Probability of boundary condition in quantum cosmology

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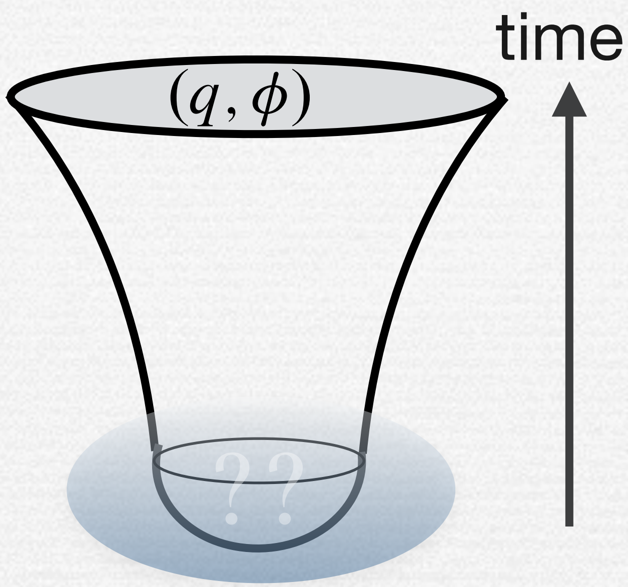
Introduction



Quantum cosmology (QC):

treat the universe as a single quantum system

- Canonical quantization of the universe

$$\hat{H}|\Psi\rangle = 0 \quad \text{Wheeler-DeWitt (WD) equation}$$

Quantum state of the universe is contained in
the wave function of the universe $\Psi[q] = \langle q|\Psi\rangle$

We expect to obtain origin and history of our universe
by analyzing the wave function of the universe

- There are several issues to be considered in QC:

problem time: how can we derive dynamics of the universe?

probability: conserved charge is not positive definite
how can we define probability?
Prediction of the wave function : relies on WKB analysis

boundary condition: how do we determine BC of WD eq.?

Boundary condition

Hartle-Hawking (HH): sum over compact Euclidean geometries

$$\Psi(q) = \int [dNdq] \exp(-S[q, N])$$

path integral dominated by regular Euclidean classical solutions

Vilenkin (V): wave function is purely outgoing at the infinity of superspace

tunneling type

(HH) prefers small values of cosmological constant

$$P(\phi) \sim \exp\left(\frac{1}{\Lambda(\phi)}\right)$$

(V) prefers large values of cosmological constant

$$P(\phi) \sim \exp\left(-\frac{1}{\Lambda(\phi)}\right)$$

Our present universe: large scale structure, isotropy of CMB

- we expect our universe has experienced **inflation** with $\mathcal{N} \geq 60$
- our universe has small value of cosmological constant

The purpose of QC is to explain these features of our universe

Purpose of this research

We want to say something about boundary conditions of WD eq. by imposing observational constraints

(HH) or (V) or others ?

- model: closed FRW universe with a massive scalar field with a cosmological constant (toy cosmological model)

- constraint: sufficient number of e-foldings of inflation

$$\mathcal{N} \geq 60$$

We investigate which type of BCs of the universe is preferable

Contents

Mini-superspace model

Mini-superspace model

A closed FRW universe + massive scalar, cosmological constant

action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R-2\Lambda) - \frac{1}{2}\int d^4x\sqrt{-g}\,\left[(\partial_\mu \Phi)^2 + m^2\Phi^2\right]$$

metric

$$ds^2 = \frac{3}{\Lambda}\left(-\frac{N^2}{q}d\lambda^2 + q\,d\Omega_3^2\right)$$

mini-superspace (q, ϕ)

Hamiltonian

$$H_T = \frac{KN}{2}\left[\frac{1}{K^2}\left(-4p_q^2 + \frac{p_\phi^2}{q^2}\right) - 1 + q(1+\mu^2\phi^2)\right] = NH$$

dimensionless parameters

$$\phi = \left(\frac{4\pi G}{3}\right)^{1/2}\Phi \qquad \mu = \left(\frac{3}{\Lambda}\right)^{1/2} m \qquad K = \frac{9\pi}{2G\Lambda}$$

Classical solutions

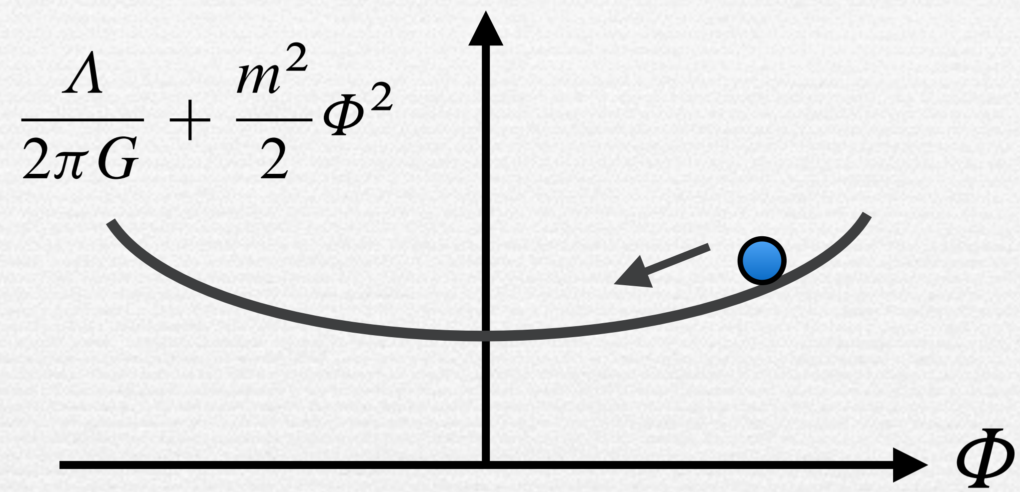
Hamiltonian constraint

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\Lambda}{3a^2} = \frac{\Lambda}{3} + \frac{4\pi G}{3}\left(\dot{\Phi}^2 + m^2\Phi^2\right)$$

scalar field eq.

$$\ddot{\Phi} + 3\left(\frac{\dot{a}}{a}\right)\dot{\Phi} + m^2\Phi = 0$$

inflaton potential



Inflationary solution

slow roll condition

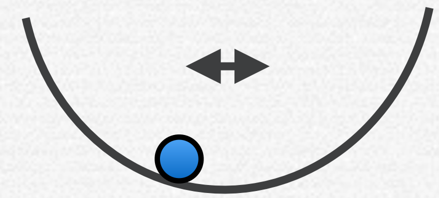
$$|\ddot{\Phi}| \lesssim \left(\frac{\dot{a}}{a}\right)|\dot{\Phi}|, \quad \dot{\Phi}^2 \lesssim m^2\Phi^2, \quad \frac{\Lambda}{3} \lesssim \frac{4\pi G}{3}m^2\Phi^2$$

