma )
{

  int ellT = par->ellT;
  int ellP = par->ellP;
  int ellN = par->ellN;
  int Neq = par->Neq1;

  double a = tpar->a;

  double HH = tpar->HH;
  double dtau = tpar->dtau;
  double R = tpar->R;

  double OmegaM = tpar->OmegaM;
  double OmegaR = tpar->OmegaR;
  double OmegaB = tpar->OmegaB;
  double OmegaC = tpar->OmegaC;
  double OmegaN = tpar->OmegaN;
  double OmegaP = tpar->OmegaP;
```

# Evolution of fluctuations




CMBFAST : Seljak, Zaldarriaga, APJ469 (1996) 437
CAMB : Lewis, Challinor, APJ538 (2000) 473
CLASS II : Blas, Lesgourgues, Tram, JCAP 1107 (2011) 034
CosmoLib : Huang, JCAP 1206 (2012) 012

They are described by the Bose-Einstein/Fermi-Dirac distribution functions:

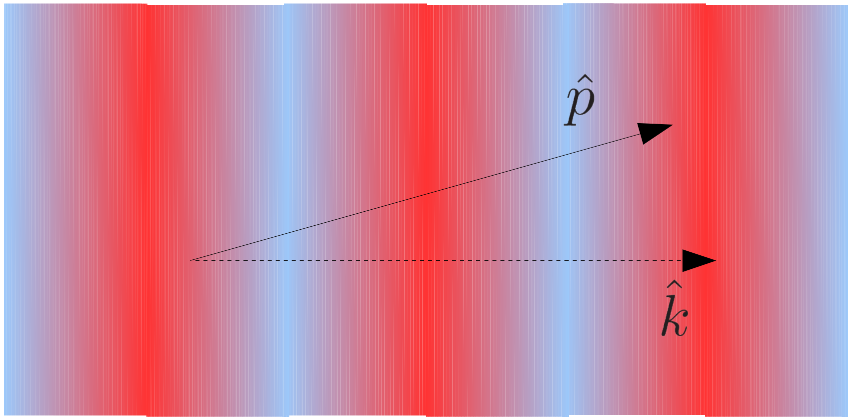
$$f_{\gamma,\nu}(\mathbf{x},\mathbf{p},\eta) = \left[\exp\left\{\frac{p}{T_{\gamma,\nu}(\eta)\left[1+\Theta_{\rm T,N}(\mathbf{x},\hat{p},\eta)\right]}\right\} \pm 1\right]^{-1}$$

Temperature fluctuation has three independent variables

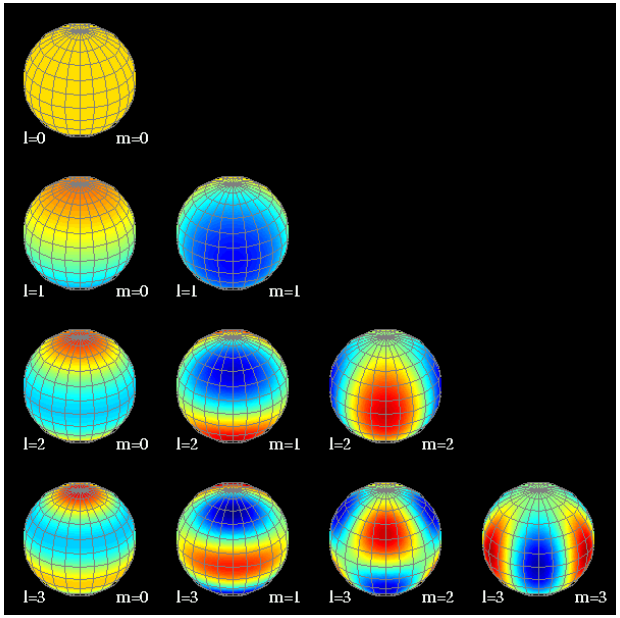
$$\Theta(\mathbf{x},\hat{p},\eta) \longrightarrow \Theta(\mathbf{k},\hat{p},\eta) \longrightarrow \Theta(k,\mu,\eta) \longrightarrow \Theta_\ell(k,\eta)$$

Fourier trans. Directional cosine Multipole exp.

$$\mu \equiv \frac{\mathbf{k}\cdot\hat{p}}{k}$$




http://spud.spa.umn.edu/~pryke




Multipole expansion

$$\Theta_\ell(k,\eta) = \frac{1}{2(-i)^\ell}\int_{-1}^{1}\mathcal{P}_\ell(\mu)\Theta(k,\mu,\eta)\,d\mu$$

# Photons/Neutrinos

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Distribution function satisfies the Boltzmann equation :

$$\frac{df}{d\eta} = C[f] \longrightarrow \begin{cases} \underbrace{\dot{\Theta}_{\mathrm{T}} + ik\mu\Theta_{\mathrm{T}} + \dot{\Phi} + ik\mu\Psi}_{\text{Liouville term}} = \underbrace{-\dot{\tau}\left[\Theta_0 - \Theta + \mu v_b - \frac{1}{2}\mathcal{P}_2(\mu)\Pi\right]}_{\text{Collision term}} \\ \dot{\Theta}_{\mathrm{P}} + ik\mu\Theta_{\mathrm{P}} = -\dot{\tau}\left[-\Theta_{\mathrm{P}} + \frac{1}{2}(1-\mathcal{P}_2(\mu))\Pi\right] \\ \dot{\Theta}_{\mathrm{N}} + ik\mu\Theta_{\mathrm{N}} + \dot{\Phi} + ik\mu\Psi = 0 \end{cases}$$

$$\Pi = \Theta_{T2} + \Theta_{P0} + \Theta_{P2}$$

**S. Dodelson, "*Modern Cosmology*"**

The efficiency of collision term is controlled by $\dot{\tau}(\eta)$ , time-derivative of optical depth

$$\dot{\tau}(\eta) = -n_e(\eta)\sigma_T a$$

$\sigma_T$ : Thomson cross-section

(cf. Rayleigh scattering adds ~1% contribution)

**Alipour, Sigurdson, Hirata, arXiv:1410.6484**

| | |
|---|---|
| Deep in radiation dominant epoch | Extremely large $\dot{\tau}(\eta) \longrightarrow$ tight-coupling |
| Last-scattering surface | $\dot{\tau}(\eta)$ suddenly decays |
| Late-time Universe | No collision, free-streaming |
| Reionisation | $\dot{\tau}(\eta)$ is revived, but not so significant |

## Polarisation

According to Dodelson's textbook, polarisation strength is given by $\Theta_\mathrm{P}$, relating with the Stokes parameter Q and U. Furthermore, they relates with E and B modes as

Lin, Wandelt, arXiv:astro-ph/0409734

$$\Theta_E(k,\mu,\eta) = -\frac{1}{2}\left[\flat^2\Theta_P(k,\mu,\eta) + \sharp^2\Theta_P^*(k,\mu,\eta)\right]$$

$$\Theta_B(k,\mu,\eta) = -\frac{1}{2i}\left[\flat^2\Theta_P(k,\mu,\eta) - \sharp^2\Theta_P^*(k,\mu,\eta)\right]$$

( $\flat, \sharp$ : spin raising/lowerling operators)

Here we treat $\Theta_\mathrm{P}$ as nothing but another intrinsic degree-of-freedom of photons in solving Boltzmann equations for $\Theta_\mathrm{T}$.

## Massive neutrino

Don't take care, but it'd be possible to implement.

Boltzmann equations for CDM/baryons + photon coupling. $\xrightarrow{\text{Fluid approx.}}$ Relativistic Fluid equations for CDM and baryons with source terms.

= Taking momenta of dist.func.

$$n_{c,b}(\mathbf{x},\eta) = \int \frac{d^3p}{(2\pi)^3} f_{c,b} = n^{(0)}_{c,b}(1+\underline{\delta_{c,b}}) \qquad n^{(0)}_{c,b}\underline{v^i_{c,b}}(\mathbf{x},\eta) = \int \frac{d^3p}{(2\pi)^3}\frac{p\hat{p}^i}{E} f_{c,b}$$

$\downarrow$ Fourier transform

CDM
baryon

$$\begin{cases} \dot{\delta}_\mathrm{c} = -ikv_\mathrm{c} - 3\dot{\Phi} \\ \dot{\delta}_\mathrm{b} = -ikv_\mathrm{b} - 3\dot{\Phi} \\ \dot{v}_\mathrm{c} = -\mathcal{H}v_\mathrm{c} - ik\Psi \\ \dot{v}_\mathrm{b} = -\mathcal{H}v_\mathrm{b} - ik\Psi + \dfrac{\dot{\tau}}{R}(v_\mathrm{b} + 3i\Theta_{\mathrm{T}1}) \end{cases} \qquad R(\eta) = \frac{3\rho_B}{4\rho_\gamma}$$

Conformal Newton gauge : $ds^2 = -a^2(1+2\Psi)d\eta^2 + a^2(1+2\Phi)dx^2$

Perturbed Einstein equations read...

