Liouville integrability of Hamiltonian systems and spacetime symmetry

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Hamilton formalism

- Many dynamical systems in physics are described in this framework.

- A dynamical system is governed by a function of canonical coordinates $q^i$ and momenta $p_i$, called Hamiltonian $H(q^i, p_i)$.

- Equations of motion
  
  \[ \frac{dq^i}{d\tau} = \frac{\partial H}{\partial p_i} \quad \frac{dp_i}{d\tau} = -\frac{\partial H}{\partial q^i} \]
Hamilton formalism

- Poisson bracket
  \[ \{A, B\}_P := \frac{\partial A}{\partial q^i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q^i} \]

- Conserved quantity (First integral)
  \[
  \frac{dF}{d\tau} = \frac{\partial F}{\partial q^i} \frac{dq^i}{d\tau} + \frac{\partial F}{\partial p_i} \frac{dp_i}{d\tau} \\
  = \frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q^i} = \{F, H\}_P
  \]

\(F\) is a conserved quantity \iff \(\{F, H\}_P = 0\)


**Hamilton formalism**

- **Liouville integrability**

If there exist $D$ independent Poisson-commuting constants $\alpha_i$ (including Hamiltonian) in a $D$-dim Hamiltonian system,

$$\{\alpha_i, \alpha_j\}_P = 0, \quad (i, j = 1, \ldots, D, \quad \alpha_D = H)$$

then the system is said to be completely integrable.

Namely, one can prove that there exists a canonical transf. $(x, p) \rightarrow (\varphi, I(\alpha))$, and then easily solve the Hamilton’s eq.:

$$\dot{\varphi}^\mu = \frac{\partial H'}{\partial I_\mu}, \quad \dot{I}_\mu = -\frac{\partial H'}{\partial \varphi^\mu}.$$
Hamiltonian

\[ H = \frac{1}{2} \sum_{i,j} g^{ij}(q) p_i p_j + V(q) \]

\((q^i, p_i) : \text{canonical coordinates}\)

- \(g_{ij}(q) : \text{metric}\)

\[ ds^2 = g_{ij} dq^i dq^j \]

- \(V(q) : \text{potential}\)

\(V \neq 0\) \quad \text{Natural Hamiltonian}

\(V = 0\) \quad \text{Geodesic Hamiltonian}
The purpose of this talk

To show a systematic approach for investigating polynomial conserved quantities for any natural Hamiltonian system
Keywords

① Geometrisation
Any natural Hamiltonian system can be translated to the geodesic problem in a corresponding spacetime.

② Prolongation
Equations describing spacetime symmetry can be translated to a first-order linear PDE system.
Geodesic problem and spacetime symmetry
Spacetime symmetry

Isometry

Killing equation
\[ \nabla (\xi^a) = 0 \]

\[ F \equiv \xi_a p^a = g_{ab} \xi^a p^b \]

\[ (\because p^a \nabla_a F = 0) \]
Spacetime symmetry  ↔  Conserved quantities along geodesics
For a geodesic Hamiltonian $H = g^{\mu\nu}p_\mu p_\nu$, when $F$ is a $n$-order homogeneous polynomial in $p_\mu$,

$$F = K_{a_1\cdots a_n}(x) p_{a_1} \cdots p_{a_n}$$

then we find that

$$\{F, H\} = 0 \iff \nabla_{(a} K_{b_1\cdots b_n)} = 0$$

Killing-Stackel Eq.
Spacetime symmetry $\leftrightarrow$ Conserved quantities along geodesics

Isometry (Killing vector) $\leftrightarrow$ first-order polynomial

Hidden symmetry (Killing tensor) $\leftrightarrow$ higher-order polynomial
Spacetime symmetry

• Killing vector fields:
  \[ \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \]

• Killing-Stackel tensors [Stackel 1895]
  \[ \nabla_{(\mu} K_{\nu_1 \nu_2 \ldots \nu_n)} = 0 \]
  \[ K_{(\mu_1 \mu_2 \ldots \mu_n)} = K_{\mu_1 \mu_2 \ldots \mu_n} \]

• Killing-Yano tensors [Yano 1952]
  \[ \nabla_{(\mu} f_{\nu_1) \nu_2 \ldots \nu_n} = 0 \]
  \[ f_{[\mu_1 \mu_2 \ldots \mu_n]} = f_{\mu_1 \mu_2 \ldots \mu_n} \]
<table>
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<tr>
<th>vector fields</th>
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<td>Tachibana 1969, Kashiwada 1968</td>
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Why spacetime symmetry?