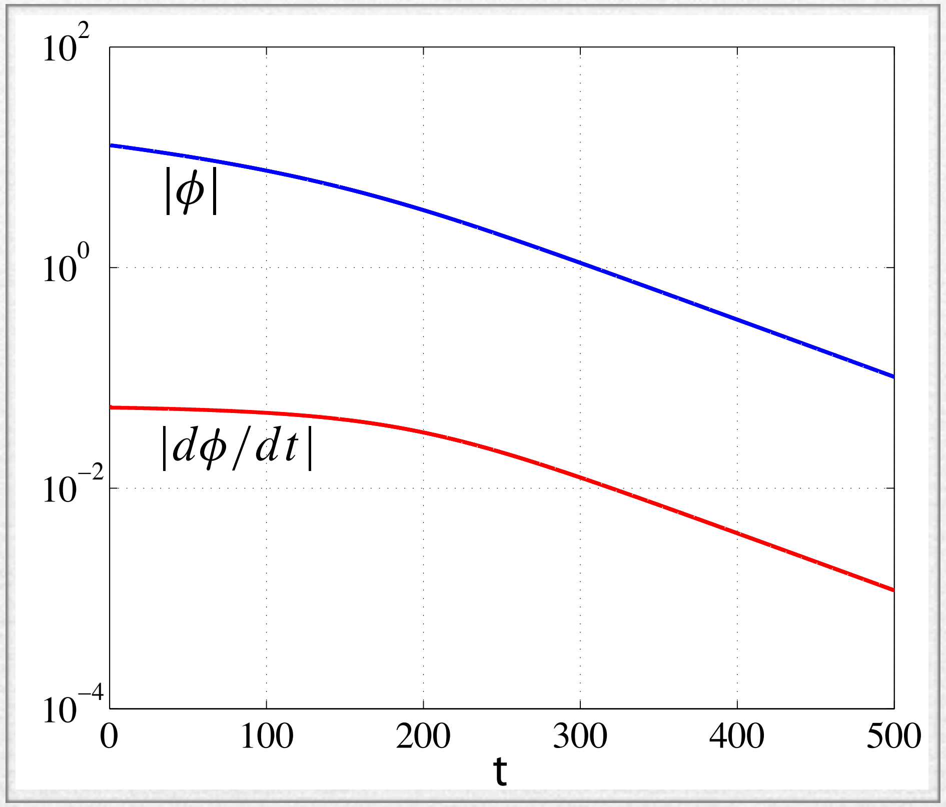
→ A universe expands with acceleration $\ddot{a} > 0$

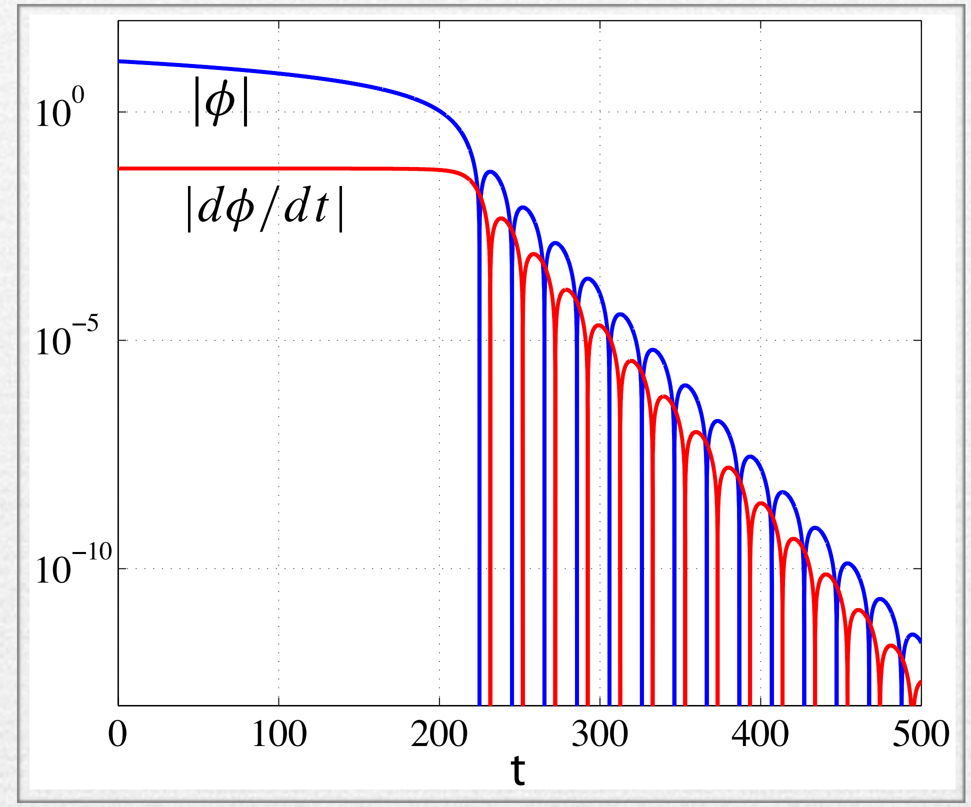
duration of inflation depends on initial values of ϕ

e-foldings $\mathcal{N} = \ln\left(\frac{a_{\mathrm{f}}}{a_{\mathrm{i}}}\right)$ ← predicted by the wave function of the universe small mass $\mu < 3,$

large mass $\mu > 3$







small m^2/Λ

slow roll ➡ over damp

large m^2/Λ

slow roll ➡ damped oscillation

The universe continues accelerated expansion forever
due to the cosmological constant in this model

Wheeler-DeWitt equation

Hamiltonian constraint

$$H(q, p_q, \phi, p_\phi) = 0$$

$$p_a \to -i\frac{\partial}{\partial q}, \quad p_\phi \to -i\frac{\partial}{\partial \phi}$$

$$\left[\frac{1}{2K^2}\left(4\frac{\partial^2}{\partial q^2} - \frac{1}{q^2}\frac{\partial^2}{\partial \phi^2}\right) - \frac{1}{2} + qV(\phi)\right]\Psi(q, \phi) = 0$$

$$V(\phi) \equiv \frac{1}{2} + \frac{\mu^2}{2}\phi^2 \qquad K = \frac{9\pi}{2G\Lambda} \qquad \mu = \left(\frac{3}{\Lambda}\right)^{1/2} m$$

As we cannot solve this equation analytically, we obtain the wave function numerically.

Two dimensional wave equation and can be solved with suitable BCs

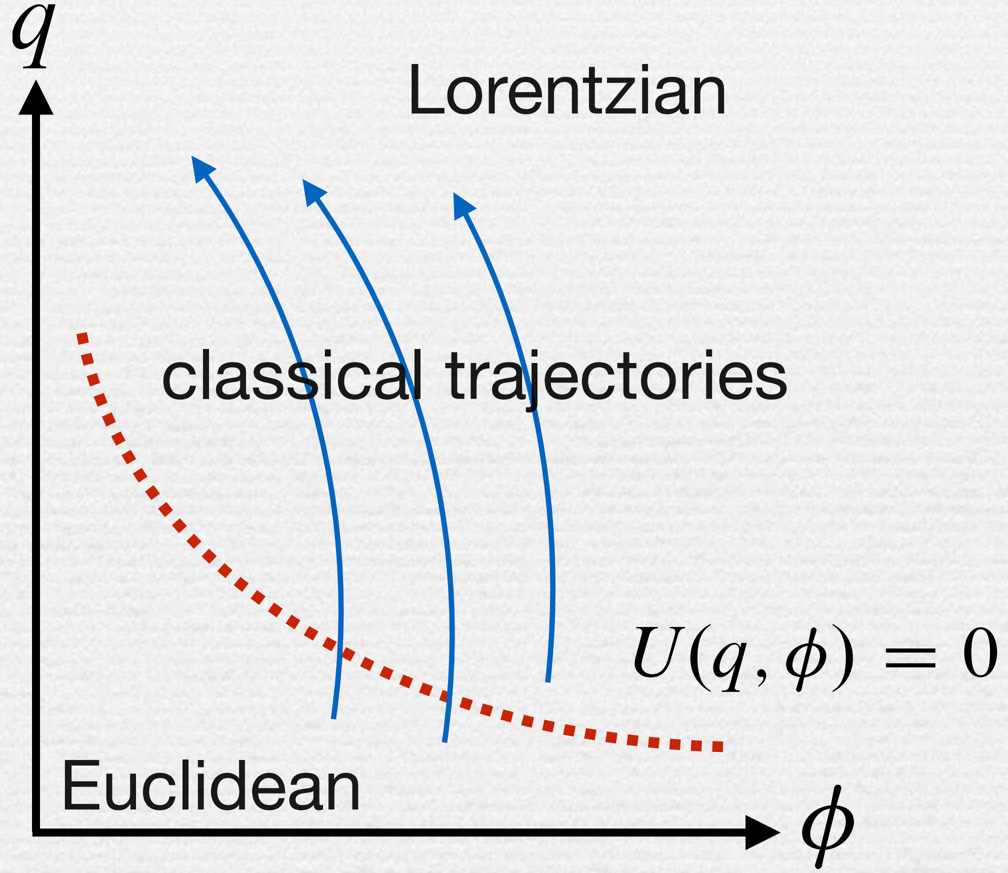
Structure of mini-superspace (q, ϕ)

scale factor matter field

WD equation (KG type eq.)

$$\left[-\frac{1}{2}G^{AB}\partial_A\partial_B + U(q,\phi)\right]\Psi(q,\phi) = 0$$

$$G^{AB} = \mathrm{diag}(-4, \frac{1}{q^2})$$
$$U(q,\phi) = -\frac{1}{2} + qV(\phi)$$



Lorentzian region: $U(q,\phi) > 0$

$q \gg 1 \qquad \Psi \sim e^{iS}$

Wave function has WKB form

"semi-classical" universe

Euclidean region: $U(q,\phi) < 0$

classically forbidden region

$q \ll 1 \qquad \Psi \sim e^{-S_E}$

"quantum" universe

How can wave functions predict classical trajectories (universe)?

