Dark Energy and Modified Gravities

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Preliminary

Universe can be regarded as isotropic and homogeneous in the scale larger than the clusters of galaxies
$\Rightarrow$ **Friedmann-Robertson-Walker (FRW) metric:**

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a(t)^2 \sum_{i,j=1}^{3} \tilde{g}_{ij} dx^i dx^j \, .$$

$a(t)$**: scale factor,** $\tilde{g}_{ij}$**: spacial metric**
$\tilde{R}_{ij} = 2K\tilde{g}_{ij}$ **(**$\tilde{R}_{ij}$**: Ricci curvature given by** $\tilde{g}_{ij}$**)**
$K = 1$**: unit sphere,** $K = -1$**: unit hyperboloid,** $K = 0$**: flat space**
K **is not always an integer.**

$$\left\{ \begin{array}{cl} da(t)/dt > 0 & : \textbf{expanding universe} \\ d^2a(t)/dt^2 > 0 & : \textbf{accelerating expansion} \end{array} \right.$$

Assume the Universe is filled with perfect fluids.

1st FRW equation: (t,t) component of the Einstein eq.

$$0 = -\frac{3}{\kappa^2}H^2 - \frac{3K}{\kappa^2 a^2} + \rho\,, \quad \kappa^2 \equiv 8\pi G$$

2nd FRW equation: (i,j) component

$$0 = \frac{1}{\kappa^2}\left(2\frac{dH}{dt} + 3H^2\right) + \frac{K}{\kappa^2 a^2} + p\,,$$

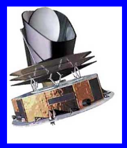
ρ**: energy density,** p**: pressure,** $H \equiv (1/a)\,da(t)/dt$**: Hubble rate**

The Hubble constant H_0: the present value of H.

$H_0 \sim 70\,\mathrm{km\,s^{-1}Mpc^{-1}} \sim 10^{-33}\,\mathrm{eV}$ **in the unit $\hbar = c = 1$.**

Cosmic Microwave Background Radiation (CMB) $\Rightarrow K \sim 0$

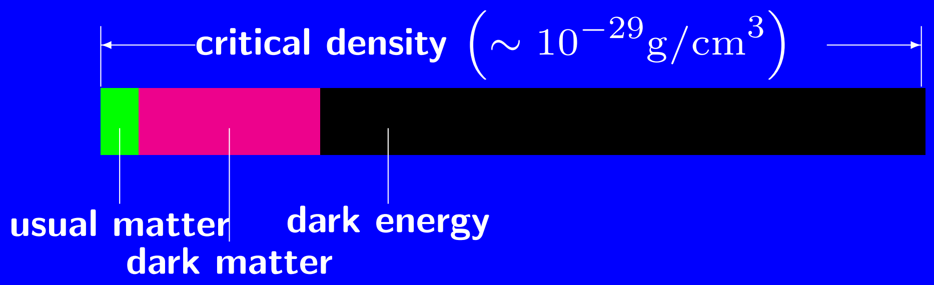
$$\rho \sim \rho_c \equiv \frac{3}{\kappa^2} H_0^2 \sim \left(10^{-3}\,\mathrm{eV}\right)^4 \sim 10^{-29}\mathrm{g/cm}^3 \,.$$



Planck satellite

ρ_c: **critical density. Flat universe** $\Rightarrow \rho \sim \rho_c$

Density of usual matter $\sim 4.9\%$ **, dark matter** $\sim 26.8\%$ **of** ρ_c
$\Longrightarrow$ **something unknown** $\sim 68.3\% \cdots$ **dark energy**



Type Ia Supernovae

$\Rightarrow$ **accelerating expansion started about 5 billion years ago.**

1st and 2nd FRW equations $\Rightarrow$

$$\frac{1}{a}\frac{d^2a(t)}{dt^2} = \frac{dH}{dt} + H^2 = -\frac{\kappa^2}{6}(\rho + 3p)\ .$$

accelerating expansion $\Rightarrow p < -\rho/3$

Dark energy: large negative pressure

Equation of state (EoS) parameter: $w \equiv \frac{p}{\rho}$

- **Dark energy: $w \sim -1$**
- **Radiation: $w = 1/3$,**
- **Usual matter, cold dark matter (CDM): $w \sim 0$ (dust),**
- **Cosmological constant: $w = -1$**

Dark energy = Cosmological constant??