• Conserved quantities along geodesics
• Integrability of EOMs for matter fields
  Klein-Gordon and Dirac equations
• Classification of spacetimes
  Stationary, axially symmetric, Bianchi type, etc.
• Application to Hamiltonian dynamics
Geometrisation
Basic idea

The dynamical trajectories of a Hamiltonian system of the form

$$H = \frac{1}{2} \sum_{i,k} g^{ik}(q) p_i p_k + U(q),$$

can be seen as geodesics of a corresponding configuration space, or of enlargement of it, under some constraints.
Examples

① Maupertuis’ principle

② Canonical transformations
  Ex. ②－1 3D Kepler problem
  Ex. ②－2 N=3 open Toda

③ Eisenhart’s lifts
  Ex. ③－1 Eisenhart lift
  Ex. ③－2 Generalised Eisenhart lift
  Ex. ③－3 Light-like Eisenhart lift
Maupertuis’ principle

Maupertuis 1744, 1746, 1756

One obtains an integral equation that determines the path followed by a physical system without specifying the time parameterization of that path.

\[ x^a(t) \]

\[ x^\mu(\tau) \]
Maupertuis’ principle

Maupertuis 1744, 1746, 1756

One obtains an integral equation that determines the path followed by a physical system without specifying the time parameterization of that path.

\[ q = q(t), \quad p = p(t) \]

Action

\[ S = \int (p_i dq^i - H(q, p) dt) \]

Abbreviated action

\[ S_0(E) = \int p_i dq^i \]
Jacobi’s formulation

Lagrangian

\[ L = \frac{1}{2} \sum_{i,j} g_{ij} \dot{q}^i \dot{q}^j - U(q) \]

 abbreviated action

\[ S_0 \equiv \int \sum_i p_i dq^i = \int \sum_{i,k} g_{ik} \frac{dq^k}{dt} dq^i = \int \sqrt{2(E - U) \sum_{i,k} g_{ik} dq^i dq^k} \]

\[ \tilde{g}_{ik} = (E - U) g_{ik} \]

\[ S_0 = \int \sqrt{2 \sum_{i,k} \tilde{g}_{ik} dq^i dq^k} \]
Jacobi’s formulation

\[ L = \frac{1}{2} \sum_{i,k} g_{ik} \dot{q}^i \dot{q}^k - U(q) , \quad E = \frac{1}{2} \sum_{i,k} g_{ik} \dot{q}^i \dot{q}^k + U(q) , \]

\[ S_0 = \int \sqrt{2(E - U)} \sum_{i,k} g_{ik} dq^i dq^k . \]

\[ \bar{L} = \frac{1}{2} \sum_{i,k} \bar{g}_{ik} \dot{q}^i \dot{q}^k , \quad \bar{E} = \frac{1}{2} \sum_{i,k} \bar{g}_{ik} \dot{q}^i \dot{q}^k , \]

\[ \bar{S}_0 = \int \sqrt{2\bar{E}} \sum_{i,k} \bar{g}_{ik} dq^i dq^k . \]
Jacobi’s formulation

**Theorem** Given a dynamical system on a manifold \((M, g_{ik})\) i.e., a dynamical system whose Lagrangian is

\[
L = \frac{1}{2} \sum_{i,k} g_{ik} \dot{q}^i \dot{q}^k - U(q),
\]

then it is always possible to find a conformal transformation of the metric (Jacobi metric)

\[
\tilde{g}_{ik} = (E - U) g_{ik},
\]

such that the geodesics of \((M, \tilde{g}_{ik})\) with the energy \(\tilde{E} = 1\) are equivalent to the trajectories of the original dynamical system.
Comparison of Hamiltonians