$$G^{(1)}_{00} = \frac{8\pi}{M_{\rm pl}^2} T^{(1)}_{00} \longrightarrow \dot{\Phi} = -\frac{k^2}{3\mathcal{H}}\Phi + \mathcal{H}\Psi + \frac{\mathcal{H}_0^2}{2\mathcal{H}}\delta\Omega_0$$

$$\delta\Omega_0 = \Omega_{\rm c}\delta_{\rm c} + \Omega_{\rm b}\delta_{\rm b} + 4\Omega_\gamma\Theta_{\rm T0} + 4\Omega_\nu\Theta_{\rm N0}$$

$$\Omega_i \equiv \frac{\rho_i}{\rho_c}$$

$$G^{(1),ij}_{ij} = \frac{8\pi}{M_{\rm pl}^2} T^{(1),ij}_{ij} \longrightarrow \Psi = -\Phi - \frac{12\mathcal{H}_0^2}{k^2}\Omega_{\rm r}\Theta_{r,2}$$ (non-dynamical)

$$\Omega_{\rm r}\Theta_{r,2} = \Omega_\gamma\Theta_{\rm T2} + \Omega_\nu\Theta_{\rm N2}$$

NOTE : CAMB, CMBFAST use sychronous gauge : $ds^2 = -a^2 d\eta^2 + a^2(\delta_{ij} + h_{ij})dx^2$

# 1st-order perturbation equations

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Photon temperature

$$\begin{cases} \dot{\Theta}_0^{\mathrm{T}} = -k\Theta_1^{\mathrm{T}} - \dot{\Phi} \\ \dot{\Theta}_1^{\mathrm{T}} = \frac{1}{3}k(-2\Theta_2^{\mathrm{T}} + \Theta_0^{\mathrm{T}}) + \dot{\tau}\left(\Theta_1^{\mathrm{T}} + \frac{1}{3}v_b\right) + \frac{1}{3}k\Psi \\ \dot{\Theta}_2^{\mathrm{T}} = \frac{1}{5}k(-3\Theta_3^{\mathrm{T}} + 2\Theta_1^{\mathrm{T}}) + \dot{\tau}\left(\Theta_2^{\mathrm{T}} - \frac{1}{10}\Pi\right) \\ \dot{\Theta}_\ell^{\mathrm{T}} = \frac{1}{2\ell+1}k\left[-(\ell+1)\Theta_{\ell+1}^{\mathrm{T}} + \ell\Theta_{\ell-1}^{\mathrm{T}}\right] + \dot{\tau}\Theta_\ell^{\mathrm{T}} \end{cases}$$

Photon polarisation

$$\begin{cases} \dot{\Theta}_0^{\mathrm{P}} = -k\Theta_1^{\mathrm{P}} + \dot{\tau}\left(\Theta_0^{\mathrm{P}} - \frac{1}{2}\Pi\right) \\ \dot{\Theta}_1^{\mathrm{P}} = \frac{1}{3}k(-2\Theta_2^{\mathrm{P}} + \Theta_0^{\mathrm{P}}) + \dot{\tau}\Theta_1^{\mathrm{P}} \\ \dot{\Theta}_2^{\mathrm{P}} = \frac{1}{5}k(-3\Theta_3^{\mathrm{P}} + 2\Theta_1^{\mathrm{P}}) + \dot{\tau}\left(\Theta_2^{\mathrm{P}} - \frac{1}{10}\Pi\right) \\ \dot{\Theta}_\ell^{\mathrm{P}} = \frac{1}{2\ell+1}k\left[-(\ell+1)\Theta_{\ell+1}^{\mathrm{P}} + \ell\Theta_{\ell-1}^{\mathrm{P}}\right] + \dot{\tau}\Theta_\ell^{\mathrm{P}} \end{cases}$$

Massless neutrino temperature

$$\begin{cases} \dot{\Theta}_{\mathrm{N}0} = -k\Theta_{\mathrm{N}1} - \dot{\Phi} \\ \dot{\Theta}_1^{\mathrm{N}} = \frac{1}{3}k(-2\Theta_2^{\mathrm{N}} + \Theta_0^{\mathrm{N}}) + \frac{1}{3}k\Psi \\ \dot{\Theta}_2^{\mathrm{N}} = \frac{1}{5}k(-3\Theta_3^{\mathrm{N}} + 2\Theta_1^{\mathrm{N}}) \\ \dot{\Theta}_\ell^{\mathrm{N}} = \frac{1}{2\ell+1}k\left[-(\ell+1)\Theta_{\ell+1}^{\mathrm{N}} + \ell\Theta_{\ell-1}^{\mathrm{N}}\right] \end{cases}$$

CDM, baryon

$$\begin{cases} \dot{\delta}_{\mathrm{c}} = -ikv_{\mathrm{c}} - 3\dot{\Phi} \\ \dot{\delta}_{\mathrm{b}} = -ikv_{\mathrm{b}} - 3\dot{\Phi} \\ \dot{v}_{\mathrm{c}} = -\mathcal{H}v_{\mathrm{c}} - ik\Psi \\ \dot{v}_{\mathrm{b}} = -\mathcal{H}v_{\mathrm{b}} - ik\Psi + \frac{\dot{\tau}}{R}(v_{\mathrm{b}} + 3i\Theta_{\mathrm{T}1}) \end{cases}$$

Gravity
(conformal Newtonian gauge)

$$\dot{\Phi} = -\frac{k^2}{3\mathcal{H}}\Phi + \mathcal{H}\Psi + \frac{\mathcal{H}_0^2}{2\mathcal{H}}\delta\Omega_0$$

# Line-of-sight integral

Takashi Hiramatsu

Large Boltzmann hierarchy, say $\ell \lesssim 2000$, is required, but it is too hard to calculate...

Up to last-scattering-surface,

$$\Theta_\ell^{\mathrm{T,P}} \sim \frac{k\eta}{2\tau}\Theta_{\ell-1}^{\mathrm{T,P}}$$

$$\Theta_\ell^{\mathrm{N}} \sim \frac{k\eta}{2}\Theta_{\ell-1}^{\mathrm{N}}$$

superhorizon scale

$$\frac{k}{\mathcal{H}} \approx k\eta \ll 1$$

tight-coupling

$$\mathcal{H} \ll \dot{\tau}$$

So, to guarantee the accuracy of $\Theta_{\ell\leq 2}, \Phi, \delta_b, \delta_c, v_b, v_c$

$$\ell_{\max} \sim 15$$

After LSS, we use the integral representation, *line-of-sight formula*.

**Seljak, Zaldarriaga, APJ 469 (1996) 437**

# Line-of-sight integral

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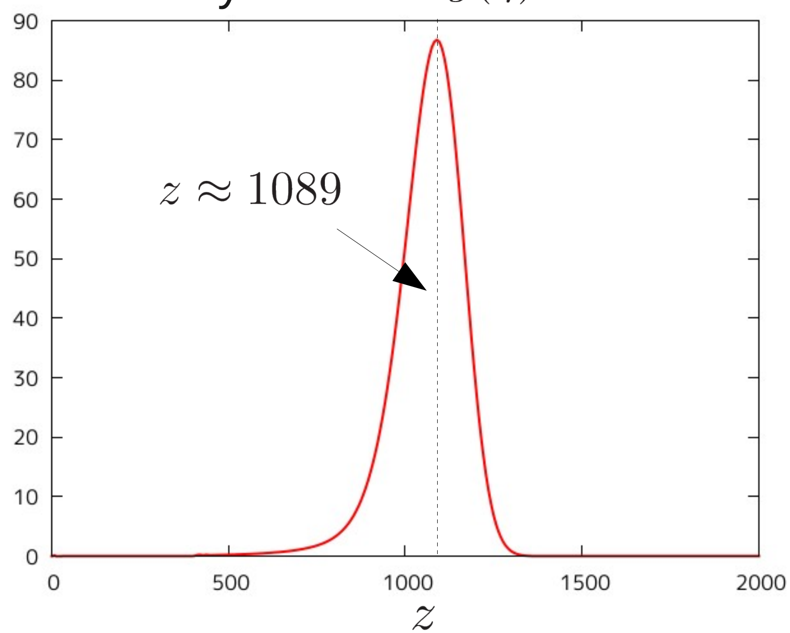
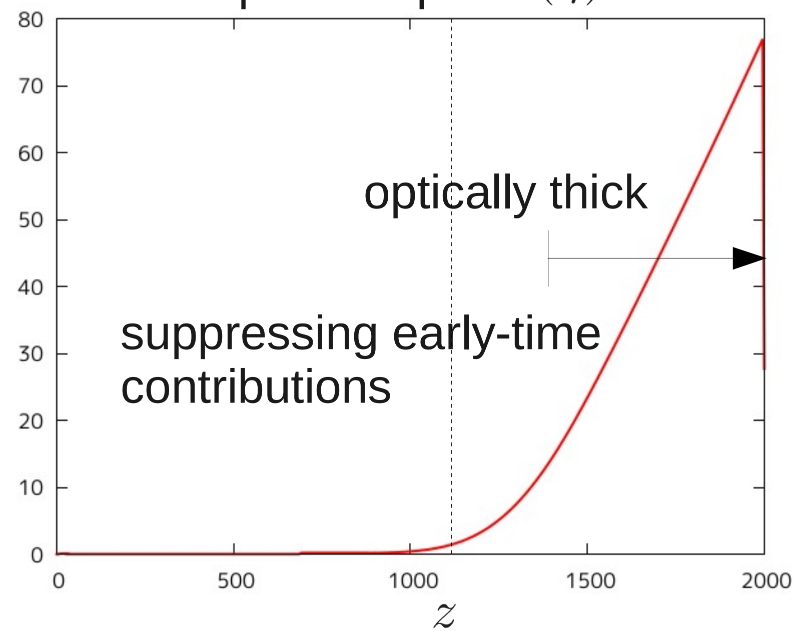
$$\Theta_\ell(k,\eta_0) = \int_0^{\eta_0} d\eta\, S(k,\eta) j_\ell[k(\eta_0-\eta)]$$

$$S(k,\eta) = g(\eta)\left[\Psi + \left(\Theta_0 + \frac{1}{4}\Pi\right)\right] + \frac{i}{k}\frac{d}{d\eta}\left[g(\eta)v_b(k,\eta)\right] + \frac{3}{4k^2}\frac{d^2}{d\eta^2}\left[g(\eta)\Pi\right]$$