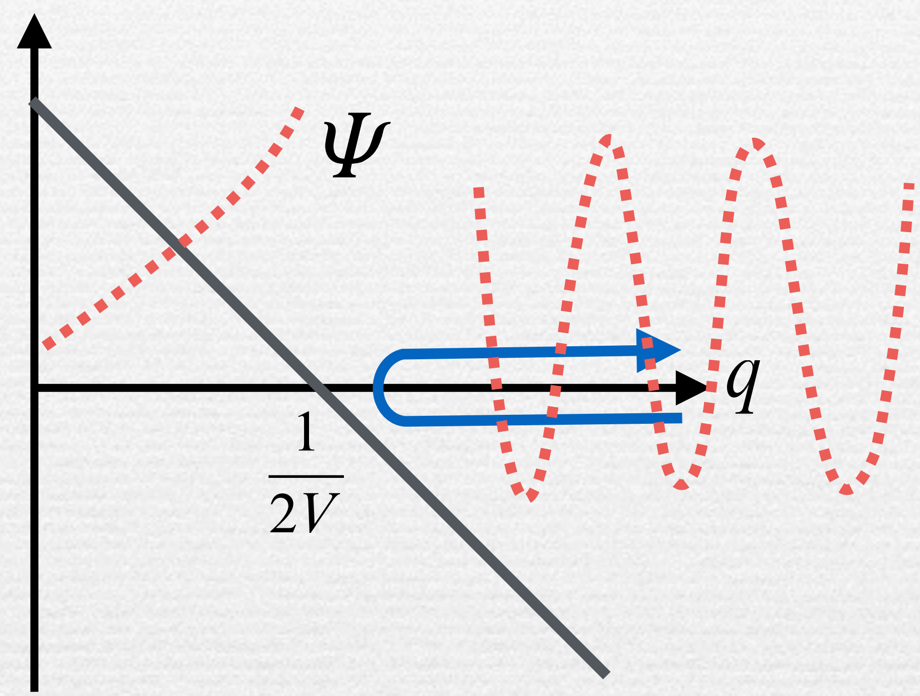
de Sitter case $V(\phi) = \mathrm{const.}$

$$\left(-8\frac{d^2}{dq^2} + \underbrace{1 - 2qV}_{\text{potential}}\right)\Psi(q) = 0$$

Schroedinger eq. with zero energy

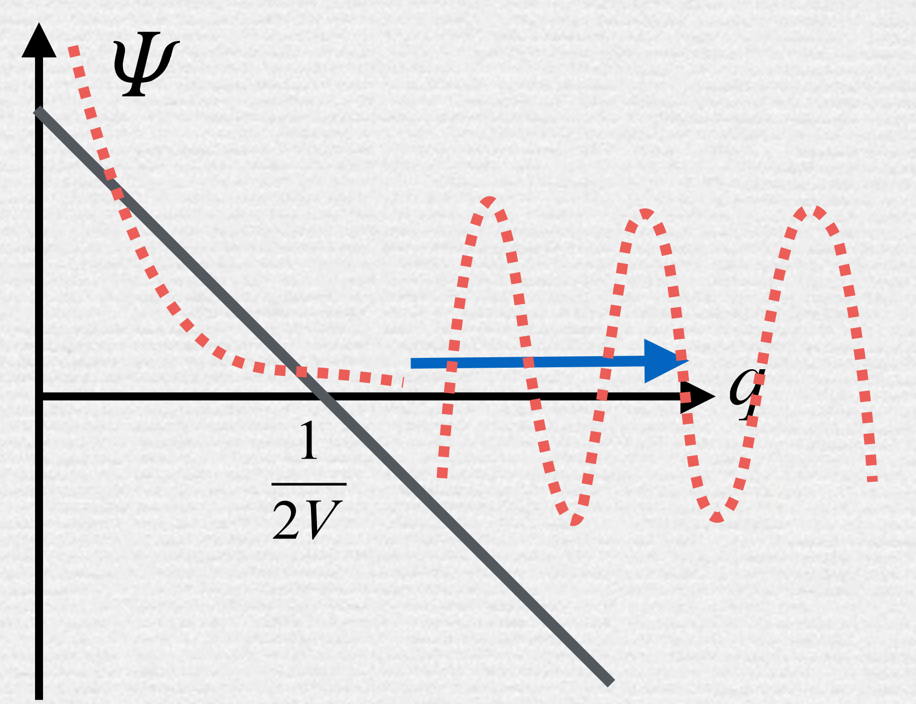
(HH): superposition of expanding and contracting universes

$$\Psi \sim e^{+q/2} \qquad \overset{q \gg 1}{\Psi \sim e^{iS} + e^{-iS}}$$



(V): purely outgoing wave tunneling type

$$\Psi \sim e^{-q/2} \qquad \overset{q \gg 1}{\Psi \sim e^{iS}}$$



de Sitter case $V(\phi) = \text{const.}$

Halliwell, Louko 1989

- General solutions of WD eq. in terms of Airy function

$$G(q|q_0) = c_1 \mathrm{Ai}(z_0)\mathrm{Ai}(z) + c_2 \mathrm{Bi}(z_0)\mathrm{Bi}(z) + c_3(\mathrm{Ai}(z_0)\mathrm{Bi}(z) + \mathrm{Bi}(z_0)\mathrm{Ai}(z))$$

(wave functions as transition amplitude from $q_0 \to q$)

$$z = z(q) = \left(\frac{4V}{K}\right)^{-2/3}(1-2qV), \quad z_0 = z(0) = \left(\frac{4V}{K}\right)^{-2/3}$$

typical wave functions

$$\boxed{\begin{aligned} \Psi_{\mathrm{HH}} &= \Psi_2 + \Psi_3 \\ \Psi_{\mathrm{V}} &= \Psi_1 + i\Psi_3 \end{aligned}}$$

$$\sim \exp\left(+\frac{K}{6V}\right)\cos S_0$$

$$\sim \exp\left(-\frac{K}{6V}\right)\exp(-iS_0)$$

$$\Psi_1 \equiv (2V)^{-1/3}\mathrm{Ai}(z_0)\mathrm{Ai}(z)$$

$$\Psi_2 \equiv (2V)^{-1/3}\mathrm{Bi}(z_0)\mathrm{Ai}(z)$$

$$\Psi_3 \equiv (2V)^{-1/3}\mathrm{Ai}(z_0)\mathrm{Bi}(z)$$

$$S_0(q,\phi) = \frac{K}{6V(\phi)}(2V(\phi)q-1)^{3/2} - \frac{\pi}{4}$$

(HH) and (V) can be represented using three functions (solutions)