**Natural Hamiltonian**

\[ H = \frac{1}{2} g^{ik} p_i p_k + U \]

with \( H = E \)

**Jacobi’s Hamiltonian**

\[ \tilde{H} = \frac{1}{2} \tilde{g}^{ik} p_i p_k \quad \text{with} \quad \tilde{H} = 1 \]

\[ \tilde{g}_{ik} = (E - U) g_{ik} \]

\[ E = H_2 + U \]

\[ 1 = \frac{H_2}{E - U} \]
Comparison of first integrals

- Natural Hamiltonian
  \[ H = H_2 + U \]
  \[ \exists K \ s.t. \ \{ H, K \} = 0 . \]

- Jacobi’s Hamiltonian
  \[ \tilde{H} = \frac{H_2}{E - U} \]
  \[ \exists \tilde{K} \ s.t. \ \{ \tilde{H}, \tilde{K} \} = 0 . \]

For instance,

\[ K = K_2 + K_0 . \]
\[ \tilde{K} = K_2 + K_0 \tilde{H} . \]
Maupertuis’ principle

Potential system

Geodesic system
Natural Hamiltonian

Quadratic + potential

\[ H = \frac{1}{2} g^{ik}(q)p_i p_k + U(q) \]

Geodesic Hamiltonian

Homogeneously quadratic

\[ \tilde{H} = \frac{1}{2} \tilde{g}^{\mu \nu}(\tilde{q}) \tilde{p}_\mu \tilde{p}_\nu \]

Point!

We need to construct a geodesic Hamiltonian, i.e., a homogeneously quadratic Hamiltonian which reduces to the original natural Hamiltonian under some transformation or constraints.
Examples

① Maupertuis’ principle

② Canonical transformations

Ex. ②—1  3D Kepler problem
Ex. ②—2  N=3 open Toda

③ Eisenhart’s lifts

Ex. ③—1  Eisenhart lift
Ex. ③—2  Generalised Eisenhart lift
Ex. ③—3  Light-like Eisenhart lift
3D Kepler problem

\[ H = \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) - \frac{\alpha}{r} \quad \text{with} \quad H = E \]

\[ r = \sqrt{(q_1)^2 + (q_2)^2 + (q_3)^2} \]

Transformed

\[ \tilde{q}^i = p_i, \quad \tilde{p}_i = q^i \]

\[ \tilde{H} = \left( E - \frac{1}{2} \tilde{r}^2 \right)^2 (\tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2) \quad \text{with} \quad \tilde{H} = \alpha^2 \]

\[ \tilde{r} = \sqrt{(\tilde{q}_1)^2 + (\tilde{q}_2)^2 + (\tilde{q}_3)^2} \]

\[ ds^2 = \left( E - \frac{1}{2} \tilde{r}^2 \right)^{-2} [(d\tilde{q}_1)^2 + (d\tilde{q}_2)^2 + (d\tilde{q}_3)^2] \quad \text{(constant curvature } -4E \text{ )} \]
N=3 open Toda

\[ H = \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) + a_1^2 + a_2^2 \]

\[ a_1 = e^{q_1 - q_2}, \quad a_2 = e^{q_2 - q_3} \]

Transf.

\[ \tilde{q}_1 = q_1 + \ln p_1, \quad \tilde{q}_2 = q^2, \quad \tilde{q}_3 = q_3 - \ln p_3 \]

\[ \tilde{p}_1 = p_1, \quad \tilde{p}_2 = p_2, \quad \tilde{p}_3 = p_3 \]

\[ \tilde{H} = \frac{1}{2} \left\{ (1 + 2\tilde{a}_1^2)\tilde{p}_1^2 + \tilde{p}_2^2 + (1 + 2\tilde{a}_2^2)\tilde{p}_3^2 \right\} \]

\[ \tilde{a}_1 = e^{\tilde{q}_1 - \tilde{q}_2}, \quad \tilde{a}_2 = e^{\tilde{q}_2 - \tilde{q}_3} \]

\[ ds^2 = \frac{(d\tilde{q}_1)^2}{1 + 2\tilde{a}_1^2} + (d\tilde{q}_2)^2 + \frac{(d\tilde{q}_3)^2}{1 + 2\tilde{a}_2^2} \]
Examples