$$+e^{-\tau}\left[\dot{\Psi}(k,\eta) - \dot{\Phi}(k,\eta)\right]$$

visibility function $g(\eta) = -\dot{\tau}e^{-\tau}$




optical depth $\tau(\eta)$



# Line-of-sight integral

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Sourced by fluctuations at LSS ( including Sachs-Wolfe effect )

$$\Theta_\ell(k,\eta_0) = \int_0^{\eta_0} d\eta\, g(\eta)\left[\Theta_0(k,\eta) + \Psi(k,\eta) + \frac{1}{4}\Pi(k,\eta)\right] j_\ell[k(\eta_0-\eta)]$$

Monopole

$$+ \int_0^{\eta_0} d\eta\, g(\eta) v_b(k,\eta) \frac{1}{k}\frac{d}{d\eta} j_\ell[k(\eta_0-\eta)]$$

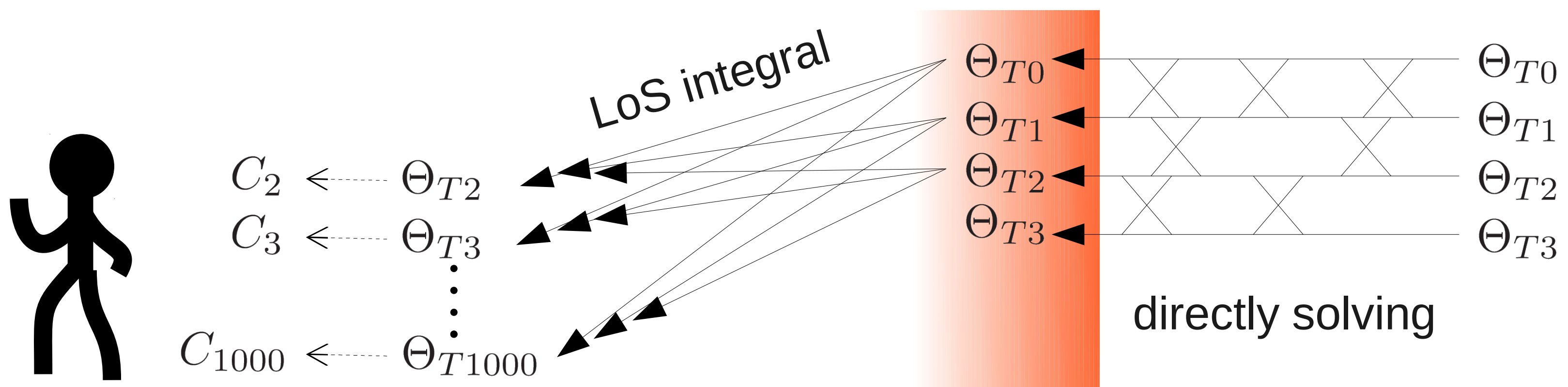
Dipole

$$+ \int_0^{\eta_0} d\eta\, g(\eta) \frac{3}{4}\Pi(k,\eta)\frac{1}{k^2}\frac{d^2}{d\eta^2} j_\ell[k(\eta_0-\eta)]$$

Quadrupole

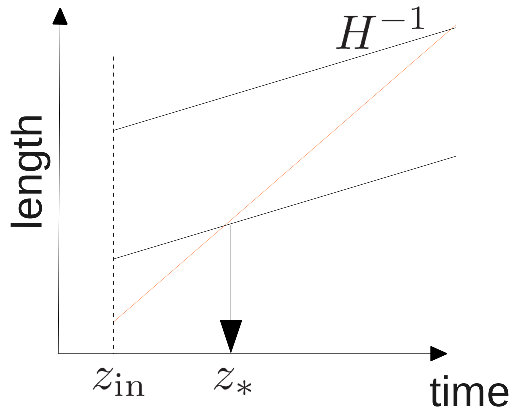
$$+ \int_0^{\eta_0} d\eta\, e^{-\tau}\left[\dot{\Psi}(k,\eta) - \dot{\Phi}(k,\eta)\right] j_\ell[k(\eta_0-\eta)]$$

Integrated Sachs-Wolfe effect



Deep in radiation dominant epoch where all modes are larger than horizon scale. We set $z_{\rm in} = 1.44 \times 10^6$

(cf. $k_{\rm max} = 1200 H_0$ which crosses the horizon at $z_* \approx 1.3 \times 10^5$ )




From the Einstein equation,

$$\dot{\Phi} = -\frac{k^2}{3\mathcal{H}}\Phi + \mathcal{H}\Psi + \frac{\mathcal{H}_0^2}{2\mathcal{H}}(\Omega_{\rm c}\delta_{\rm c} + \Omega_{\rm b}\delta_{\rm b} + 4\Omega_\gamma\Theta_{\rm T0} + 4\Omega_\nu\Theta_{\rm N0}) \qquad \left(\Omega_i = \frac{\rho_i}{\rho_c}\right)$$

unchanged potential, superhorizon, ⇩ radiation dominant

$$0 = \Psi + \frac{2\mathcal{H}_0^2}{\mathcal{H}^2}(\Omega_\gamma\Theta_{\rm T0} + \Omega_\nu\Theta_{\rm N0})$$

similarly fluctuated ⇩ radiation dominant

$$0 = \Psi + \frac{2\mathcal{H}_0^2}{\mathcal{H}^2}\Omega_R\Theta_{\rm T0} \qquad \left(\mathcal{H}^2 \approx \frac{8\pi}{3M_{\rm pl}^2}\rho_R = \mathcal{H}_0^2\Omega_R\right)$$

⇩

$$\Theta_{T0} = \Theta_{N0} = -\frac{1}{2}\Psi \qquad \left(\Theta_{T1} = \Theta_{N1} = \frac{k}{6\mathcal{H}}\Psi\right)$$

Primordial curvature perturbation (preserved on superhorizon scales)

$$\zeta = -\frac{ik_i \delta T^0{}_i H}{k^2(\rho + P)} - \Psi$$

after inflation $\quad \zeta = -\frac{3\mathcal{H}\Theta_{T1}}{k} - \Psi \longrightarrow \Psi = -\frac{2}{3}\zeta$

during inflation $\quad \zeta = -aH\frac{\delta\phi}{\dot{\phi}}$ = almost flat spectrum

$$\langle \zeta(\mathbf{k})\zeta^*(\mathbf{k})\rangle = (2\pi)^3 \delta_D^3(\mathbf{k}+\mathbf{k}')P_\zeta(k)$$

$$P_\zeta(k) = \frac{2\pi^2}{k^3}\Delta^2(k_{\rm pivot})\left(\frac{k}{k_{\rm pivot}}\right)^{n_s-1} \quad \left\{ \begin{aligned} \Delta^2(k_{\rm pivot}) &= 2.46\times 10^{-9} \\ k_{\rm pivot} &= 0.002\ {\rm Mpc}^{-1} \\ n_s &= 0.96 \end{aligned} \right.$$

# Initial conditions : Eventually, ...

We impose

$$\begin{cases} \delta_c = \delta_b = 3\Theta_{T0} = 3\Theta_{N0} = \zeta \\ v_b = v_c = -3\Theta_{T1} = -3\Theta_{N1} = \dfrac{k}{3\mathcal{H}}\zeta \\ \Phi = \dfrac{2}{3}\left(1 + \dfrac{2}{5}f_\nu\right)\zeta \\ \Psi = -\dfrac{2}{3}\zeta \end{cases}$$

neutrino fraction : $f_\nu = \rho_\nu / \rho_R$

$$\Theta_\ell \sim \frac{k\eta}{2\tau}\Theta_{\ell-1}$$

@ $z = z_{\rm in} = 1.44 \times 10^6$

Separating primordial (quantum) curvature perturbation, we focus on the *transfer functions*,

$$\Phi(k,\eta) = \mathcal{T}_\Phi(k,\eta)\zeta(k,\eta_{\rm in})$$

quantum

classical

Flat FLRW model

$$H^2 = H_0^2 \left[\Omega_M (1+z)^3 + \Omega_R (1+z)^4 + \Omega_\Lambda\right]$$

$$\begin{cases} \Omega_\Lambda = 1 - \Omega_M - \Omega_R \\ \Omega_R = \dfrac{4\pi^3 g_* T_0^4}{45 H_0^2 M_{\rm pl}^2} \\ g_* = 2 + \dfrac{7}{4}\left(\dfrac{4}{11}\right)^{4/3} N_{\rm eff} \end{cases}$$

<u>fiducial parameters</u>

$$T_0 = 2.725\ [\mathrm{K}]$$
$$N_{\rm eff} = 3.04$$
$$h = 0.7$$
$$h^2\Omega_{\rm CDM} = 0.114$$
$$h^2\Omega_{\rm B} = 0.0226$$

It can be easily extended to include non-flat case.

NOTE : we use $\sigma = \log a(\eta)$ as the time variable instead of $\eta$ in solving EB equations. Then we don't have to solve the Friedmann equation.

Number density of free electrons

$$\longrightarrow \dot{\tau} = -n_{\rm e}\sigma_{\rm T} a$$ controls ...