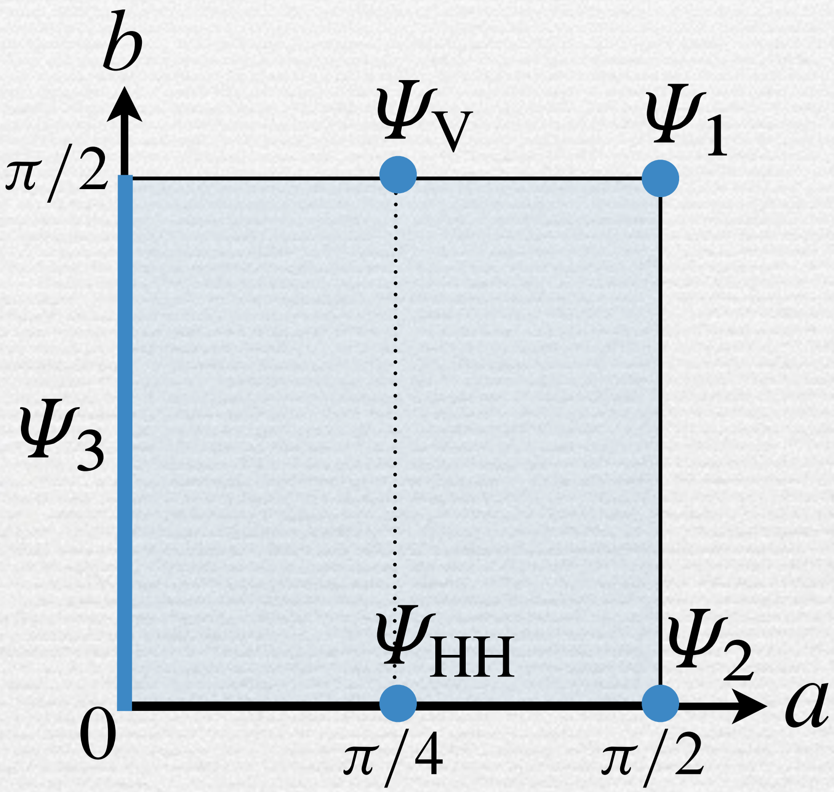
We parametrize solutions including (HH) and (V) using two real parameters

$$\Psi_C = \tan a(\cos b\, \Psi_2 - i \sin b\, \Psi_1) + \Psi_3, \qquad 0 \le a, b \le \pi/2$$

$$\Psi_1 \equiv (2V)^{-1/3}\mathrm{Ai}(z_0)\mathrm{Ai}(z)$$

$$\Psi_2 \equiv (2V)^{-1/3}\mathrm{Bi}(z_0)\mathrm{Ai}(z)$$

$$\Psi_3 \equiv (2V)^{-1/3}\mathrm{Ai}(z_0)\mathrm{Bi}(z)$$



space of BCs

wave function	parameter (a, b)	asymptotic form for $q \gg 1$
Ψ_HH	$(\frac{\pi}{4}, 0)$	$\sim \exp\left(+\frac{K}{6V}\right)\cos S_0$
Ψ_V	$(\frac{\pi}{4}, \frac{\pi}{2})$	$\sim \exp\left(-\frac{K}{6V}\right)\exp(-i S_0)$
Ψ_1	$(\frac{\pi}{2}, \frac{\pi}{2})$	$\sim \exp\left(-\frac{K}{6V}\right)\cos S_0$
Ψ_2	$(\frac{\pi}{2}, 0)$	$\sim \exp\left(+\frac{K}{6V}\right)\cos S_0$
Ψ_3	(0, any values)	$\sim -\exp\left(-\frac{K}{6V}\right)\sin S_0$

We specify BCs of WD eq. for non-constant potential case using this parametrization

WKB analysis and probability

Hartle, Hawking and Hertog 2008

$$\left[-\frac{1}{2}G^{AB}\partial_A\partial_B + U(q,\phi)\right]\Psi(q,\phi) = 0$$

WKB anzatz

$$\Psi(q^A) = C(q^A)e^{-\frac{1}{\hbar}I(q^A)}$$

phase function is complex in general

$$I = I_R - iS$$

$$O(\hbar^0): \quad -\frac{1}{2K^2}(\nabla I)^2 + U(q^A) = 0,$$

$$O(\hbar^1): \quad 2\nabla I \cdot \nabla C + C\nabla^2 I = 0,$$

conservation of current

If the condition holds

$$\frac{|\nabla I_R|^2}{|\nabla S|^2} \ll 1$$

"classicality" condition

we obtain the Hamilton-Jacobi equation

$$\frac{1}{2K^2}(\nabla S)^2 + U = 0. \qquad p_A = \frac{\partial S}{\partial q_A}$$

WKB wave function

$$\Psi(q^A) = \sum_{i=\mathrm{saddle}} C^{(i)}(q^A)e^{-I_R^{(i)}(q^A)}e^{iS^{(i)}(q^A)}$$

Conserved current of WD eq.

$$\mathcal{J}_A = \frac{i}{2}(\Psi^* \nabla_A \Psi - \Psi \nabla_A \Psi^*), \quad \nabla \cdot \mathcal{J}_A = 0.$$

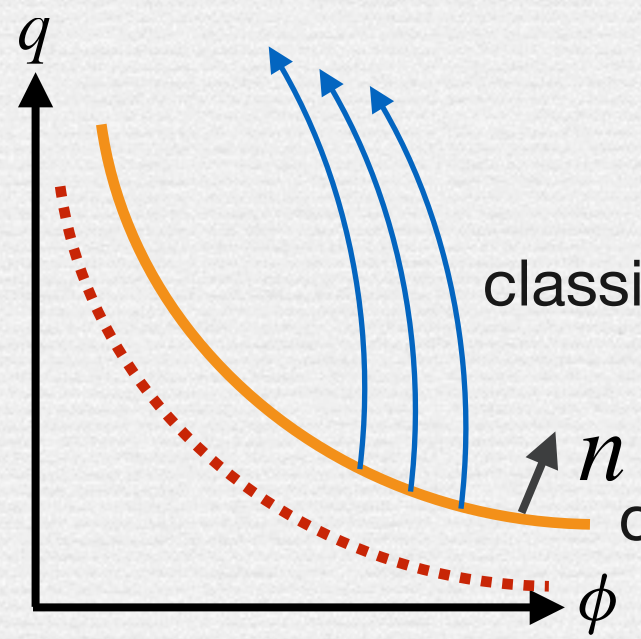
For WKB wave function,

$$J_A^{(i)} \equiv -|C^{(i)}|^2 \exp(-2I_R^{(i)}) \nabla_A S^{(i)} \qquad \nabla \cdot J^{(i)} = 0.$$

We consider a hypersurface Σ_c on which the classicality condition is satisfied.
Define probability measure for expanding universes on Σ_c by

$$\mathcal{P}(\phi) \equiv J^+ \cdot n = \underline{-|C|^2 \exp(-2I_R) \nabla_n S} \qquad p_q = -\partial_q S < 0$$