① Maupertuis’ principle

② Canonical transformations
  Ex. ②—1 3D Kepler problem
  Ex. ②—2 N=3 open Toda

③ Eisenhart’s lifts
  Ex. ③—1 Eisenhart lift
  Ex. ③—2 Generalised Eisenhart lift
  Ex. ③—3 Light-like Eisenhart lift
(Standard) Eisenhart lift

- Natural Hamiltonian on $n$-dim space $(M, g)$
  \[
  H = \frac{1}{2} g^{ik}(q) p_i p_k + U(q)
  \]

- Geodesic Hamiltonian on $(n+1)$-dim space $(M \times \mathbb{R}, g_E)$
  \[
  H_E = \frac{1}{2} g^{ik} p_i p_k + U(q) p_s^2
  \]
  \[
  = \frac{1}{2} g_E^{\mu\nu} p_{\mu} p_{\nu}
  \]

Eisenhart metric

\[
\frac{ds_E^2}{ds} = 2U(q)^{-1} ds^2 + g_{ik} dq^i dq^k
\]
Generalised Eisenhart lift

• Natural Hamiltonian on \( n \)-dim space \((M, g)\)

\[
H = \frac{1}{2} g^{ik}(q)p_ip_k + U(q), \quad U(q) = \sum_{\ell=1}^{m} a_{\ell} U_{\ell}(q)
\]

• Geodesic Hamiltonian on \((n+m)\)-dim space \((M \times \mathbb{R}^m, g_E)\)

\[
H_E = \frac{1}{2} g^{ik} p_ip_k + \sum_{\ell=1}^{m} U_{\ell}(q) p^2_{s\ell} \\
= \frac{1}{2} g^{\mu\nu}_E p_\mu p_\nu
\]

Eisenhart metric

\[
ds_E^2 = 2U_{\ell}(q)^{-1}(ds^\ell)^2 + g_{ik} dq^i dq^k
\]
Light-like Eisenhart lift

- Natural Hamiltonian on $n$-dim space $(M, g)$

$$H = \frac{1}{2} g^{ik}(q) p_i p_k + U(q)$$

- Geodesic Hamiltonian on $(n+m)$-dim spacetime $(M \times R^m, g_E)$

$$H_E = \frac{1}{2} g^{ik} p_i p_k + U(q) p_s^2 + p_s p_t$$

$$= \frac{1}{2} g^\mu_\nu p_\mu p_\nu$$

Eisenhart metric

$$ds_E^2 = -2U(q) dt^2 + 2dt ds + g_{ik} dq^i dq^k$$
Comparison
Natural Htn v.s. LL Eisenhart’s Htn

- Natural Hamiltonian

\[ H = H_2 + U \]
\[ \exists K \text{ s.t. } \{H, K\} = 0. \]

For instance,
\[ K = K_2 + K_0. \]

- Eisenhart’s Hamiltonian

\[ H_E = H_2 + UP_s^2 + p_sp_t \]
\[ \exists K' \text{ s.t. } \{H', K'\} = 0. \]

\[ K' = K_2 + K_0p_s^2. \]
Prolongation
Review I: 
Integrability conditions for systems of first order PDEs
A system of first order PDEs

\[ \frac{\partial u^\alpha}{\partial x^i} = \psi_{i\beta}(x)u^\beta \]

\[ u = (u^1, u^2, \ldots, u^N) \text{ ; unknown functions} \]
\[ x = (x^1, x^2, \ldots, x^n) \text{ ; variables} \]

Questions:

Does the solution exist? \text{ at most } N \text{ dimensions}

If exist, is the solution space finite or infinite? How many dimensions?