When EoS parameter w: constant $\Rightarrow$ conservation law:

$$\frac{d\rho}{dt} + 3H\left(\rho + p\right) = 0\,,$$

$\Rightarrow \rho = \rho_0 a^{-3(1+w)}$ $(w \neq -1)$, ρ_0: **constant of integration**

1st FRW eq. $\Rightarrow$

In case $w > -1$, $a(t) \propto t^{\frac{2}{3(1+w)}}$

In case $w < -1$, $a(t) \propto (t_0 - t)^{\frac{2}{3(1+w)}}$

When $t = t_0$, $a(t)$ **diverges: Big Rip singularity**

In case $w = -1$, $a(t) \propto a_0 \mathrm{e}^{H_0 t}$, $H_0 \equiv \frac{\rho_0 \kappa^2}{3}$, **de Sitter space-time**

Big Rip

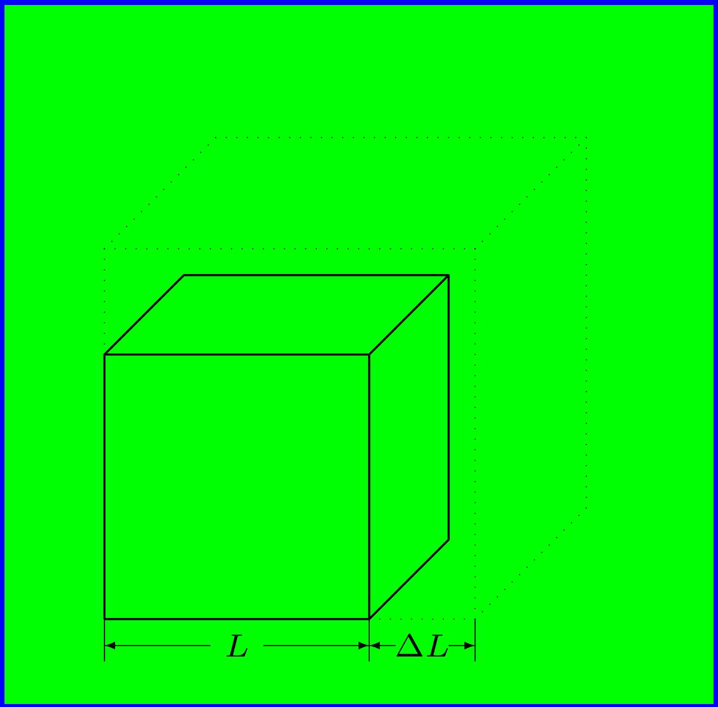
Big Rip

Big ~~L~~ip

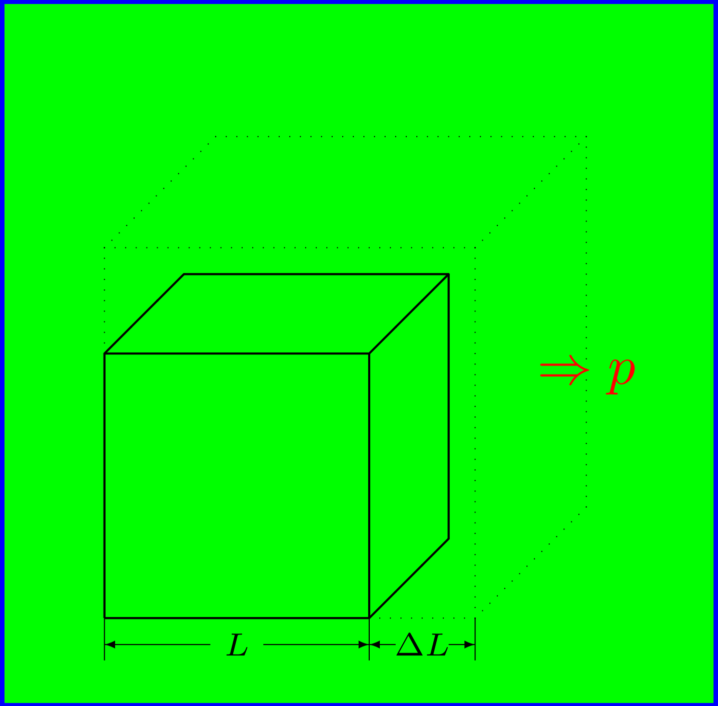
Everything, atoms, hadrons, quark, and etc. are ripped or torn.

Negative Pressure

Cube with sides $L \Rightarrow$ Each side becomes longer by ΔL



Internal energy U changes due to pressure p.



$$\Delta U \sim -p\Delta V \sim -3pL^2 \Delta L$$

If the pressure is negative, the internal energy increases by the expansion!

The total energy is NOT conserved if we take into account for gravity!