- Strength of Photon-Baryon coupling

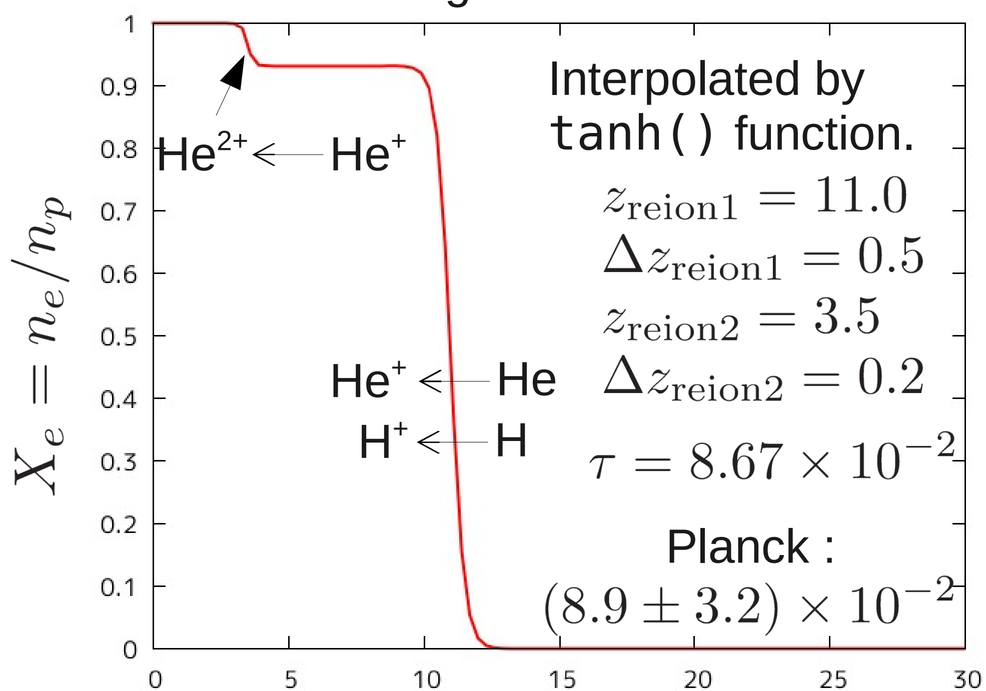
  cf. $\dot{v}_{\rm b} = -\mathcal{H}v_{\rm b} - ik\Psi + \frac{\dot{\tau}}{R}(v_{\rm b} + 3i\Theta_{\rm T1})$

- Opacity of the Universe (affects ISW)

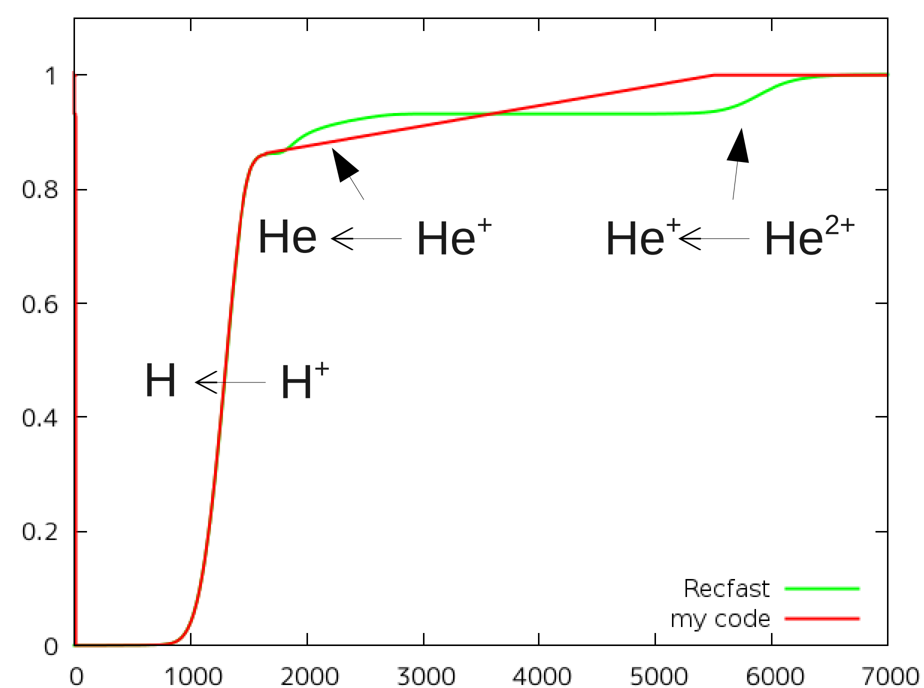
Weinberg : $$\frac{dX_e}{dT} = \frac{\alpha n}{HT}\left(1 + \frac{\beta}{\Gamma_{2s} + 8\pi H/\lambda_\alpha^3 n(1-X)}\right)^{-1}\left[X_e^2 - (1-X_e)/S\right]$$

**Peebles, APJ 153 (1968) 1; Weinberg, "*Cosmology*"**

2-stage reionisation




Peebles-Weinberg scheme



# Miscellaneous : Spherical Bessel functions

Use three different methods to maintain an accuracy of $\mathcal{O}(10^{-6})$ for $\ell, x < 10^5$

$$\begin{cases} \text{Descending reccurence :} & x > \ell \\ \text{Debye's expansion :} & x < \ell, \ell > 20 \\ \text{Taylor expansion :} & x < 0.5, \ell \leq 20 \end{cases}$$

[Descending]

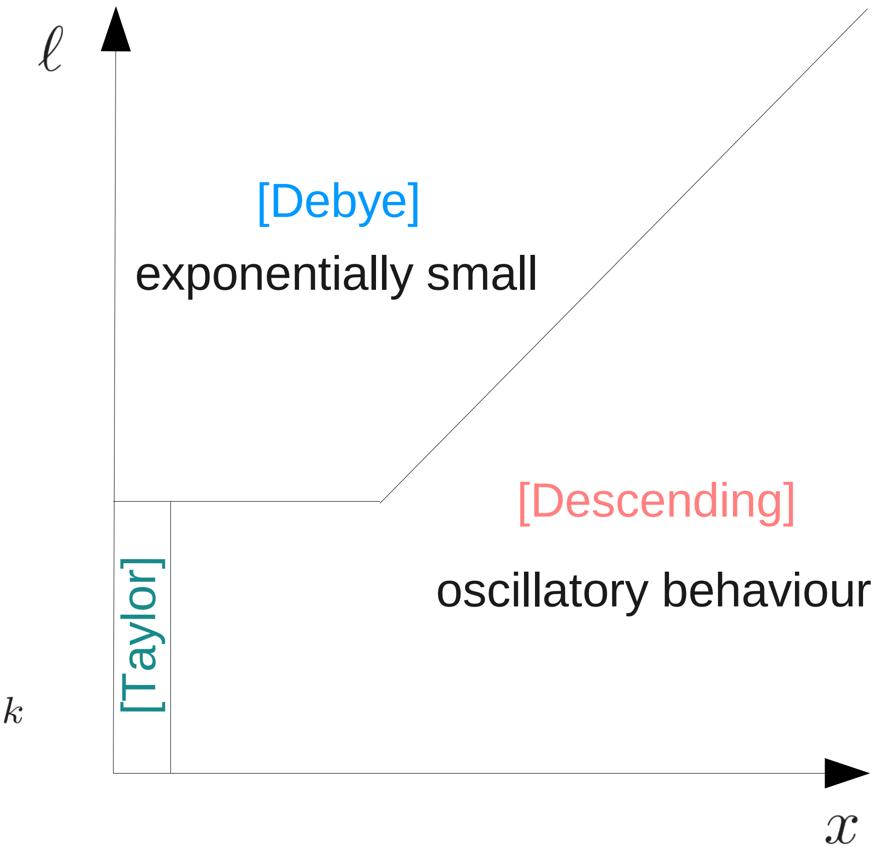
$$j_\ell(z) = \frac{2\ell+3}{z} j_{\ell+1}(z) - j_{\ell+2}(z)$$

[Debye]

$$J_\nu(z) \sim \frac{e^{\nu(\tanh\alpha - \alpha)}}{\sqrt{2\nu\pi \tanh\alpha}} \sum_{k=0}^{\infty} \frac{U_k(\coth\alpha)}{\nu^k}$$

[Taylor]

$$j_\ell(z) = \frac{\sqrt{\pi}}{2} \left(\frac{z}{2}\right)^\ell \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\ell+k+3/2)} \left(\frac{z}{2}\right)^{2k}$$



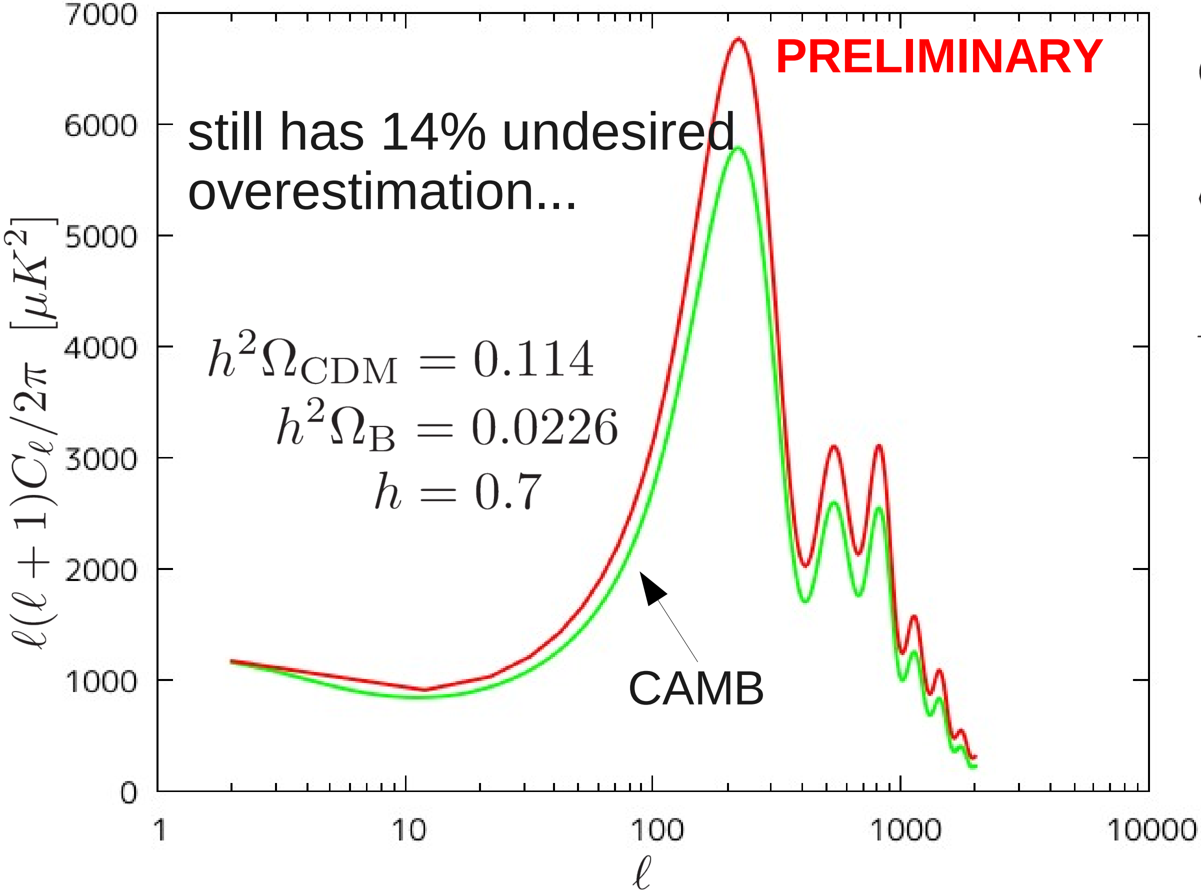

Once storing $j_\ell(x_n)$ for $0 \leq x \leq k_{\max}\eta_0$, $j_\ell\left[k(\eta_0 - \eta)\right]$ with arbitrary argument is given by the partitioned polynomial interpolation.