Explicit expressions?
Consistency conditions

\[
\frac{\partial}{\partial x^j} \frac{\partial u^\alpha}{\partial x^i} - \frac{\partial}{\partial x^i} \frac{\partial u^\alpha}{\partial x^j} = 0
\]

\[
\frac{\partial}{\partial x^j} \frac{\partial u^\alpha}{\partial x^i} = \frac{\partial \psi^\alpha_{i\beta}}{\partial x^j} u^\beta + \psi^\alpha_{i\beta} \frac{\partial u^\beta}{\partial x^j} = \frac{\partial \psi^\alpha_{i\beta}}{\partial x^j} u^\beta + \psi^\alpha_{i\beta} \psi^\beta_{j\gamma} u^\gamma
\]

\[
\frac{\partial}{\partial x^i} \frac{\partial u^\alpha}{\partial x^j} = \frac{\partial \psi^\alpha_{j\beta}}{\partial x^i} u^\beta + \psi^\alpha_{j\beta} \frac{\partial u^\beta}{\partial x^i} = \frac{\partial \psi^\alpha_{j\beta}}{\partial x^i} u^\beta + \psi^\alpha_{j\beta} \psi^\beta_{i\gamma} u^\gamma
\]

\[
\left(\frac{\partial \psi^\alpha_{i\gamma}}{\partial x^j} - \frac{\partial \psi^\alpha_{j\gamma}}{\partial x^i} + \psi^\alpha_{i\beta} \psi^\beta_{j\gamma} - \psi^\alpha_{j\beta} \psi^\beta_{i\gamma}\right) u^\gamma = 0
\]
Frobenius’ theorem

The necessary and sufficient conditions for the unique solution $u^\alpha = u^\alpha(x)$ to the system

$$\frac{\partial u^\alpha}{\partial x^i} = \psi_{i\beta} u^\beta$$

such that $u(x_0) = u_0$ to exist for any initial data $(x_0, u_0)$ is that the relation

$$\frac{\partial \psi_{i\gamma}}{\partial x^j} - \frac{\partial \psi_{j\gamma}}{\partial x^i} + \psi_{i\beta} \psi_{j\gamma} - \psi_{j\beta} \psi_{i\gamma} = 0$$

hold.
Parallel equation

\[
\frac{\partial u^\alpha}{\partial x^i} = \psi^\alpha_{i\beta}(x) u^\beta
\]

\[
\frac{\partial u^\alpha}{\partial x^i} - \psi^\alpha_{i\beta}(x) u^\beta = 0
\]

\[
D_i u^\alpha = 0
\]

where \( D_i u^\alpha := \frac{\partial u^\alpha}{\partial x^i} - \psi^\alpha_{i\beta}(x) u^\beta \)

The system can be viewed as a parallel equation for sections \( u^\alpha \) of a vector bundle \( \pi: E \to M \) of rank \( N \).
Curvature conditions

For a connection $D_i$

$$D_i u^\alpha := \frac{\partial u^\alpha}{\partial x^i} - \psi^\alpha_{i\beta}(x) u^\beta$$

the curvature of $D_i$ is defined by $(D_iD_j - D_jD_i)u^\alpha = -R_{ij\beta}^\alpha u^\beta$.

$$D_i u^\alpha = 0 \quad \rightarrow \quad R_{ij\beta}^\alpha u^\beta = 0$$

This is equivalent to the Frobenius integrability condition
Frobenius’ theorem (II)

The necessary and sufficient conditions for the unique solution \( u^\alpha = u^\alpha(x) \) to the system

\[
D_i u^\alpha = 0 \quad i = 1, \ldots, n \quad \alpha = 1, \ldots, N
\]

where

\[
D_i u^\alpha := \frac{\partial u^\alpha}{\partial x^i} - \psi_{\alpha \beta}(x)u^\beta
\]

such that \( u(x_0) = u_0 \) to exist for any initial data \((x_0, u_0)\) is that the relation

\[
R_{ij\beta}^\alpha u^\beta = 0
\]

hold.
Discussion

• If the curvature conditions hold, the general solution depends on $N$ arbitrary constants.

$$u^\alpha(x; a_i) = a_1 u_1^\alpha(x) + a_2 u_2^\alpha(x) + \cdots + a_N u_N^\alpha(x)$$