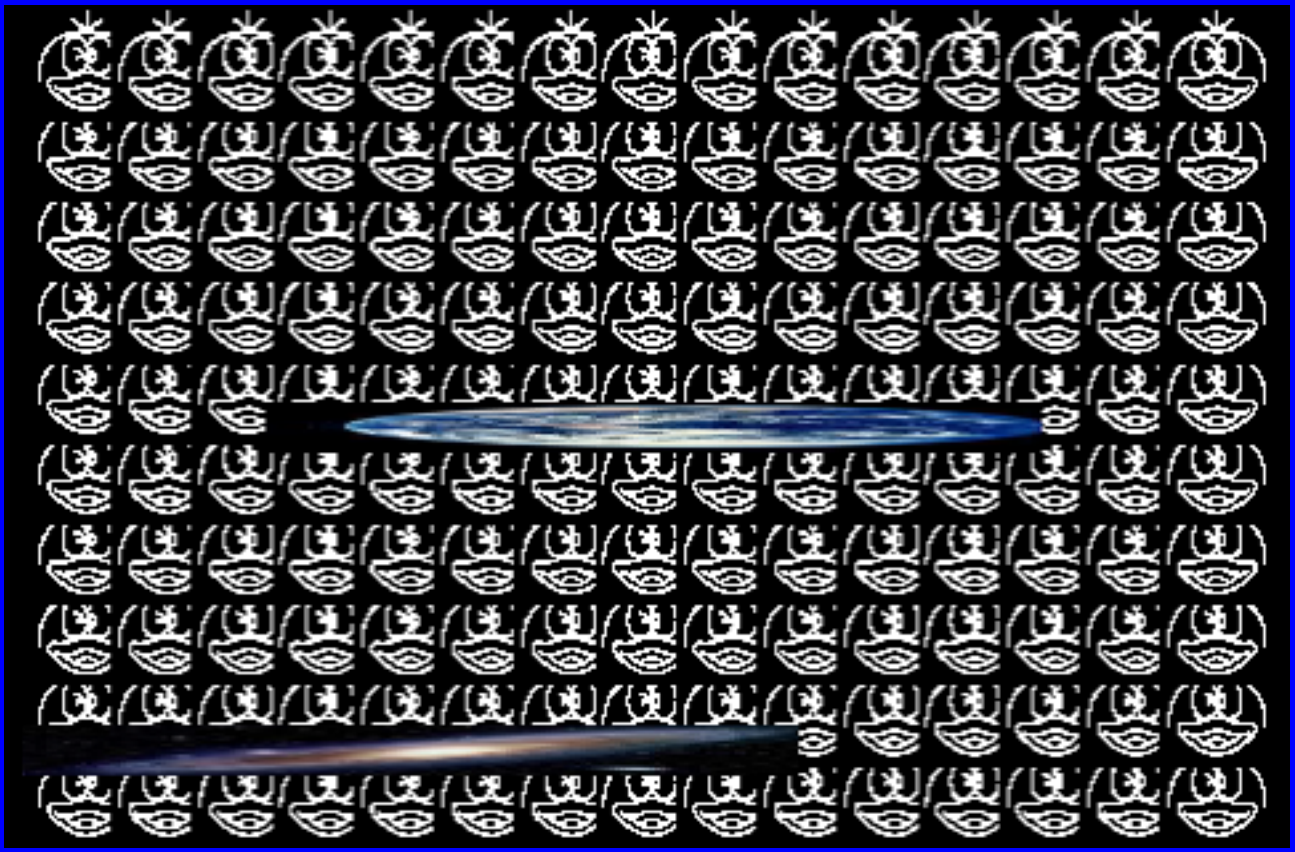
If there are phantoms in the universe, the density of the phantom increases by the expansion of the universe.

$$H = \frac{\dot{a}}{a} \propto \sqrt{\rho}$$

If the density of the phantoms increases, the ratio H of the expansion also increases.

If there is a phantom,

Phantoms grow up $\Rightarrow$ H increases



The universe is fulfilled by phantoms.


Finally the universe is ripped!

Fine-tuning problem and Coincidence problem

Fine-tuning problem, Coincidence problem:
The definitions slightly depend on persons.

A. 1st and 2nd FRW equations ($K = 0$)

$$0 = -\frac{3}{\kappa^2}H^2 + \frac{\Lambda}{2\kappa^2} + \rho_{\mathrm{matter}} \,, \quad 0 = \frac{1}{\kappa^2}\left(2\frac{dH}{dt} + 3H^2\right) - \frac{\Lambda}{2\kappa^2} + p_{\mathrm{matter}} \,,$$

Λ: cosmological constant
If the dark energy is the cosmological term, the cosmological constant is unnaturally small.

$$\Lambda \sim \left(10^{-33}\,\mathrm{eV}\right)^2 \ll M_{\mathrm{Planck}} \sim 1/\kappa \sim 10^{19}\,\mathrm{GeV} = 10^{28}\,\mathrm{eV}$$

B. Anthropic principle?

$\dfrac{\Lambda}{2\kappa^2} \sim \rho_{\mathrm{matter}}$ **(including dark matter)** **Very accidental! if Λ is a constant**

Age of the Universe: 13.7 billion years

$$\sim \left(10^{-33}\,\mathrm{eV}\right)^{-1} \sim \Lambda^{-\frac{1}{4}}$$

Present temperature of the Universe: $(3K)$

$$\sim 10^{-3}\,\mathrm{eV} \sim (\rho_{\mathrm{matter}})^{1/4} \sim \left(\frac{\Lambda}{2\kappa^2}\right)^{1/4}$$