density of trajectories crossing Σ_c

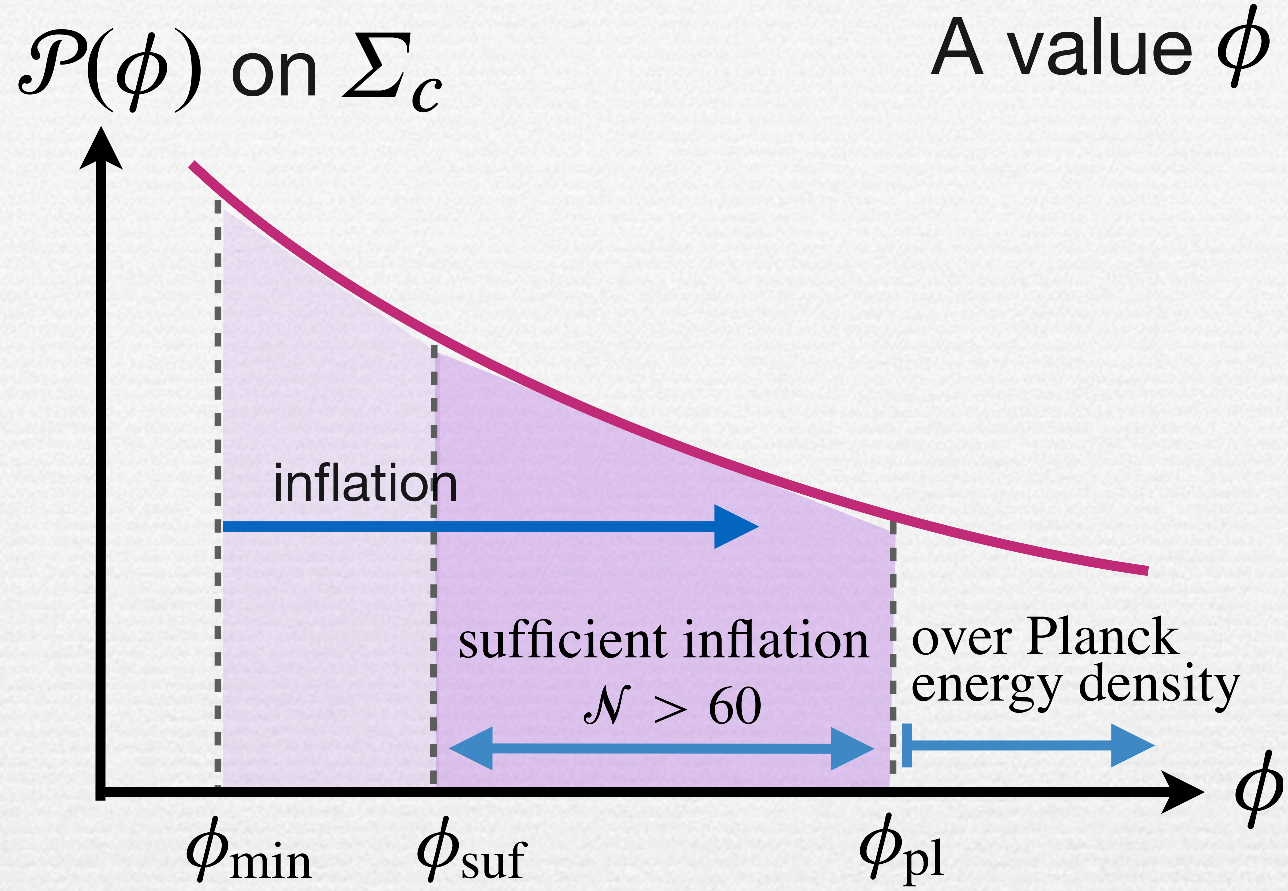


probability for ϕ

$\mathcal{P}(\phi)$

unnormalized

A value ϕ on Σ_c determines $\mathcal{N}$



Conditional probability for sufficient e-foldings of inflation

$$P_{\rm suf} \equiv P(\mathcal{N} \geq 60) = \frac{\int_{\phi_{\rm suf}}^{\phi_{\rm pl}} d\phi\, \mathcal{P}(\phi)}{\int_{\phi_{\rm min}}^{\phi_{\rm pl}} d\phi\, \mathcal{P}(\phi)}.$$

prob. of inflation with $\mathcal{N} > 60$

prob. of inflation

$\phi_{\rm min}$: end of inflation driven by scalar field

Probability of boundary conditions

Probability of boundary conditions

For a wave function with a specific BC, it is possible to obtain probability of sufficient inflation

$$P_{\mathrm{suf}} = P(S|B_i)$$

result cause

On the other hand,
probability of BC under the condition of sufficient inflation is

$$\boxed{P(B_i|S) = \frac{P(B_i)P(S|B_i)}{\sum_k P(B_k)P(S|B_k)}}$$

cause result

Bayes' theorem

$P(S|B_i)$ probability of sufficient inflation under a specific BC

$P(B_k)$ prior probability: assume uniformly distributed

cause

(principle of insufficient reason)

We represented $\{B_i\}$ using two parameters a, b

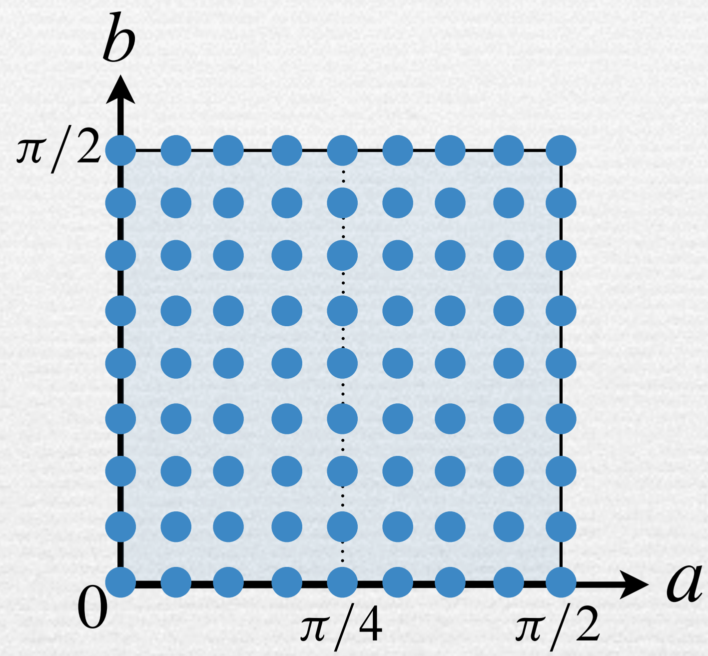
Probability of boundary condition under restriction of sufficient inflation