• If not, they give a set of algebraic equations

$$R_{ij\beta}^\alpha u^\beta = 0$$

• Differentiating these equations and eliminating the derivatives of $u^\alpha$ leads to a new set of equations

$$(D_k R_{ij\beta}^\alpha) u^\beta = 0$$

$$D_k F_{ij\beta}^\alpha := \partial_k F_{ij\beta}^\alpha - \psi_{k\gamma}^\alpha F_{ij\beta}^\gamma + F_{ij\gamma}^\alpha \psi_{k\beta}^\gamma$$
Discussion

• Proceeding in this way we get a sequence of sets of equations

\[ R_{ij\beta}^\alpha u^\beta = 0, \ (D_k R_{ij\beta}^\alpha) u^\beta = 0, \ (D_\ell D_k R_{ij\beta}^\alpha) u^\beta = 0, \ \ldots \]

• If \( p \) is the number of independent equations in the first \( K \) sets, then the general solution depends on \( N - p \) arbitrary constants.
Review II: Prolongation of PDEs
Prolongation

\[ F(x, f, \partial f, \partial \partial f, \cdots) = 0 \]

\[ \frac{\partial u^\alpha}{\partial x^i} = \psi_{i\beta}^\alpha u^\beta \]

\[ i = 1, \cdots, n \quad \alpha = 1, \cdots, N \]
Example 1

Introduce $w = u_y - v_x$

\[
\begin{align*}
    u_x &= au + bv \\
    u_y + v_x &= cu + dv \\
    v_y &= eu + fv
\end{align*}
\]

\[
\begin{align*}
    u_x &= au + bv \\
    u_y &= \frac{1}{2} (cu + dv + w) \\
    v_x &= \frac{1}{2} (cu + dv - w) \\
    v_y &= eu + fv \\
    w_x &= w_x(u, v, w) \\
    w_y &= w_y(u, v, w)
\end{align*}
\]
Example 2: Cauchy-Riemann equation

\[ u_x = v_y \]
\[ u_y = -v_x \]

Impossible to make a prolongation!

In fact, solution of this system depends on one holomorphic function.
Prolongation

\[ F(x, f, \partial f, \partial \partial f, \ldots) = 0 \]

Not always possible
When can we make a prolongation successfully?

\[ \frac{\partial u^\alpha}{\partial x^i} = \psi^\alpha_i(x, u) \]

\[ i = 1, \ldots, n \quad \alpha = 1, \ldots, N \]
Prolongation of Killing equations
Spacetime symmetry

• Killing vector fields:
  \[ \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \]

• Killing-Stackel tensors
  \[ \nabla_{(\mu} K_{\nu_1\nu_2...\nu_n)} = 0 \quad K_{(\mu_1\mu_2...\mu_n)} = K_{\mu_1\mu_2...\mu_n} \]
  [Stackel 1895]

• Killing-Yano tensors
  \[ \nabla_{(\mu} f_{\nu_1)\nu_2...\nu_n} = 0 \quad f_{[\mu_1\mu_2...\mu_n]} = f_{\mu_1\mu_2...\mu_n} \]
  [Yano 1952]
Killing vectors

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$
Killing vector equation

\[ \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \]

\[ \nabla_\mu \xi_\nu = L_{\mu \nu}, \quad L_{\mu \nu} = \nabla_{[\mu} \xi_{\nu]} \]

\[ \nabla_\mu L_{\nu \rho} = -R_{\nu \rho \mu}^{\quad \sigma} \xi_\sigma \]
\[ \nabla_\mu \xi_\nu = L_{\mu \nu}, \quad L_{\mu \nu} = L_{[\mu \nu]} \]
\[ \nabla_\mu L_{\nu \rho} = -R_{\nu \rho \mu} \sigma \xi_\sigma \]

\[ \nabla_\mu \begin{pmatrix} \xi_\nu \\ L_{\nu \rho} \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -R_{\nu \rho \mu} \sigma & 0 \end{pmatrix} \begin{pmatrix} \xi_\sigma \\ L_{\mu \nu} \end{pmatrix} = 0 \]

\[ D_\mu \hat{\xi}_A = 0 \]