# Preliminary Results

# Angular power spectrum (1st-order)

Angular power spectrum

$$C_\ell = \frac{2}{\pi}\int_0^\infty dk\, k^2 \mathcal{T}_{\Theta_\ell}(k,\eta)^2 P_\zeta(k,\eta_{\rm in})$$




$$\Theta_\ell(k,\eta) = \mathcal{T}_{\Theta_\ell}(k,\eta)\zeta(k,\eta_{\rm in})$$

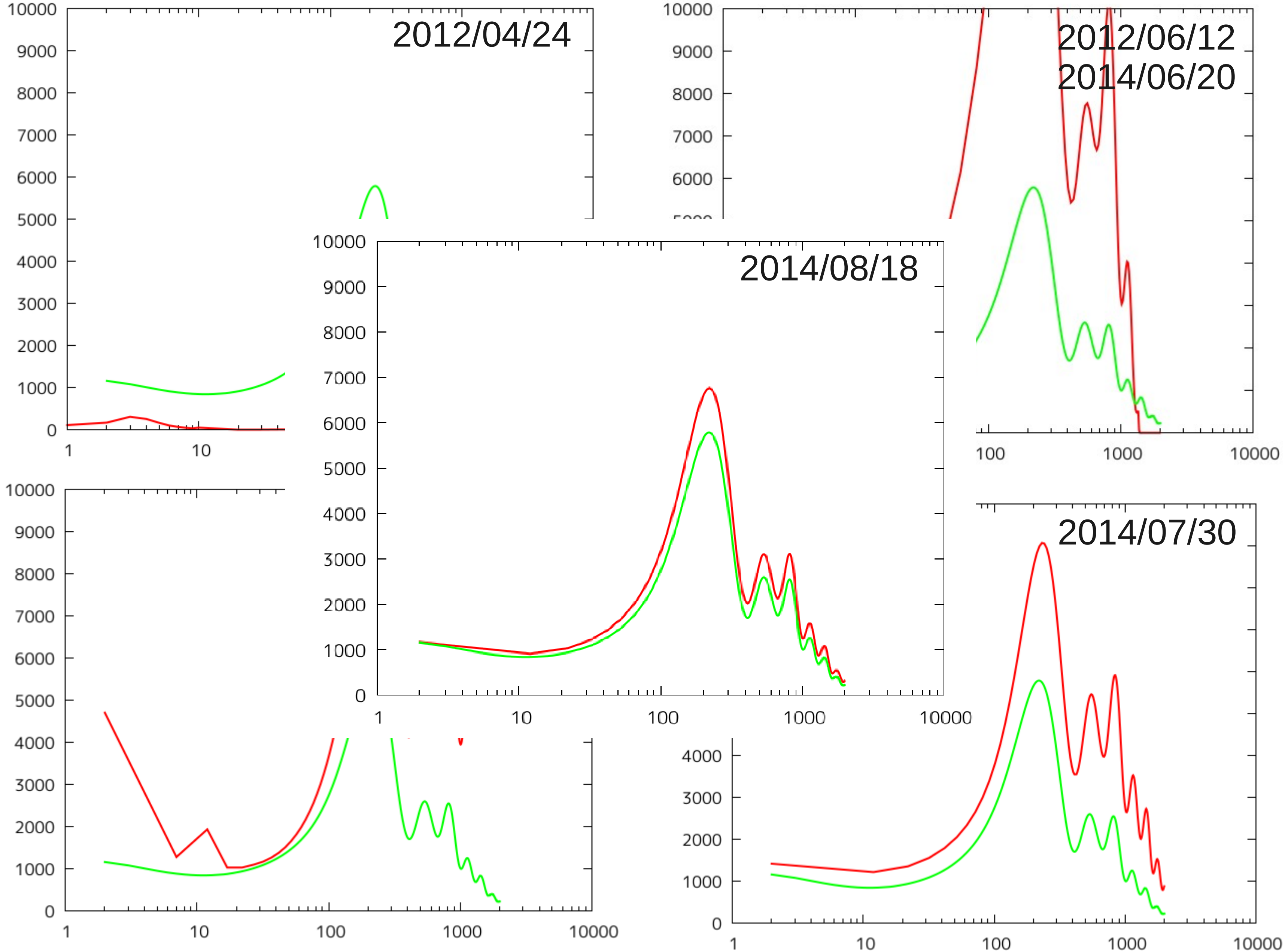
$$\langle\zeta(\mathbf{k})\zeta^*(\mathbf{k})\rangle = (2\pi)^3\delta_D^3(\mathbf{k}+\mathbf{k}')P_\zeta(k)$$

$$P_\zeta(k) = \frac{2\pi^2}{k^3}\Delta^2(k_{\rm pivot})\left(\frac{k}{k_{\rm pivot}}\right)^{n_s-1}$$

$$\Delta^2(k_{\rm pivot}) = 2.46\times 10^{-9}$$
$$k_{\rm pivot} = 0.002\ {\rm Mpc}^{-1}$$
$$n_s = 0.96$$

# Angular power spectrum (1st-order)

Takashi Hiramatsu



# 2nd-order

```
    outputTime( 38, par, "bispectrum preprocess" );

    ///////////////////////////////////////////////
    // calculating bispectra
    ///////////////////////////////////////////////

    cerr << "calculating bispectrum...";

    double * preBl_SG = new double [numBells*numBells*4];
    double * preBl_GG = new double [numBells*numBells*3];

    double * bSG_S = new double [numBells*tS.N*4];
    double * bSG_T = new double [numBells*tS.N*4];
    double * bGG_S = new double [nu
    double * bGG_T = new double [nu

    bispectrumKernelSGS( bSG_S, tS, tT, K, bells, bellRank, conformalTime, jB, Pzeta, ThetaT, blockSiz
, vis );
    bispectrumKernelSGT( bSG_T, tS, tT, K, bells, bellRank, conformalTime, jB, Pzeta, ThetaT, blockSiz
W, iHH );
    bispectrumSG( preBl_SG, bSG_S, bSG_T, numBells, tS, par->nthreads );

    bispectrumKernelGGS( bGG_S, tT, K, bells, bellRank, conformalTime, jB, Pzeta, ThetaT, blockSizeISW
    bispectrumKernelGGT( bGG_T, tT, K, bells, bellRank, conformalTime, jB, Pzeta, ThetaT, blockSizeISW
optdepth );
    bispectrumGG( preBl_GG, bGG_S, bGG_T, numBells, tT, par->nthreads );

    delete [] bSG_S;
```

# Go to 2nd-order



2nd-order contributions appear in...

- Einstein-Boltzmann equations for 2nd-order quantities sourced by [1st-order]$^2$

  CDM+Baryon+Gravity have been implemented,
  but Baryon-Photon/Gravity-Photon couplings are not considered yet.

- Line-of-sight integral sourced by [1st-order]$^2$

  Formulations have been completed by R.Saito.

  **R.Saito, Naruko, Hiramatsu, Sasaki, JCAP10(2014)051 [arXiv:1409.2464]**

  **(cf. Fidler, Koyama, Pettinari, arXiv:1409.2461)**

Existing 2nd-order Boltzmann solver

`CMBquick` — **CMBquick : Creminelli, Pitrou, Vernizzi, arXiv:1109.1822**

`SONG` — **SONG : Pettinari, arXiv:1405.2280 (thesis)**

`CosmoLib2nd` — **CosmoLib2nd : Huang, Vernizzi, arXiv:1212.3573**

## Bispectrum

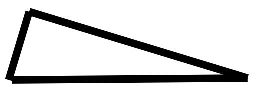
$$\langle \zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle = (2\pi)^3 B_\zeta(k_1,k_2,k_3)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)$$

Bartolo et al., Phys.Rep.402(2004)103 [arXiv:astro-ph/0406398]

## Significance and shape of non-Gaussianity

Significance : parametrised by $f_{\rm NL}$

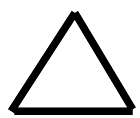
local-type



$$\zeta(\mathbf{x}) = \zeta_L(\mathbf{x}) + f_{\rm NL}(\zeta_L^2 - \langle \zeta_L^2\rangle)$$

$$\longrightarrow B_\zeta^{\rm local} = 2 f_{\rm NL}^{\rm local}\left[P_\zeta(k_1)P_\zeta(k_2) + 2\text{ perms}\right] \qquad \left(P_\zeta(k) \propto \frac{1}{k^{4-n_s}}\right)$$

equilateral-type



$$B_\zeta^{\rm equil} = 6 f_{\rm NL}^{\rm equil}\left[-P_\zeta(k_1)P_\zeta(k_2) + 2\text{ perms} + \left\{P_\zeta(k_1)^{1/3}P_\zeta(k_2)^{2/3}P_\zeta(k_3) + 5\text{ perms}\right\}\right]$$

orthogonal-type

$$B_\zeta^{\rm ortho} = 6 f_{\rm NL}^{\rm ortho}\left[-3P_\zeta(k_1)P_\zeta(k_2) + 2\text{ perms} + 3\left\{P_\zeta(k_1)^{1/3}P_\zeta(k_2)^{2/3}P_\zeta(k_3) + 5\text{ perms}\right\}\right]$$

folded-type

$$B_\zeta^{\rm folded} = 6 f_{\rm NL}^{\rm folded}\left[P_\zeta(k_1)P_\zeta(k_2) + 2\text{ perms} - \left\{P_\zeta(k_1)^{1/3}P_\zeta(k_2)^{2/3}P_\zeta(k_3) + 5\text{ perms}\right\}\right]$$

+ a variety of non-separable types

Planck collaboration, A&A 571(2014) A24 [arXiv:1303.5084]

# Non-Gaussianity

## Results not well converged

CMBquick $f_{\rm NL}^{\rm local}, f_{\rm NL}^{\rm equil} \sim 5(\to 3.7)$

Pitrou, Uzan, Bernardeau, JCAP07(2010)003 [arXiv:1103.0481]

SONG $f_{\rm NL}^{\rm local}, f_{\rm NL}^{\rm equil}, f_{\rm NL}^{\rm ortho} \approx 0.51, 4.2, -1.4$

Pettinari, Fidler, Crittenden, Koyama, Wands, JCAP04(2013)003 [arXiv:1302.0832]

CosmoLib2nd $f_{\rm NL}^{\rm local} \approx 0.73$

Huang, Vernizzi, PRD89(2014)021302 [arXiv:1311.6105]

## Observational constraints by Planck

$$f_{\rm NL}^{\rm local} = 2.7 \pm 5.8$$
$$f_{\rm NL}^{\rm equil} = -42 \pm 75$$
$$f_{\rm NL}^{\rm ortho} = -25 \pm 39$$

(68% confidence level)

Planck collaboration, A&A 571(2014) A24 [arXiv:1303.5084]

# Expansion up to 2nd-order

Takashi Hiramatsu

Poisson gauge + neglecting 1st-order $B_i$ and $h_{ij}$

$$ds^2 = -a^2 e^{2\Psi} d\eta^2 - 2a^2 B_i dx^i d\eta + a^2 \left(e^{2\Phi}\delta_{ij} + h_{ij}\right)$$

$$\partial_i B^i = 0 \qquad B_i = B^{(x)} e_i^{(x)} + B^{(y)} e_i^{(y)}$$
$$\partial_i h^{ij} = 0 \qquad h_{ij} = h^{(+)} e_{ij}^{(+)} + h^{(\times)} e_{ij}^{(\times)}$$
$$\delta_{ij} h^{ij} = 0$$

Expanding up to 2and-order

$$\Psi = \Psi^{(1)} + \Psi^{(2)} + \cdots \qquad \delta_i = \delta_i^{(1)} + \delta_i^{(2)} + \cdots$$
$$\Phi = \Phi^{(1)} + \Phi^{(2)} + \cdots \qquad v_i = v_i^{(1)} + v_i^{(2)} + \cdots$$
$$B^{(p)} = 0 + B^{(p)(2)} + \cdots \qquad (i = b, c)$$
$$h^{(p)} = 0 + h^{(p)(2)} + \cdots$$

# 2nd-order perturbation equtaions

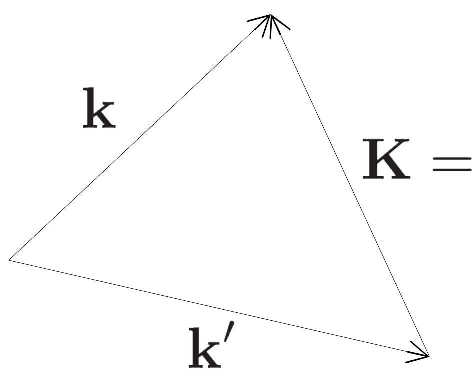
Takashi Hiramatsu

CDM + gravity (contribution from radiation is work in progress....)