$$\boxed{P(a,b|S) = \frac{P(S|a,b)}{\int da' db' P(S|a',b')}}$$

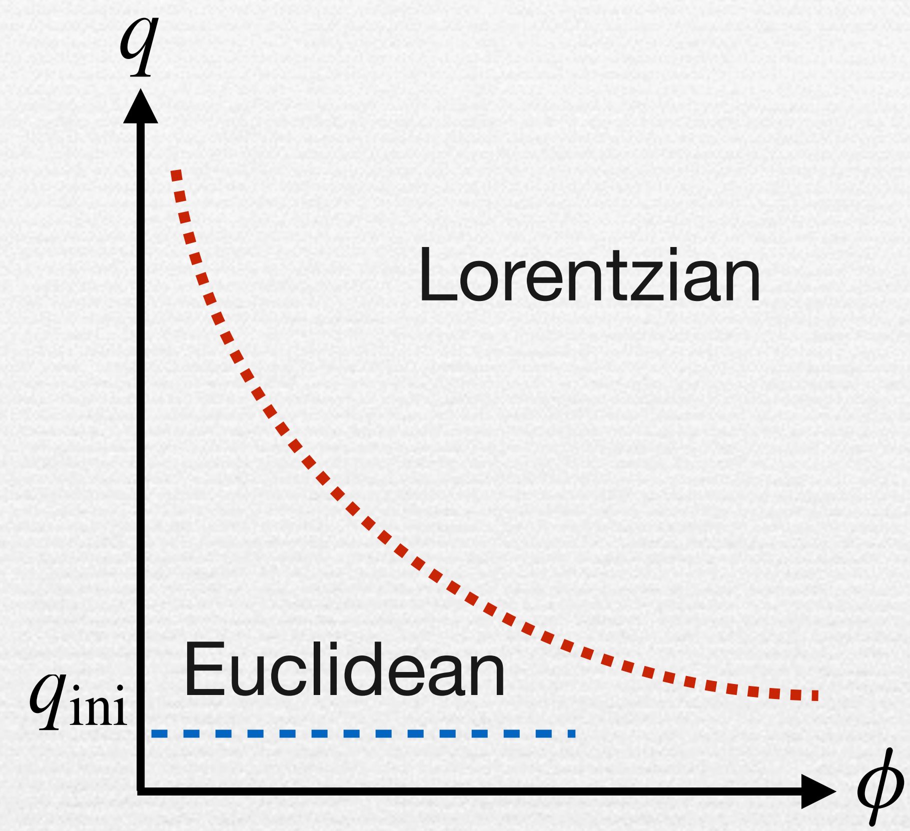
Numerical analysis of the wave function

H.Suenobu & YN
arXiv:1603.08172

We solved the WD equation numerically
and obtained wave functions for 9x9 BCs



space of BCs



2000x200 grids in mini-superspace

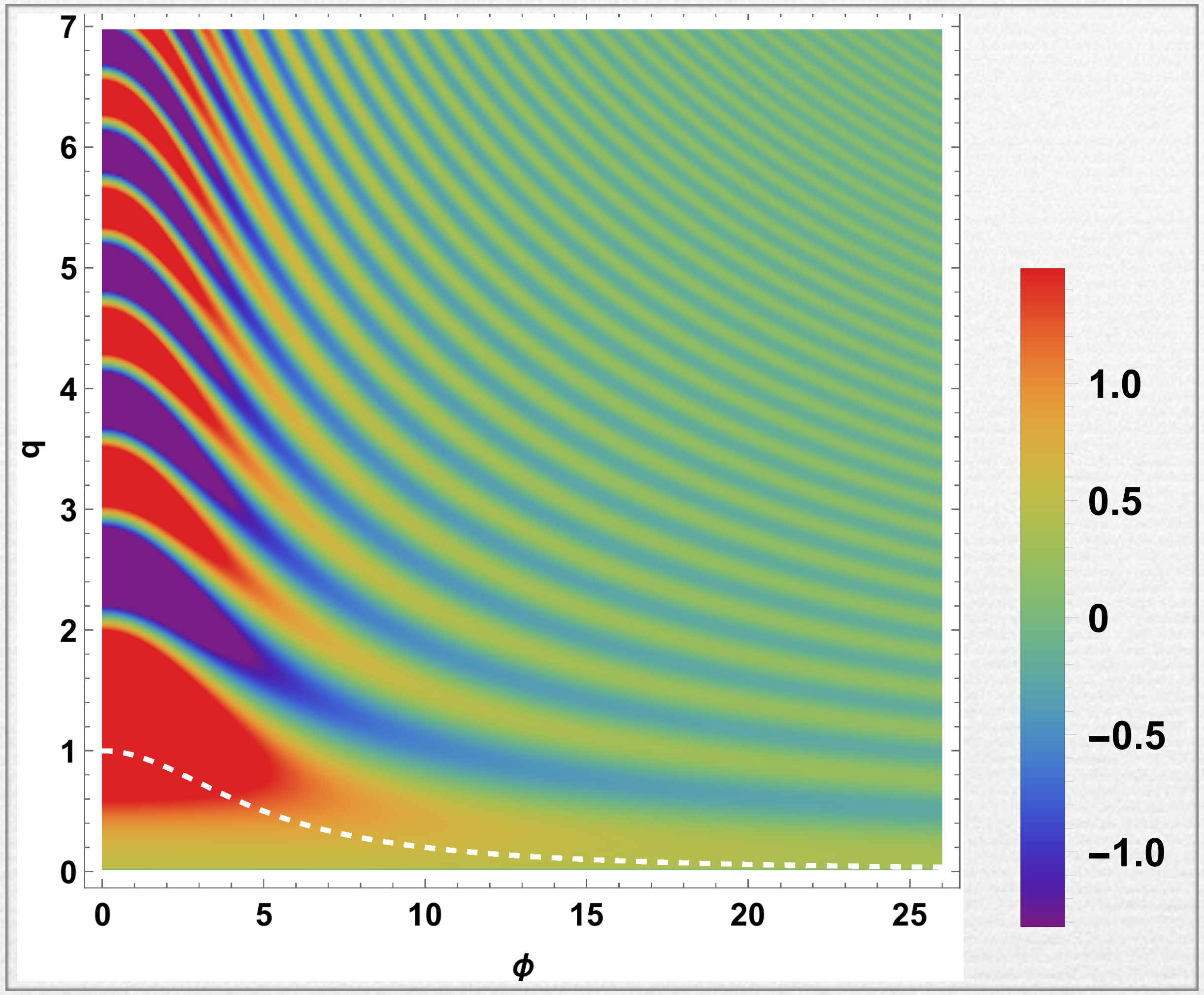
5-step Adams-Bashforth method

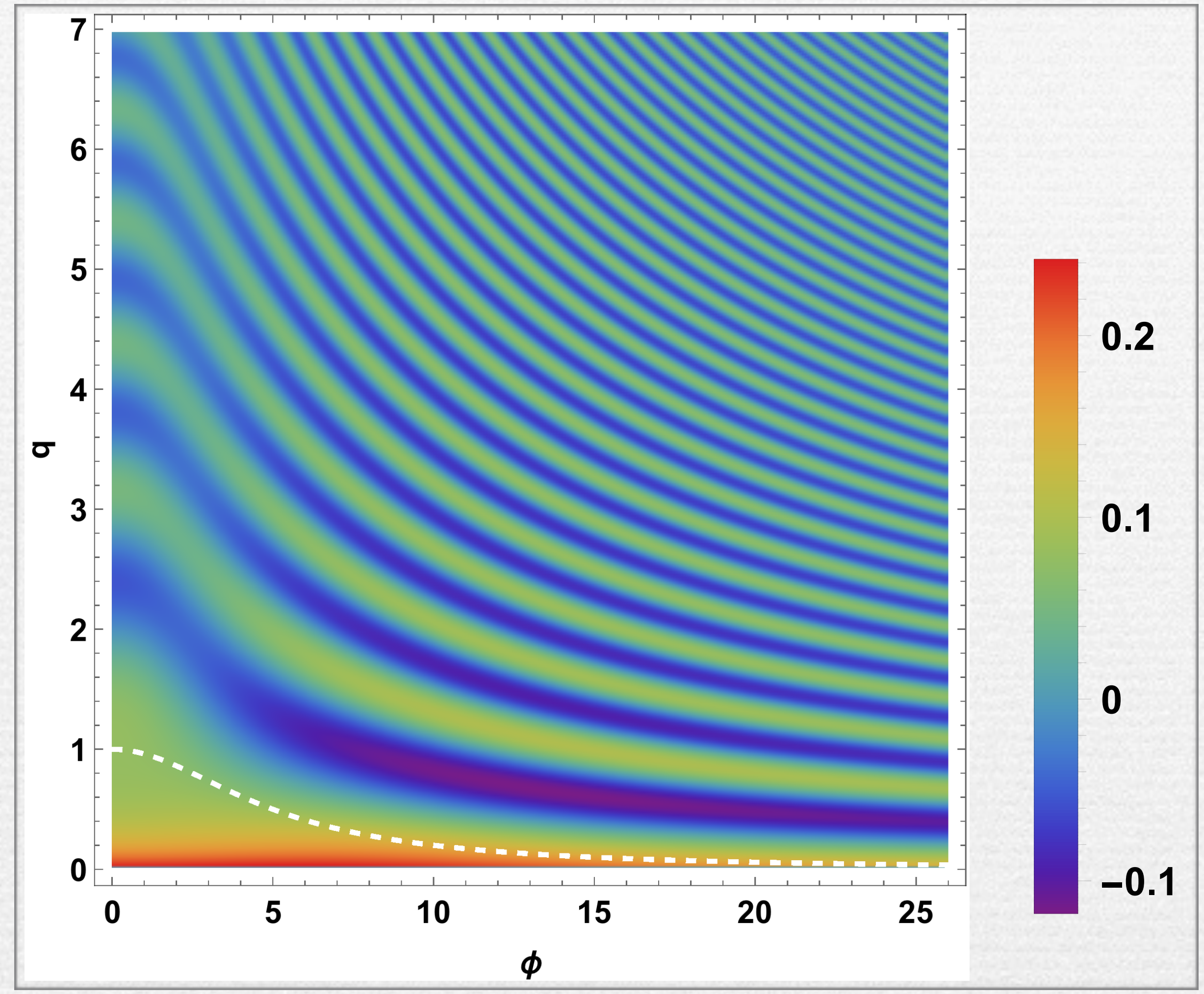
Boundary condition for WD equation: exact solution for constant potential