- \[ \hat{\xi}_A = (\xi_\mu, L_{\mu \nu}) : \text{a section of } \Lambda^1(M) \oplus \Lambda^2(M) \]
- \[ D_\mu : \text{connection on } \Lambda^1(M) \oplus \Lambda^2(M) \]
\[ D_\mu \hat{\xi}_A \equiv \nabla_\mu \begin{pmatrix} \xi_\nu \\ L_{\nu \rho} \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -R_{\nu \rho \mu} \sigma & 0 \end{pmatrix} \begin{pmatrix} \xi_\sigma \\ L_{\mu \nu} \end{pmatrix} \]
Point ① Prolongation

Killing vectors $\iff$ parallel sections of $\Lambda^1(M) \oplus \Lambda^2(M)$

$\xi^\mu$ \quad $\xi_A = \left( \begin{array}{c} \xi_\mu \\ \nabla_{[\mu} \xi_{\nu]} \end{array} \right)$ \quad s.t. \quad $D_\mu \hat{\xi}_A = 0$

Parallel equation

The number of linearly independent sections of $\Lambda^1(M) \oplus \Lambda^2(M)$ is bound by the rank of the vector bundle.

$$N = \binom{n}{1} + \binom{n}{2} = \frac{n(n + 1)}{2}$$
Maximally symmetric spaces

Spaces that have the maximum number of KVs
\[ \iff \text{constant curvature spaces} \]

\[
\begin{array}{|c|c|}
\hline
n & N = \frac{n(n + 1)}{2} \\
\hline
2 & 3 \\
3 & 6 \\
4 & 10 \\
5 & 15 \\
\ldots & \ldots \\
\hline
\end{array}
\]
Point ② Curvature conditions

\[
[D_\mu, D_\nu] \hat{\xi}_A = 0 \quad [D_\mu, [D_\nu, D_\rho]] \hat{\xi}_A = 0 \\
[D_\mu, [D_\nu, D_\rho, D_\sigma]] \hat{\xi}_A = 0 \\
\ldots
\]

# of parallel sections = rank of Ep − # of curv. cond.
Killing-Yano tensors

\[ \nabla_{(\mu f_{\nu_1})\nu_2\ldots\nu_n} = 0 \]

\[ f_{[\mu_1\mu_2\ldots\mu_n]} = f_{\mu_1\mu_2\ldots\mu_n} \]
KY tensor equation

\[ \nabla_{(\mu f_{\nu})\rho} = 0 \quad f_{\mu\nu} = -f_{\nu\mu} \]

\[ \nabla_{\mu} f_{\nu\rho} = \nabla_{[\mu f_{\nu\rho}]} \]

\[ \nabla_{\mu} (\nabla_{[\nu f_{\rho\sigma}]}) = -R_{[\nu\rho|\mu}^{\alpha} f_{\alpha|\sigma]} \]
Rank-2

\[ \nabla_{\mu} f_{\nu\rho} = \nabla_{[\mu f_{\nu\rho}]} \]

\[ \nabla_{\mu}(\nabla_{[\nu f_{\rho\sigma}]}) = -R_{[\nu\rho|\mu}^{\alpha f_{\alpha|\sigma]} \]

Rank-p

\[ \nabla_{\mu} f_{\nu_1...\nu_p} = \nabla_{[\mu f_{\nu_1...\nu_p}]} \]

\[ \nabla_{\mu}(\nabla_{[\nu f_{\rho_1...\rho_p}]}) = -R_{[\nu\rho_1|\mu}^{\alpha f_{\alpha|\rho_2...\rho_p]} \]
Prolongation of KY tensors

rank-\( p \) KY tensors \( \iff \) parallel sections of \( E^p \)

\[ E^p = \Lambda^p (M) \oplus \Lambda^{p+1}(M) \]

\[ \text{rank}(E^p) = \binom{n+1}{p+1} \]
The number of KY tensors in maximally symmetric spaces

\[ N = \binom{n + 1}{p + 1} \]

<table>
<thead>
<tr>
<th></th>
<th>p=1</th>
<th>p=2</th>
<th>p=3</th>
<th>p=4</th>
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<td>15</td>
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Semmelmann 2002
Examples in four dimensions