$\Rightarrow$ **Dark energy might be dynamical?**

C. Initial condition?
If the dark energy is a perfect fluid whose EoS parameter $w \sim -1$,

$$\rho_{\rm DE} = \rho_{\rm DE\,0} a^{-3(1+w)} \sim \rho_{\rm DE\,0}$$

Usual matter or CDM (dust with $w = 0$)

$$\rho_{\rm matter} = \rho_{\rm matter\,0} a^{-3}$$

Ratio of densities of the dark energy to usual matter and dark matter

$$\rho_{\rm DE} / \rho_{\rm matter} \sim \left(\rho_{\rm DE\,0} / \rho_{\rm matter\,0} \right) a^{-3}$$

In order that $\rho_{\mathrm{DE}\,0} \sim \rho_{\mathrm{matter}\,0}$ in the present Universe,

because the ratio is given by $\rho_{\mathrm{DE}}/\rho_{\mathrm{matter}} \sim a^{-3}$,
When transparent to radiation ($a \sim 10^{-3}$), for example:

$$\rho_{\mathrm{DE}}/\rho_{\mathrm{matter}} \sim 10^{-9}$$

We need to fine-tune the initial condition of the ratio.

There might be a model where the dark matter interacts with dark energy and there is a transition between them?
The EoS parameter of the dark energy changes dynamically depending on the expansion (tracker model)?

D. If the dark energy is the vacuum energy,

the quantum corrections from the matter diverge $\sim \Lambda_{\rm cutoff}^4$**.**

$$\rho_{\rm vacuum} = \frac{1}{(2\pi)^3}\int d^4k \frac{1}{2}\sqrt{k^2+m^2} \sim \Lambda_{\rm cutoff}^4 \, .$$

$\Lambda_{\rm cutoff}$**: cutoff scale**

If the supersymmetry is restored in the high energy,
the vacuum energy by the quantum corrections $\sim \Lambda_{\rm cutoff}^2 \Lambda_{\cancel{\rm SUSY}}^2$

$$\rho_{\rm vacuum} = \frac{1}{(2\pi)^3}\int d^4k \frac{1}{2}\left(\sqrt{k^2+m_{\rm boson}^2} - \sqrt{k^2+m_{\rm fermion}^2}\right) \sim \Lambda_{\rm cutoff}^2 \Lambda_{\cancel{\rm SUSY}}^2 \, .$$

$\Lambda_{\cancel{\rm SUSY}}$**: the scale of the supersymmetry breaking.**
$\Lambda_{\cancel{\rm SUSY}}^2 = m_{\rm boson}^2 - m_{\rm fermion}^2$**.**

If we use the counter term in order to obtain the very small vacuum energy $\left(10^{-3}\,\mathrm{eV}\right)^4$, we need very very fine-tuning and extremely unnatural.

Maybe we do not understand quantum gravity?

The Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu}$$

What we consider the dark energy as a perfect fluid filling the Universe corresponds to the modification of the energy momentum tensor $T_{\mu\nu}$ of matters, which appears in the r.h.s. in the Einstein equation. On the other hand, there are many models to consider the modification of the Einstein tensor in the l.h.s., which are called modified gravity models.

$F(R)$ gravity, scalar-tensor theory (Brans-Dicke type model), Gauss-Bonnet gravity, $F(G)$ gravity, massive gravity, bigravity...

Recently there have been remarkable progresses in the study of massive gravity and bigravity.

Modified gravities

Scalar-tensor theory

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right\}\ , \quad V(\phi)\sim\phi^{-n}\text{: } \textbf{tracker}\,.$$

Brans-Dicke type

$$S = \int d^4x\sqrt{-g}\left\{f(\phi)R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right\}\ .$$

k-essence (X-matter)

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R - F\left(\partial_\mu\phi\partial^\mu\phi\right)\right\}\ ,$$

$$F\left(\partial_\mu\phi\partial^\mu\phi\right)\ \rightarrow\ F\left(\partial_\mu\phi\partial^\mu\phi,\phi\right)\text{: } \textbf{generalized } k\textbf{-essence}\,.$$

Scalar-Gauss-Bonnet gravity

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) - f(\phi)G\right\} ,$$
$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} .$$

$F(R)$ **gravity**

$$S = \int d^4x\sqrt{-g}F(R) .$$

$F(G)$ **gravity**

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R + F(G)\right\} .$$

Galileon model, Born-Infeld gravity, massive gravity, bigravity...