$$\Psi^{(2)} + \Phi^{(2)} = \mathcal{Q}_\Psi\left[\Phi^{(1)}\Phi^{(1)}, \Psi^{(1)}\Psi^{(1)}, \Phi^{(1)}\Psi^{(1)}, V^{(1)}V^{(1)}\right]$$

'Q'uadratic terms of 1st-order quantities

$$\begin{aligned}
&= c_{1,1}(k,k',K)\Phi^{(1)}(k')\Phi^{(1)}(K) \\
&\quad + c_{1,2}(k,k',K)\Psi^{(1)}(k')\Psi^{(1)}(K) \\
&\quad + c_{1,3}(k,k',K)\Phi^{(1)}(k')\Psi^{(1)}(K) \\
&\quad + \kappa^2 a^2 \rho^{(0)}(1+w)c_{1,4}(k,k',K)V^{(1)}(k')V^{(1)}(K) \\
&\quad - \frac{3}{2k^4}\kappa^2 a^2 \mathcal{F}\left[\widehat{T}^{R(2)i}{}_{j}{}^{,j}{}_{,i}\right]
\end{aligned}$$




$\mathbf{K} = \mathbf{k} - \mathbf{k}'$

$c_{i.j}(k,k',K)$ is a fractional expression like $c_{1,1} = \dfrac{3(K^2 - k'^2)^2 - k^2(3K^2 - k'^2) + 2k^4}{4k^4}$

# 2nd-order perturbation equtaions

CDM + gravity (contribution from radiation is work in progress....)

$$\Phi^{(2)\prime} - \mathcal{H}\Psi^{(2)} + \frac{k^2}{3\mathcal{H}}\Phi^{(2)} - \frac{\kappa^2 a^2}{6\mathcal{H}}\rho^{(0)}\delta^{(2)} = \mathcal{Q}_\Phi\left[\Psi^{(1)}\Psi^{(1)}, \Phi^{(1)\prime}\Psi^{(1)}, \Phi^{(1)\prime}\Phi^{(1)\prime}, \Phi^{(1)\prime}\Phi^{(1)}, V^{(1)}V^{(1)}\right] - \frac{\kappa^2 a^2}{6\mathcal{H}} T^{R(2)0}{}_0$$

$$B^{(A)\prime} + 2\mathcal{H}B^{(A)} = \mathcal{Q}_B\left[\Phi^{(1)}\Phi^{(1)}, \Psi^{(1)}\Psi^{(1)}, \Phi^{(1)}\Psi^{(1)}, \Psi^{(1)}\Phi^{(1)}, V^{(1)}V^{(1)}\right] + \frac{2i\kappa^2 a^2}{k^2} e^i(\hat{k})\mathcal{F}\left[\widehat{T}^{R(2)}{}_{ij}{}^{,j}\right]$$

$$h^{(A)\prime} + 2\mathcal{H}h^{(A)} + k^2 h^{(A)} = \mathcal{Q}_h\left[\Phi^{(1)}\Phi^{(1)}, \Psi^{(1)}\Psi^{(1)}, \Phi^{(1)}\Psi^{(1)}, \Psi^{(1)}\Phi^{(1)}, V^{(1)}V^{(1)}\right] + 2\kappa^2 a^2 e^{ij}(\hat{k})\mathcal{F}\left[\widehat{T}^{R(2)}{}_{ij}\right]$$

$$V_c^{(2)\prime} + \mathcal{H}V_c^{(2)} + k\Psi^{(2)} = \mathcal{Q}_{V_c}\left[V_c^{(1)}\delta^{(1)\prime}, V_c^{(1)}\Phi^{(1)\prime}, V_c^{(1)}\Phi^{(1)}, V_c^{(1)}\Psi^{(1)}, V_c^{(1)}\delta^{(1)}, \Psi^{(1)}\delta^{(1)}, V_c^{(1)\prime}\delta^{(1)}, V_c^{(1)\prime}\Phi^{(1)}, V_c^{(1)\prime}\Psi^{(1)}, V_c^{(1)\prime}V_c^{(1)}\right]$$

$$\delta_c^{(2)\prime} + 3\Phi^{(2)\prime} - kV_c^{(2)} = \mathcal{Q}_{\delta_c}\left[\delta_c^{(1)}\Phi^{(1)\prime}, \delta_c^{(1)}V_c^{(1)}, V_c^{(1)\prime}V_c^{(1)}, V_c^{(1)}V_c^{(1)}, \Psi^{(1)}V_c^{(1)}, \Phi^{(1)}V_c^{(1)}\right]$$

CDM+Baryon+Gravity have been implemented,
but Baryon-Photon/Gravity-Photon couplings are not considered yet.

# 2nd-order line(curve)-of-sight formula

R.Saito, Naruko, Hiramatsu, Sasaki, JCAP10(2014)051 [arXiv:1409.2464]

$$\delta I^{(\mathrm{II})} = \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} \mathcal{T}^{(\mathrm{II})}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{n}_{\mathrm{obs}}) \zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)$$

$$\mathcal{T}^{(\mathrm{II})}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{n}_{\mathrm{obs}}) = F(\mathbf{n}_{\mathrm{obs}}) \int_0^{\eta_0} d\eta' \, F_S(\hat{k}_1) S(k_1, \eta') e^{i\mathbf{k}_1 \cdot \mathbf{n}_{\mathrm{obs}}(\eta_0 - \eta')} \times \int d\eta_1 \, F_T(\hat{k}_2) T(k_2, \eta_1, \eta') e^{i\mathbf{k}_2 \cdot \mathbf{n}_{\mathrm{obs}}(\eta_0 - \eta_1)}$$