parameter a, b

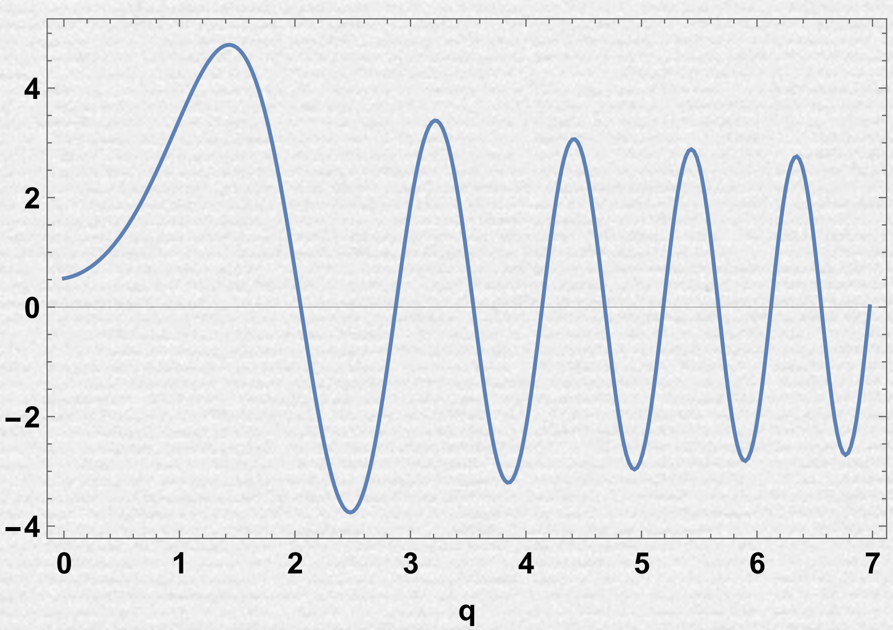
$$\Psi(q_{\rm ini}, \phi) = \Psi_C(q_{\rm ini}, \phi), \qquad \partial_q \Psi(q_{\rm ini}, \phi) = \partial_q \Psi_C(q_{\rm ini}, \phi)$$

$\Psi(\phi, q)$ (HH) $\mu = 0.2$ $\mathrm{Im}[\Psi(\phi, q)]$ (V)