$F(R)$ gravity

The Einstein-Hilbert action:

$$S_{\rm EH} = \int d^4x \sqrt{-g} \left(\frac{R}{2\kappa^2} - \frac{\Lambda}{2\kappa^2} + \mathcal{L}_{\rm matter} \right) \,,$$

The action of $F(R)$ gravity:

$$S_{F(R)} = \int d^4x \sqrt{-g} \left(\frac{F(R)}{2\kappa^2} + \mathcal{L}_{\rm matter} \right) \,.$$

$F(R)$ **: a function of the scalar curvature** R**.**

Effective EoS parameter: $w_{\rm eff} = -1 - \dfrac{2\dot{H}}{3H^2}\,.$

$F(R)$ **equation:**

$$\frac{1}{2}g_{\mu\nu}F(R) - R_{\mu\nu}F'(R) - g_{\mu\nu}\Box F'(R) + \nabla_\mu\nabla_\nu F'(R) = -\frac{\kappa^2}{2}T_{\mathrm{matter}\,\mu\nu}\,.$$

Assuming the FRW universe ($K=0$),

$$0 = -\frac{F(R)}{2} + 3\left(H^2 + \dot H\right)F'(R) - 18\left(4H^2\dot H + H\ddot H\right)F''(R) + \kappa^2\rho_{\mathrm{matter}}\,,$$

$$\begin{aligned} 0 =& \frac{F(R)}{2} - \left(\dot H + 3H^2\right)F'(R) + 6\left(8H^2\dot H + 4\dot H^2 + 6H\ddot H + \dddot H\right)F''(R) \\ &+ 36\left(4H\dot H + \ddot H\right)F'''(R) + \kappa^2 p_{\mathrm{matter}}\,. \end{aligned}$$

Exact solution:

If we assume the Ricci curvature is covariantly constant ($\nabla_\rho R_{\mu\nu} = 0$),

$$R_{\mu\nu} = (R_0/4) g_{\mu\nu} \,, \quad (R = R_0 \ \textbf{constant})$$

$$\Rightarrow \ \textbf{Algebraic equation:} \quad 0 = 2F(R_0) - R_0 F'(R_0) \,.$$

If there is a solution, the (anti-)de Sitter space-time, Schwarzschild-(anti-)de Sitter space-time, and Kerr-(anti-)de Sitter space-time are exact vacuum solutions ($T_{\mu\nu} = 0$).

When $F(R) \propto f_0 R^m$**,**

$$0 = f_0 \left\{ -\frac{1}{2} \left(6\dot{H} + 12H^2 \right)^m + 3m \left(\dot{H} + H^2 \right) \left(6\dot{H} + 12H^2 \right)^{m-1} \right.$$
$$\left. -3mH \frac{d}{dt} \left\{ \left(6\dot{H} + 12H^2 \right)^{m-1} \right\} \right\} + \kappa^2 \rho_0 a^{-3(1+w)} .$$

In case $\rho_0 = 0$**:**

$$H = \frac{-\frac{(m-1)(2m-1)}{m-2}}{t} \quad \Rightarrow \quad w_{\mathrm{eff}} = -\frac{6m^2 - 7m - 1}{3(m-1)(2m-1)} .$$

When we include a matter with EoS parameter w,

$$a = a_0 t^{h_0}\,, \qquad h_0 \equiv \frac{2m}{3(1+w)}\,,$$

$$a_0 \equiv \left[-\frac{3 f_0 h_0}{\kappa^2 \rho_0} \left(-6h_0 + 12h_0^2\right)^{m-1} \left\{ \left(1-2m\right)\left(1-m\right) - (2-m)h_0 \right\} \right]^{-\frac{1}{3(1+w)}} .$$

Effective EoS parameter:

$$w_{\mathrm{eff}} = -1 + \frac{w+1}{m}\,.$$

Scalar-tensor expression of $F(R)$ gravity

Auxiliary field $A \Rightarrow$

$$S = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left\{ F'(A) \left(R - A \right) + F(A) \right\} .$$

Variation of $A \Rightarrow A = R \Rightarrow$ **original action**
Scale transformation: $g_{\mu\nu} \to \mathrm{e}^\sigma g_{\mu\nu}$, $\sigma \equiv - \ln F'(A)$

$$\begin{aligned} \Rightarrow \ S_E =& \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left(R - \frac{3}{2} g^{\rho\sigma} \partial_\rho \sigma \partial_\sigma \sigma - V(\sigma) \right) , \\ V(\sigma) =& \mathrm{e}^\sigma \mathcal{G}\left(\mathrm{e}^{-\sigma}\right) - \mathrm{e}^{2\sigma} f\left(\mathcal{G}\left(\mathrm{e}^{-\sigma}\right)\right) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} . \end{aligned}$$

Action in the Einstein frame
$\mathcal{G}\left(\mathrm{e}^{-\sigma}\right)$ **:** $\sigma = - \ln F'(A) \Rightarrow A = \mathcal{G}\left(\mathrm{e}^{-\sigma}\right)$

Via the metric, there appears an interaction between the scalar field σ and matter.
Mass of σ:

$$m_\sigma^2 \equiv \frac{3}{2}\frac{d^2V(\sigma)}{d\sigma^2} = \frac{3}{2}\left\{\frac{A}{F'(A)} - \frac{4F(A)}{\left(F'(A)\right)^2} + \frac{1}{F''(A)}\right\} .$$

If m_σ is small, by the propagation of σ, there appears a large correction to the Newton law.
Naively $m_\sigma \sim H$, very small.