<table>
<thead>
<tr>
<th>4D metrics</th>
<th>$p = 1$</th>
<th>$p = 2$</th>
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<tbody>
<tr>
<td>Maximally symmetric</td>
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<td>10</td>
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<td>Plebanski-Demianski</td>
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<td>Kerr</td>
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<td>Schwazschild</td>
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<tr>
<td>Self-dual Taub-NUT</td>
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<tr>
<td>Eguchi-Hanson</td>
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TH-Yasui 2014
Examples in five dimensions

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<tbody>
<tr>
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<td>15</td>
<td>6</td>
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<tr>
<td>Myers-Perry</td>
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<td>0</td>
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<tr>
<td>Emparan-Realll</td>
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<tr>
<td>Kerr string</td>
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<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

TH-Yasui 2014
Killing-Stackel tensors

\[ \nabla_{(\mu} K_{\nu_1 \nu_2 \ldots \nu_n)} = 0 \]

\[ K_{(\mu_1 \mu_2 \ldots \mu_n)} = K_{\mu_1 \mu_2 \ldots \mu_n} \]
\[ \nabla_{(\mu} K_{\nu\rho)} = 0 \quad K_{\mu\nu} = K_{\nu\mu} \]

\[ \nabla_\mu K_{\nu\rho} = \frac{2}{3} \left( \nabla_{[\mu} K_{\nu]\rho} + \nabla_{[\mu} K_{\rho]\nu} \right) \]

\[ \nabla_\mu \left( \nabla_{[\nu} K_{\rho]}\sigma \right) = -R_{\nu\rho(\mu} K^{\alpha} \alpha_{\sigma]} - R_{(\mu[\nu\rho]} K^{\alpha} \alpha_{\sigma]} \\
- \frac{1}{4} R_{\nu\rho[\mu} K^{\alpha} \alpha_{\sigma]} - \frac{1}{2} R_{(\mu[\nu\rho]} K^{\alpha} \alpha_{\sigma]} + \phi_{[\mu[\nu\rho]} K^{\alpha} \alpha_{\sigma]} + \phi_{[\mu[\nu\rho]} K^{\alpha} \alpha_{\sigma]} + \phi_{[\mu[\nu\rho]} K^{\alpha} \alpha_{\sigma]} \]

where \( \phi_{\mu\nu\rho\sigma} \equiv \nabla_{(\mu} \nabla_{\nu)} K_{\rho\sigma} \)

\[ \nabla_\mu \left( \phi_{[\nu[\rho\sigma]} K^{\kappa]} \right) = (R_1 \cdot K^{**})_{\mu\nu\rho\sigma} K^{\kappa} + (R_2 \cdot \nabla^{[*K_*]*})_{\mu\nu\rho\sigma} K^{\kappa} \]
Prolongation of KS tensors

rank-\(p\) KS tensors \iff parallel sections of \(E^p\)

\[
E^p = \bigoplus \begin{array}{ccc} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{array} \bigoplus \begin{array}{ccc} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{array}
\]

\[
\text{rank}(E^p) = \frac{1}{n} \left( n + p \right) \binom{n + p - 1}{p}
\]
The number of KS tensors in maximally symmetric spaces

\[ N = \frac{1}{n} \binom{n + p}{p + 1} \binom{n + p}{p} - 1 \]

Barbance 1973, Michel et al 2012

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</table>
On-going tasks

• Analysis of curvature conditions
  - Compute the curvature conditions
  - Construct the package of Mathematica which compute and solve the curvature conditions
  - Investigate the curvature conditions for various metrics

Conjecture  No non-trivial quadratic constant for geodesic motion in the Kerr spacetime exists, with the exception of Carter constant.
Foresight into the future

- CKY and CKS
  Cotton tensor, Bach tensor, Q-curvature, conformal geometry

- PDE theory
  Prolongation

- Differential geometry
  Generalised gradients, Weitzenbock formula, twisted Dirac

- Hamiltonian dynamics
  Integrable systems, Chaos, Lax pairs, Painleve systems

- GR, SUGRA, ...  
  Exact solutions, strings, branes