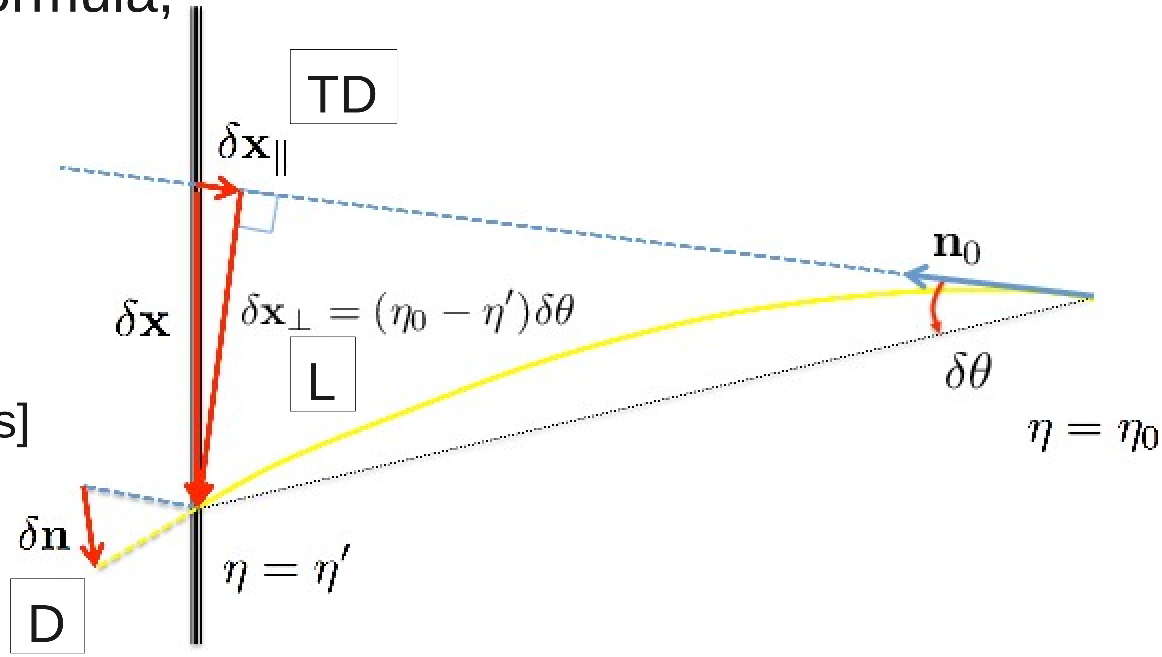
We found 7 combinations in this formula,

[Fluc. on LSS] x [Fluc. on way to us]

Source x ISW
Source x Lensing
Source x Time-delay
Source x Deflection

[Fluc. on way to us] x [Fluc. on way to us]

ISW x ISW
ISW x Lensing
ISW x Time-delay



# 2nd-order line(curve)-of-sight formula



Spergel, Goldberg, PRD59(1999)103001 [astro-ph/9811252]
Goldberg, Spergel, PRD59(1999)103002 [astro-ph/9811251]
Seljak, Zaldarriaga, PRD60(1999)043504 [astro-ph/9811123]
Planck collaboration, A&A 571(2014) A24 [arXiv:1303.5084]

## Source x Lensing

$$S(k_1,\eta') = 4k_1 g(\eta')\left[\Theta_{T0}+\Psi\right] + 4\frac{d}{d\eta'}\left(\frac{g(\eta')v_{\rm b}}{k_1}\right) + 4\mathcal{P}_2\left(\frac{1}{ik_1}\frac{d}{d\eta'}\right)\left[g(\eta')\Pi\right]$$

$$T(k_2,\eta_1,\eta') = k_2(\eta_1-\eta')\left[\Psi(k_2,\eta_1)-\Phi(k_2,\eta_1)\right]$$

$$F(\mathbf{n}_{\rm obs})F_S(\hat{k}_1)F_T(\hat{k}_2) = -\sum_{\lambda=\pm}(i\epsilon^\lambda\cdot\hat{k}_1)(i\epsilon^\lambda\cdot\hat{k}_2)$$

## Bispectrum

Spin-weighted Gaunt integral

$$B^{m_1m_2m_3}_{\ell_1\ell_2\ell_3} = 2[1+(-1)^{\ell_1+\ell_2+\ell_3}]\mathcal{G}^{m_1m_2m_3;(+1)(-1)0}_{\ell_1\ell_2\ell_3}\int_0^{\eta_0} d\eta'\, b^S_{\ell_1}(\eta')b^T_{\ell_2}(\eta') + 2\ \text{sym.}$$

$$b^S_{\ell_1}(\eta') = \frac{2}{\pi}\sqrt{\frac{\ell_1(\ell_1+1)}{2}}\int dk_1\, k_1^2 P_\zeta(k_1)\mathcal{T}_{\Theta_{\ell_1}}(k_1)\frac{S(k_1,\eta')}{k_1(\eta_0-\eta')}j_{\ell_1}[k_1(\eta_0-\eta')]$$

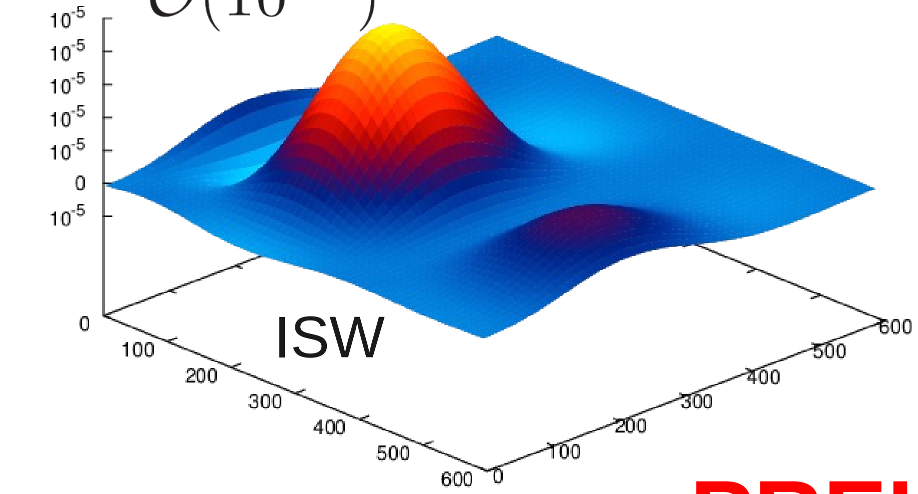
$$b^T_{\ell_2}(\eta') = \frac{2}{\pi}\sqrt{\frac{\ell_2(\ell_2+1)}{2}}\int_{\eta'}^{\eta_0} d\eta_1 \int dk_2\, k_2^2 P_\zeta(k_2)\mathcal{T}_{\Theta_{\ell_2}}(k_1)\frac{T(k_2,\eta_1,\eta')}{k_2(\eta_0-\eta_1)}j_{\ell_2}[k_2(\eta_0-\eta_1)]$$

All 7 combinations have been implemented in my code.
(except for multiplying by Gaunt integral)

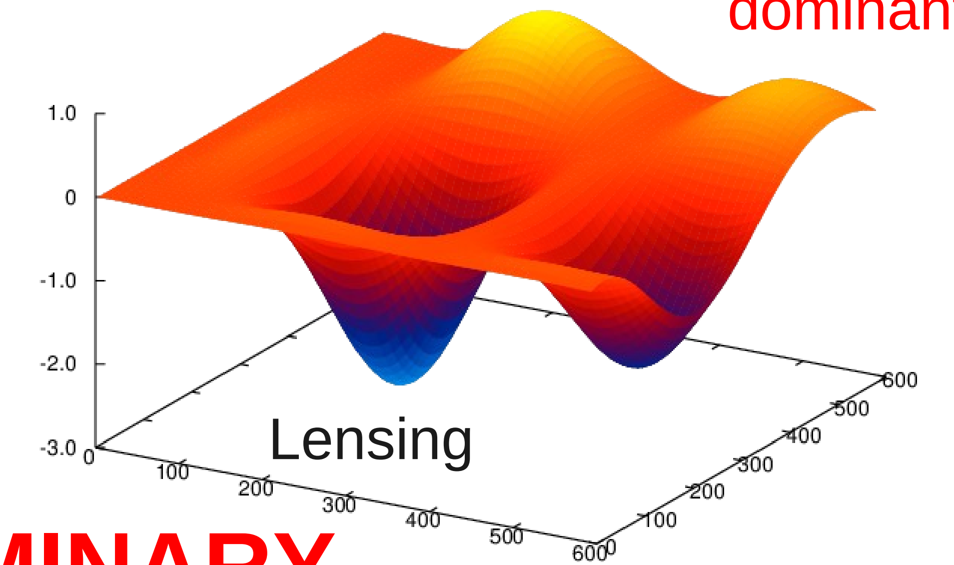
# Contributions from [Source] x [Gravity]

$$\ell_1(\ell_1+1)\ell_2(\ell_1+2)b_{\ell_1\ell_2}\times 10^{10}$$

$\mathcal{O}(10^{-5})$

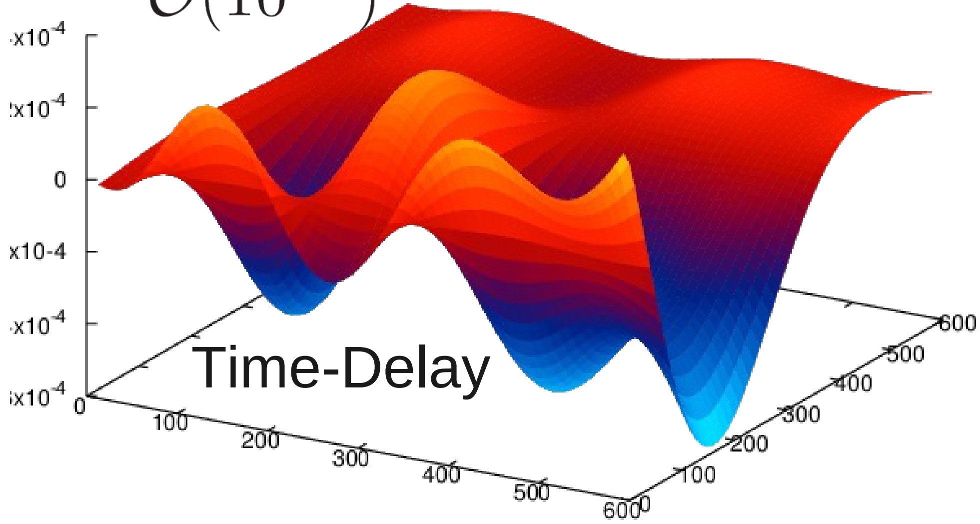



dominant

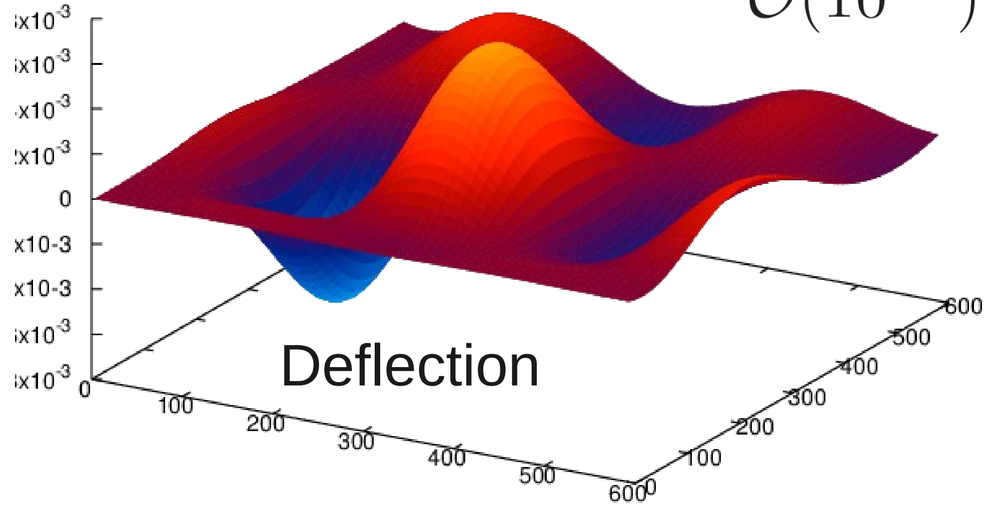



**PRELIMINARY**

$\mathcal{O}(10^{-4})$



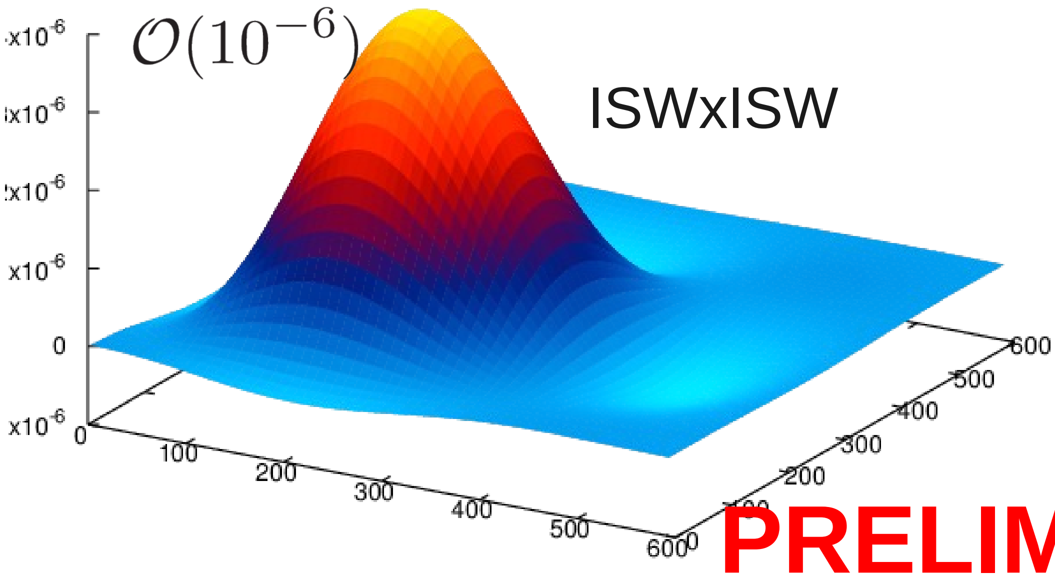

$\mathcal{O}(10^{-3})$

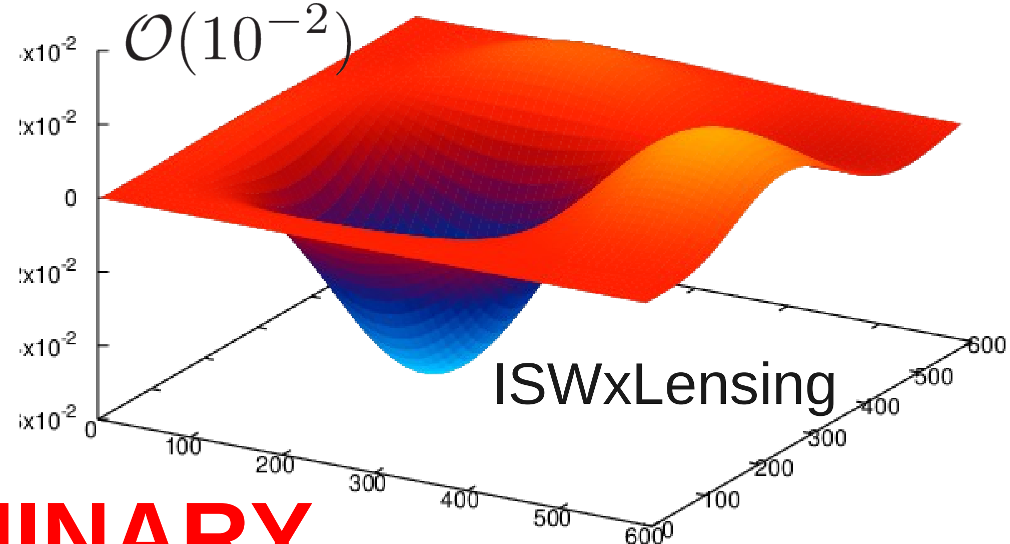


# Contributions from [Gravity] x [Gravity]

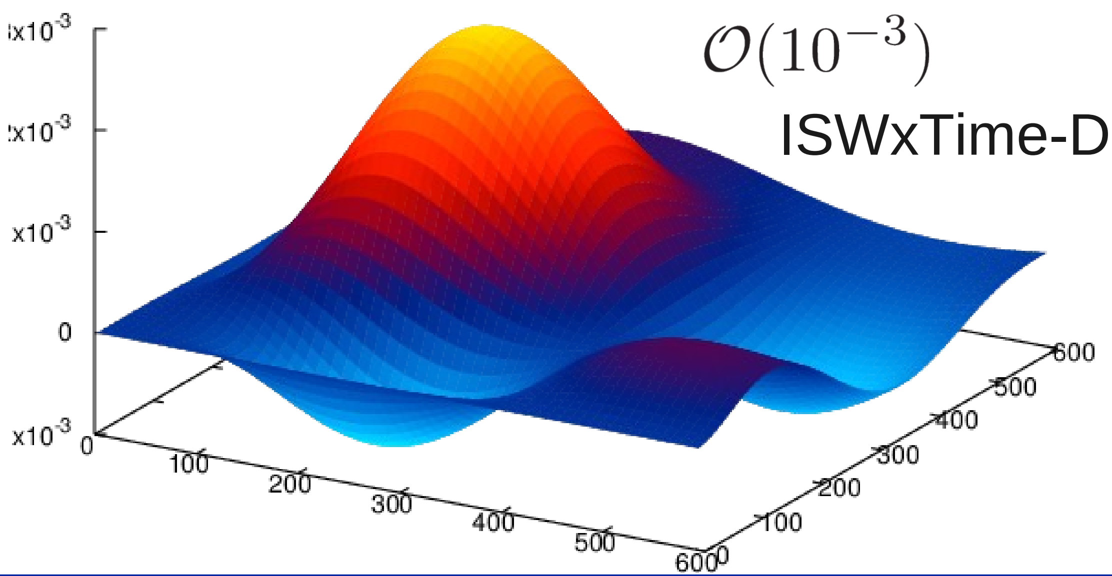
Takashi Hiramatsu

$$\ell_1(\ell_1+1)\ell_2(\ell_1+2)b_{\ell_1\ell_2}\times 10^{10}$$




**PRELIMINARY**



# Intrinsic bispectrum



## 'Full' bispectrum

$$B^{m_1 m_2 m_3}_{\ell_1 \ell_2 \ell_3} = \langle a_{\ell_1 m_1} a_{\ell_2 m_2} a_{\ell_3 m_3} \rangle = \mathcal{G}^{m_1 m_2 m_3;\lambda,-\lambda,0}_{\ell_1 \ell_2 \ell_3} b_{\ell_1 \ell_2 \ell_3}$$

$$\mathcal{G}^{m_1 m_2 m_3; s_1 s_2 s_3}_{\ell_1 \ell_2 \ell_3} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -s_1 & -s_2 & -s_3 \end{pmatrix}$$

Wigner's 3j-symbol ~ Clebsch-Gordan coeffs.