W. Hu and I. Sawicki, "Models of f(R) Cosmic Acceleration that Evade Solar-System Tests,"
Phys. Rev. D 76, 064004 (2007) [arXiv:0705.1158 [astro-ph]].
Model using Chameleon mechanism.

There is a problem that the curvature easily becomes large in the Hu-Sawicki model.
The model contains a small parameter $m \sim 10^{-33}\,\mathrm{eV}$
When $R \gg m^2$**,**

$$F(R) = R - c_1 m^2 + \frac{c_2 m^{2n+2}}{R^n} + \mathcal{O}\left(R^{-2n}\right) .$$

c_1, c_2, n: **dimensionless constants.**

$$V(\sigma) \sim \frac{c_1 m^2}{A^2} .$$

$R = A \to \infty \Leftrightarrow$ **small value of the potential**
$\Rightarrow$ **large curvature appears easily.**

Assume that when R is large, $F(R)$ behaves as

$$F(R) \sim F_0 R^{\epsilon} \,.$$

F_0、ϵ**: positive constants,** $\epsilon > 1$。

Potential $V(\sigma)$

$$V(\sigma) \sim \frac{\epsilon - 1}{\epsilon^2 F_0 A^{\epsilon - 2}} \,.$$

If $1 < \epsilon < 2$**, when** $R = A \to \infty$**,** $V(\sigma) \to +\infty$**.**
$\Rightarrow$ **large curvature cannot appear easily.**

If $\epsilon = 2.$**,** $R \to \infty$ で $V(\sigma) \to 1/F_0$
$\Rightarrow$ **If** $1/F_0$ **is large enough, the large curvature could not appear.**

Anti-gravity problem

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left\{F'(A)\left(R-A\right)+F(A)\right\}\ .$$

$F'(R)<0 \Leftrightarrow$ **anti-gravity** $\Rightarrow$ **We require** $F'(R)>0$**.**

$$\frac{dV(\sigma)}{dA} = \frac{F''(A)}{F'(A)^3}\left(-AF'(A)+2F(A)\right)\ ,$$

Condition that the scalar field σ **can have a stationary value:**

$$0 = -AF'(A)+2F(A)\,,$$

identical with the condition that de Sitter space-time is a solution.

When the above condition for the de Sitter space-time is satisfied,

$$m_\sigma^2 = \frac{3}{2F'(A)}\left(-A + \frac{F'(A)}{F''(A)}\right) ,$$

if $F'(R) > 0$ **and**

$$-A + \frac{F'(A)}{F''(A)} > 0 \, ,$$

we have $m_\sigma^2 > 0 \Leftrightarrow$ **at the minimum of the potential.**

If

$$-A + \frac{F'(A)}{F''(A)} < 0 \, ,$$

the maximum of the potential

$\Rightarrow$ **de Sitter space-time solution is unstable.**

In case $-A + \frac{F'(A)}{F''(A)} = 0$**,**

$$0 = -RF'(R) + 2F(R) \;\Leftrightarrow\; 0 = \frac{d}{dR}\left(\frac{F(R)}{R^2}\right) .$$

let a solution $R = R_0$ **and assume** $F(R)$ **behaves as**

$$\frac{F(R)}{R^2} = f_0 + f(R)\left(R - R_0\right)^n .$$

f_0**: constant,** n**: an integer greater than** 2**, assume** $f(R_0) \neq 0$**.**
Assume $F(0) = 0$**,** $F'(R) > 0 \Rightarrow F(R) > 0 \ (R > 0)$**,** $f_0 > 0$

$n = 2$

$$-R_0 + \frac{F'(R_0)}{F''(R_0)} = -\frac{f(R_0)A_0}{f_0 + f(A_0)} ,$$

$\Rightarrow$ **If** $-f_0 < f(A_0) < 0$**, de Sitter space-time is a stable solution.**

$n \geq 3$

$$-R_0 + \frac{F'(R_0)}{F''(R_0)} = 0\,.$$

Check the sign of m_σ^2 around $R \sim R_0$:

$$m_\sigma^2 \sim -\frac{3n(n-1)R_0^2 f(R_0)}{2f_0^2}\left(R - R_0\right)^{n-2}\,.$$

When n even, if $f(R_0) < 0$ stable, if $f(R_0) > 0$ unstable.
When n odd, always unstable.
When $f(R_0) < 0$ $(f(R_0) > 0)$,
If $R > R_0$, $m_\sigma^2 > 0$ $\left(m_\sigma^2 < 0\right)$, if $R < R_0$, $m_\sigma^2 < 0$ $\left(m_\sigma^2 > 0\right)$
$\Rightarrow$
When $f(R_0) < 0$, R decreases.
When $f(R_0) > 0$, R increases.

In general, the time development of the universe in the Einstein frame is different from that in the original frame (the Jordan frame) because the time coordinate is also transformed under the scale transformation.