## Azimuthal-angle Averaged bispectrum

$$B_{\ell_1 \ell_2 \ell_3} = \sum_{m_1, m_2, m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B^{m_1 m_2 m_3}_{\ell_1 \ell_2 \ell_3}$$

$$= \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -\lambda & \lambda & 0 \end{pmatrix} b_{\ell_1 \ell_2 \ell_3}$$

## Reduced bispectrum (in the case of COS)

$$b_{\ell_1 \ell_2} \equiv \int_0^{\eta_0} d\eta' \, b^S_{\ell_1}(\eta') b^T_{\ell_2}(\eta')$$

$$b_{\ell_1 \ell_2 \ell_3} = b_{\ell_1 \ell_2} + b_{\ell_2 \ell_1} + b_{\ell_2 \ell_3} + b_{\ell_3 \ell_2} + b_{\ell_3 \ell_1} + b_{\ell_1 \ell_3}$$

Komatsu, Spergel, PRD63 (2001) 063002

Using $\chi^2$ fitting

$$\chi^2 \equiv \sum_{2\le\ell_1\le\ell_2\le\ell_3}^{\ell_{\max}} \frac{\left(B_{\ell_1\ell_2\ell_3} - \sum_i f_{\rm NL}^{(i)} B^{(i)}_{\ell_1\ell_2\ell_3}\right)^2}{\sigma^2_{\ell_1\ell_2\ell_3}}$$

Variance is calculated by six-point function of $a_{\ell m}$

$$\sigma^2_{\ell_1\ell_2\ell_3} \equiv \langle B^2_{\ell_1\ell_2\ell_3}\rangle - \langle B_{\ell_1\ell_2\ell_3}\rangle^2 \approx \overline{C}_{\ell_1}\overline{C}_{\ell_2}\overline{C}_{\ell_3}\Delta_{\ell_1\ell_2\ell_3}$$

$$\overline{C}_\ell \equiv C_\ell + N_\ell \longleftarrow \text{Signal + Noise power}$$

$f_{\rm NL}$ minimising $\chi^2$

$$\partial\chi^2/\partial f_{\rm NL}^{(i)} = 0 \longrightarrow F^{ij} f_{\rm NL}^{(j)} = F^{Bi}$$

$$F^{ij} \equiv \sum_{2\le\ell_1\le\ell_2\le\ell_3} \frac{B^{(i)}_{\ell_1\ell_2\ell_3} B^{(j)}_{\ell_1\ell_2\ell_3}}{\sigma^2_{\ell_1\ell_2\ell_3}}$$

$$f_{\rm NL}^{(j)} = F^{Bi}(F^{-1})^{ij} \qquad \left(\frac{S}{N}\right)^{(i)} = \frac{1}{\sqrt{(F^{-1})^{ii}}}$$

I implemented this scheme, but so many bugs still live in my code...

# Summary : overview of my code



- Full scratch development, completely independent of existing codes
- C++
- Parallelised by OpenMP
- Time evolution : 1-stage 2nd-order implicit Runge-Kutta (Gauss-Legendre) method (implementing up to 4th-order schemes)
- Line-of-sight Integration : Trapezoidal/Simpson's rule
- Interpolation scheme : Polynomial approximation (up to $\mathcal{O}(h^5)$)
- Ready for implementing a variety of recombination/reionisation simulators
- Fast evaluation of spherical Bessel functions, and (specific) Gaunt integral

# Summary : current status



- We are now suffered from a small mismatch between results of our code and CAMB at the 1st-order.

- We implemented 2nd-order perturbations only for gravity and matter. The implementation of radiation part would be straightforwardly done (hopefully, ) if the mismatch problem is resolved.

- We also implemented “curve”-of-sight formulas for scalar contributions of temperature fluctuations.

- Bispectrum estimator has been implemented, and bug-fixing now...

# Summary : to-do and application ?



To-do

- Implement pure 2nd-order equations for radiation
- Bug-fixing bispectrum estimator
- Resolving 14% over-estimation of 1st-order power spectrum

NOTE : reduced to sub-% level on 21 Nov

Applications ?

- 2nd-order gravitational waves
- [1st-order]$^2$ for polarisation
- [Scalar] x [Tensor] & [Tensor] x [Tensor]
- y-distortion to photon's distribution function