For example, in the Einstein frame, the time development of the universe corresponding to the fluid with $w < -1$ (phantom) cannot be realized but the development can be realized in the Jordan frame. The Big Rip time $t = t_0$ in the Jordan frame is transformed into the infinite future $t \to \infty$.

Reconstruction of $F(R)$

Usually we start from a given model and investigate the development of the universe etc. by using the given equations. Here we consider the inverse, that is, for a given development of the universe, we construct a model which reproduces the development, which we call "reconstruction".

Reconstruction using cosmological time

Rewrite the action as follows,

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left(P(\phi) R + Q(\phi) \right) + \mathcal{L}_{\mathrm{matter}} \right\} .$$

P、Q**: functions of the auxiliary scalar field** ϕ**.**
Variation of $\phi \Rightarrow 0 = P'(\phi)R + Q'(\phi) \Rightarrow \phi = \phi(R)$。

$$S = \int d^4x \sqrt{-g} \left\{ \frac{F(R)}{2\kappa^2} + \mathcal{L}_{\mathrm{matter}} \right\} , \quad F(R) \equiv P(\phi(R))R + Q(\phi(R)) .$$

Variation of $g_{\mu\nu} \Rightarrow \sim$ FRW equations

$$0 = -6H^2 P(\phi) - Q(\phi) - 6H\frac{dP(\phi(t))}{dt} + 2\kappa^2 \rho_{\rm matter}\, ,$$
$$0 = \left(4\dot H + 6H^2\right) P(\phi) + Q(\phi) + 2\frac{d^2 P(\phi(t))}{dt} + 4H\frac{dP(\phi(t))}{dt} + 2\kappa^2 p_{\rm matter}\, .$$

Deleting $Q(\phi)$,

$$0 = 2\frac{d^2 P(\phi(t))}{dt^2} - 2H\frac{dP(\phi(t))}{dt} + 4\dot H P(\phi) + 2\kappa^2 \left(p_{\rm matter} + \rho_{\rm matter}\right)\, .$$

We can always redefine the scalar field ϕ, we can choose $\phi = t$.
In many cases, $\rho_{\rm matter}$ and $p_{\rm matter}$ are given by the sum of the contributions from the matters with constant w_i.

By using a function $g(t)$, we write the scale factor as $a = a_0 \mathrm{e}^{g(t)}$ (a_0: constant)
$\Rightarrow$ 2nd order differential equation:

$$\begin{aligned} 0 =& 2\frac{d^2 P(\phi)}{d\phi^2} - 2g'(\phi)\frac{dP(\phi))}{d\phi} + 4g''(\phi)P(\phi) \\ &+ 2\kappa^2 \sum_i (1+w_i)\,\rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)g(\phi)} \,. \end{aligned}$$

In principle, we can find $P(\phi)$ by solving this differential equation.

$$Q(\phi) = -6\left(g'(\phi)\right)^2 P(\phi) - 6g'(\phi)\frac{dP(\phi)}{d\phi} + 2\kappa^2 \sum_i \rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)g(\phi)} \,.$$

For arbitrary development given by $a = a_0 \mathrm{e}^{g(t)}$, we can find the form of $F(R)$ which generates the development.

Example: Cosmological constant + matter with w (ΛCDM for $w = 0$):

$$a = a_0 \mathrm{e}^{g(t)}\,, \qquad g(t) = \frac{2}{3(1+w)} \ln\left(\alpha \sinh\left(\frac{3(1+w)}{2l}(t - t_s)\right)\right)\,.$$

t_s**: constant of the integration.** $\alpha^2 \equiv \frac{1}{3}\kappa^2 l^2 \rho_0 a_0^{-3(1+w)}$**,**
l**: length parameter (cosmological constant:** $\Lambda = 6/l^2$**)**

$$\begin{aligned}\Rightarrow\quad 0 =& 2\frac{d^2 P(\phi)}{d\phi^2} - \frac{2}{l}\coth\left(\frac{3(1+w)}{2l}(\phi - t_s)\right)\frac{dP(\phi)}{d\phi} \\ & - \frac{6(1+w)}{l^2}\sinh^{-2}\left(\frac{3(1+w)}{2l}(\phi - t_s)\right)P(\phi) \\ & + \frac{4}{3}\rho_{r0} a_0^{-4}\left(\alpha\sinh\left(\frac{3(1+w)}{2l}(\phi - t_s)\right)\right)^{-8/3(1+w)} \\ & + \rho_{d0} a_0^{-3}\left(\alpha\sinh\left(\frac{3(1+w)}{2l}(\phi - t_s)\right)\right)^{-2/(1+w)}\,.\end{aligned}$$

Neglect the contribution from matter and change of the variable from ϕ to z:

$$z \equiv -\sinh^{-2}\left(\frac{3(1+w)}{2l}(t-t_s)\right) ,$$

$\Rightarrow$ **Gauss' hypergeometric differential equation:**

$$0 = z(1-z)\frac{d^2P}{dz^2} + \left[\tilde{\gamma} - \left(\tilde{\alpha} + \tilde{\beta} + 1\right) z\right] \frac{dP}{dz} - \tilde{\alpha}\tilde{\beta}P ,$$

$$\tilde{\gamma} \equiv 4 + \frac{1}{3(1+w)} , \ \tilde{\alpha} + \tilde{\beta} + 1 \equiv 6 + \frac{1}{3(1+w)} , \ \tilde{\alpha}\tilde{\beta} \equiv -\frac{1}{3(1+w)} .$$

Solution: hypergeometric function

$$P = P_0 F(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}; z) \equiv P_0 \frac{\Gamma(\tilde{\gamma})}{\Gamma(\tilde{\alpha})\Gamma(\tilde{\beta})} \sum_{n=0}^{\infty} \frac{\Gamma(\tilde{\alpha}+n)\Gamma(\beta+n)}{\Gamma(\tilde{\gamma}+n)} \frac{z^n}{n!} \,.$$

Γ**: gamma function**

One more linearly independent solution:
$(1-z)^{\tilde{\gamma}-\tilde{\alpha}-\tilde{\beta}} F(\tilde{\gamma}-\tilde{\alpha}, \tilde{\gamma}-\tilde{\beta}, \tilde{\gamma}; z)$**, which we forget just for the simplicity.**

$$Q = -\frac{6(1-z)P_0}{l^2} F(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}; z) - \frac{3(1+w)z(1-z)P_0}{l^2(13+12w)} F(\tilde{\alpha}+1, \tilde{\beta}+1, \tilde{\gamma}+1; z) \,.$$

Reconstruction by using e-folding

Reconstruction using e-folding instead of the cosmological time t.
e-folding : $N = \ln \frac{a}{a_0}$, redshift z : $\mathrm{e}^{-N} = \frac{a_0}{a} = 1 + z$.
Redshift z: directly related with the observation

By using $\frac{d}{dt} = H \frac{d}{dN}$、 $\frac{d^2}{dt^2} = H^2 \frac{d^2}{dN^2} + H \frac{dH}{dN} \frac{d}{dN}$ etc.

$$
\begin{aligned}
0 = & -\frac{F(R)}{2} + 3\left(H^2 + HH'\right) F'(R) \\
& - 18\left(4H^3H' + H^2\left(H'\right)^2 + H^3H''\right) F''(R) + \kappa^2 \rho_{\mathrm{matter}} \,.
\end{aligned}
$$

$H' \equiv dH/dN$、 $H'' \equiv d^2H/dN^2$。

Assume the Hubble rate H as a function $g(N)$ of N:

$$H = g(N) = g\left(-\ln\left(1+z\right)\right) \,.$$

Scalar curvature: $R = 6g'(N)g(N) + 12g(N)^2 \Rightarrow N = N(R)$。

$$\begin{aligned} 0 =& -18\left(4g\left(N\left(R\right)\right)^3 g'\left(N\left(R\right)\right) + g\left(N\left(R\right)\right)^2 g'\left(N\left(R\right)\right)^2 \right. \\ & \left. + g\left(N\left(R\right)\right)^3 g''\left(N\left(R\right)\right)\right)\frac{d^2F(R)}{dR^2} \\ & + 3\left(g\left(N\left(R\right)\right)^2 + g'\left(N\left(R\right)\right)g\left(N\left(R\right)\right)\right)\frac{dF(R)}{dR} - \frac{F(R)}{2} \\ & + \kappa^2\sum_i \rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)N(R)} \,. \end{aligned}$$

Differential equation w.r.t. $F(R)$ for the variable R.

$$G(N) \equiv g\left(N\right)^2 = H^2 \Rightarrow$$

$$\begin{aligned}
0 = & -9G\left(N\left(R\right)\right)\left(4G'\left(N\left(R\right)\right) + G''\left(N\left(R\right)\right)\right)\frac{d^2F(R)}{dR^2} \\
& + \left(3G\left(N\left(R\right)\right) + \frac{3}{2}G'\left(N\left(R\right)\right)\right)\frac{dF(R)}{dR} \\
& - \frac{F(R)}{2} + \kappa^2 \sum_i \rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)N(R)} \,.
\end{aligned}$$

$R = 3G'(N) + 12G(N)$。

Summary

- The present status of the dark energy and $F(R)$ is reviewed.
- $F(R)$ gravity: Iterations of the discoveries of the problem and their solutions.
- Relation with the higher dimensional models like superstring?

  Example: Low energy effective theory a la Brans-Dicke?

$$S = \int d^4x \sqrt{-g} \left\{ \frac{\mathrm{e}^{\phi}}{2\kappa^2} R - \frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi) + L_{\mathrm{matter}} \right\} .$$

  Neglecting kinetic term, varying ϕ, solving the equation with respect to ϕ as a function of R, and deleting ϕ, we obtain $F(R)$ model.

Dark energy

Too many problems, too many models.

???????

Anyway problem