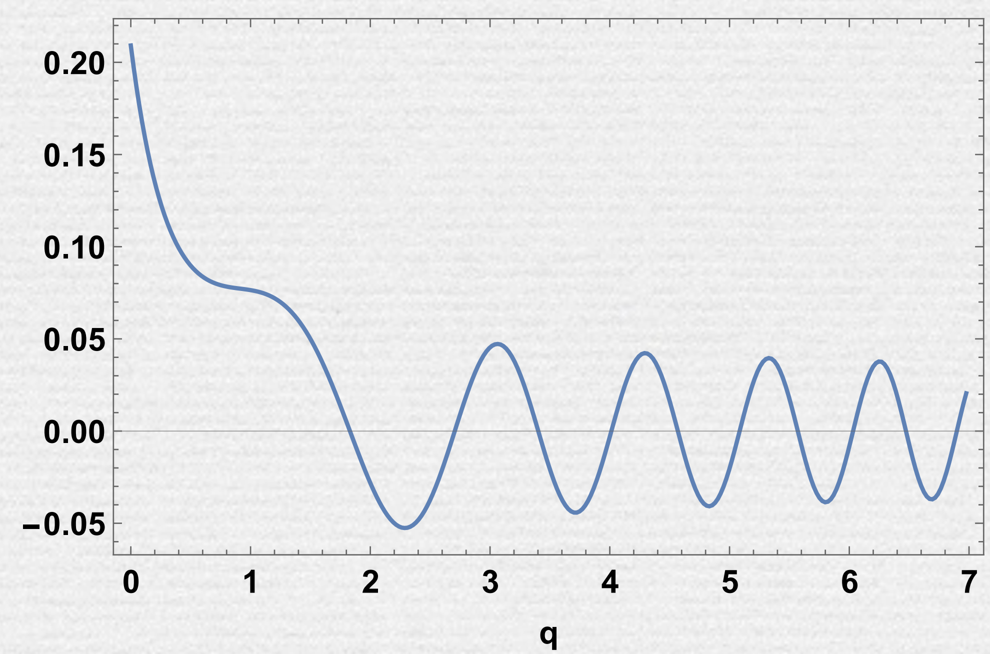




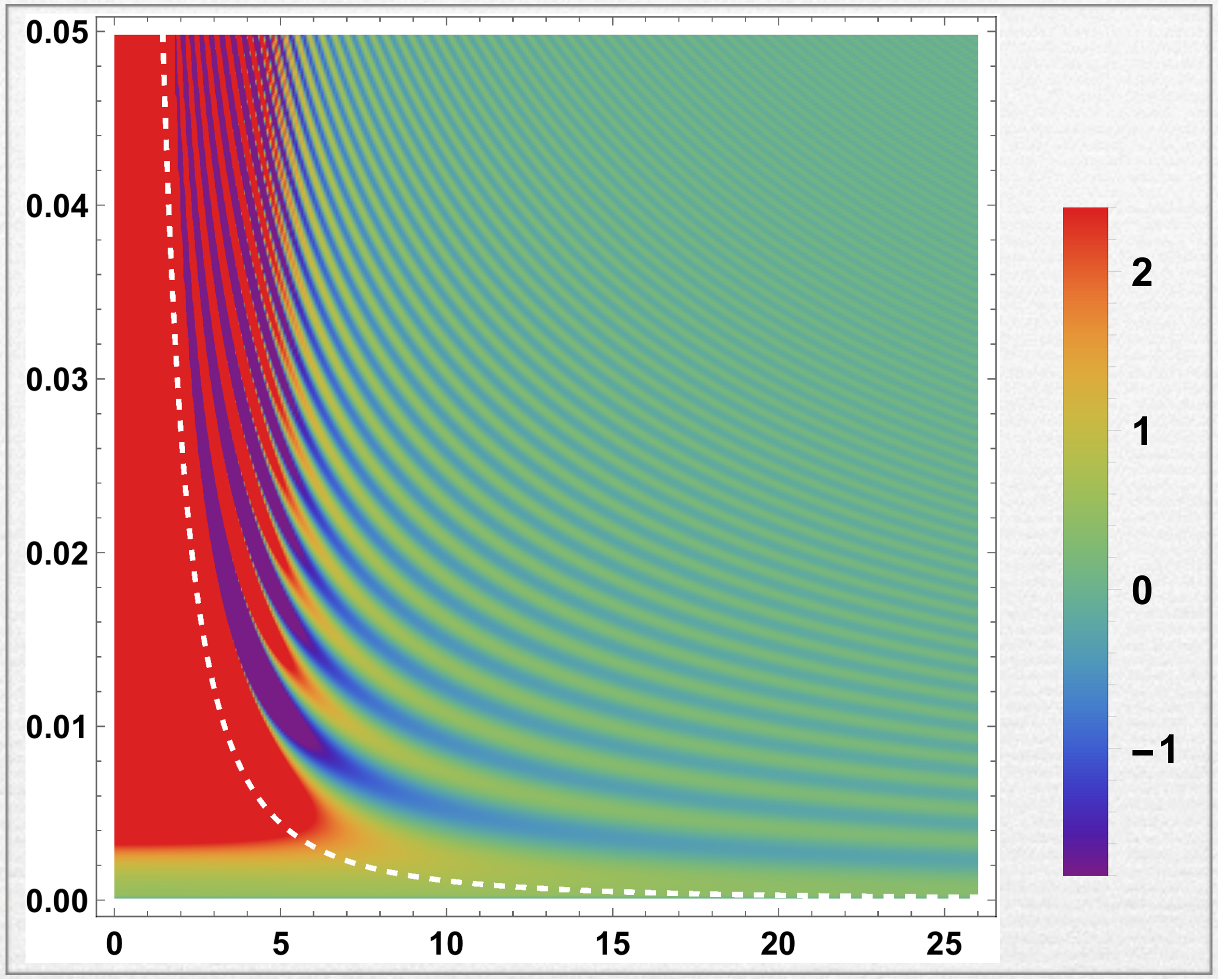
slice at $\phi = 10$

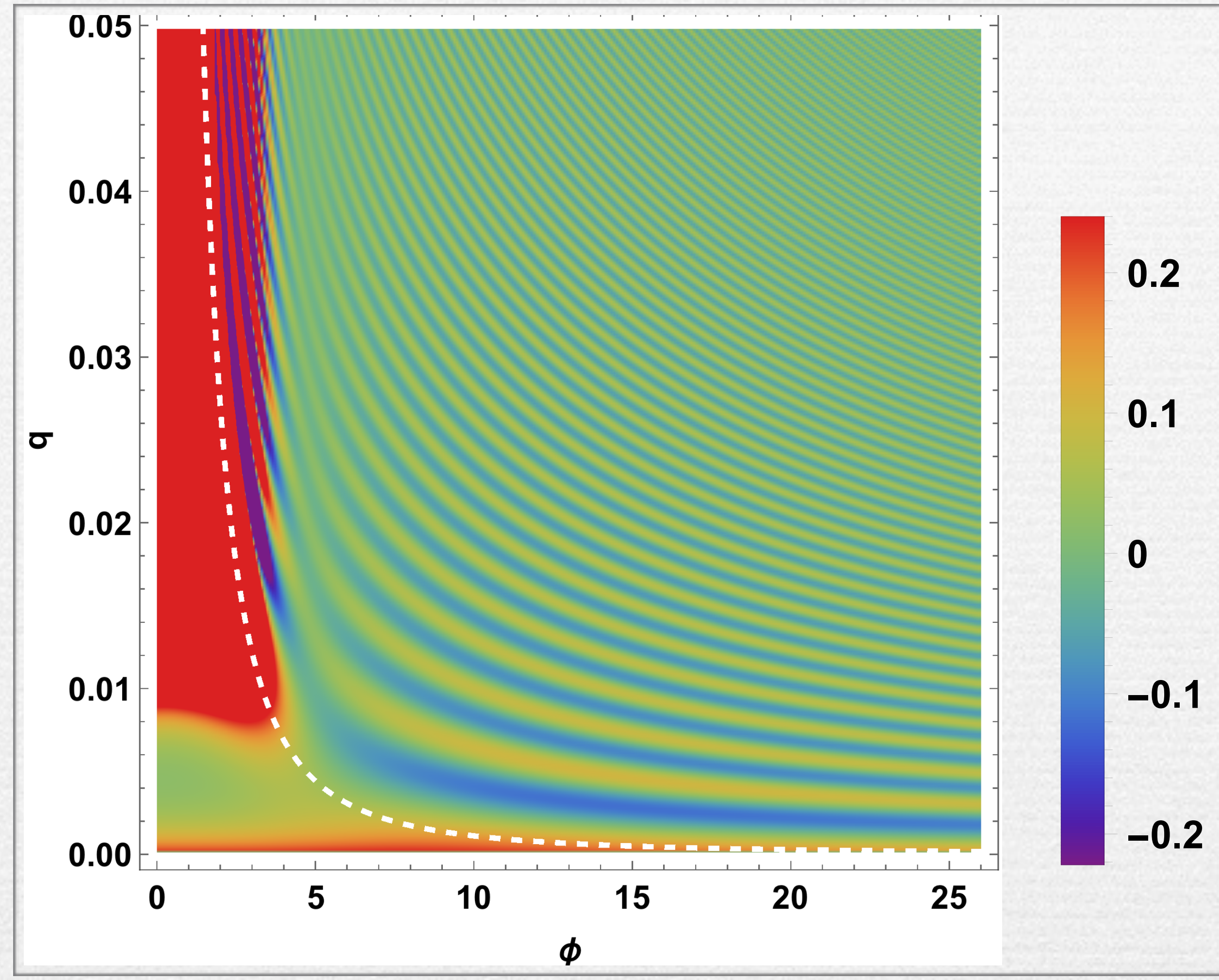


slice at $\phi = 10$

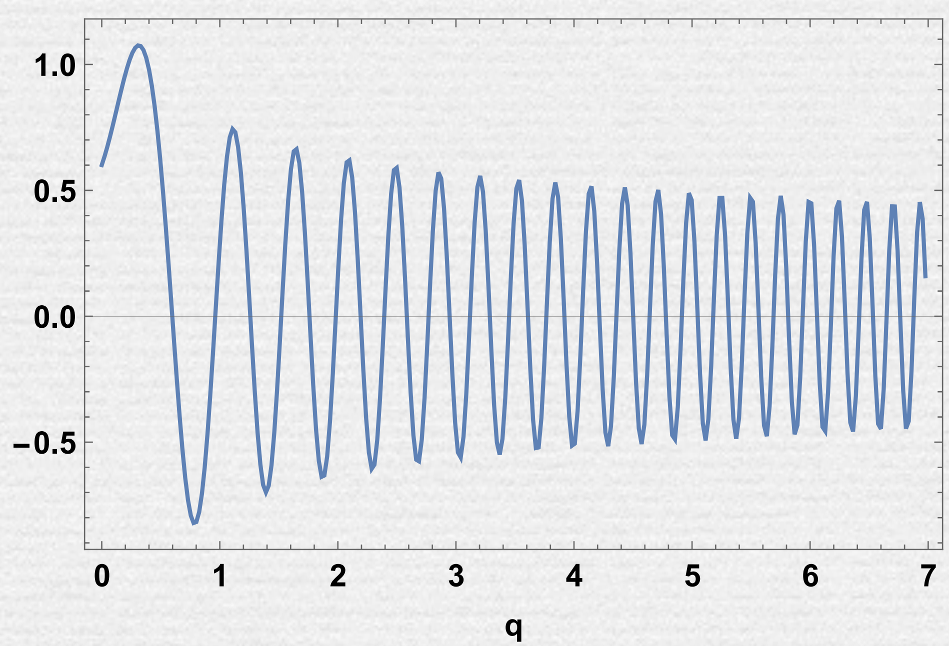


$\Psi(\phi, q)$ (HH) $\mu = 3$ $\mathrm{Im}[\Psi(\phi, q)]$ (V)

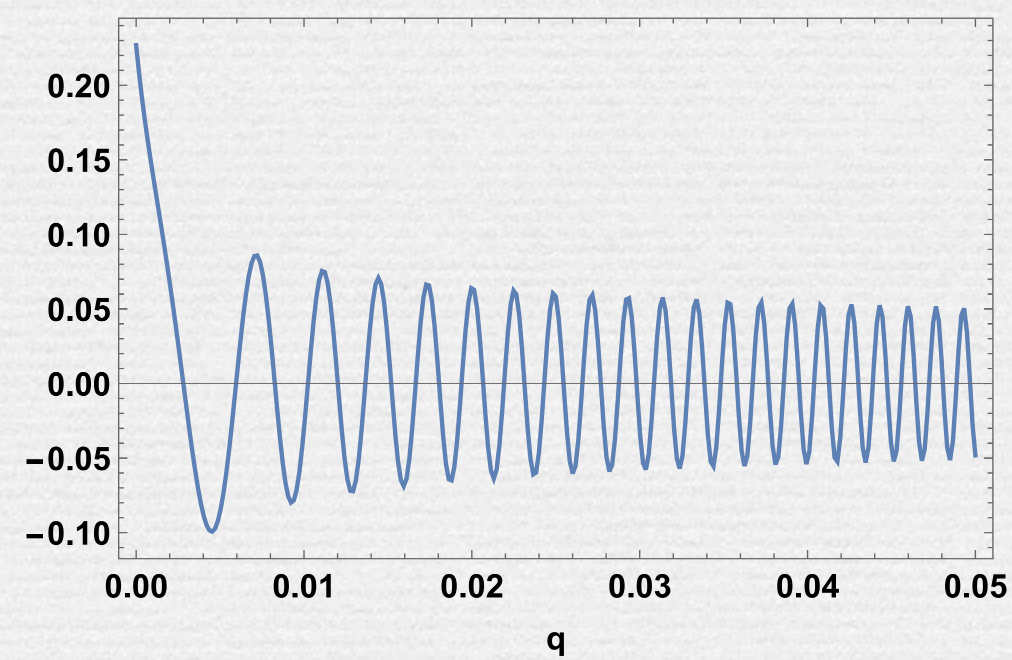




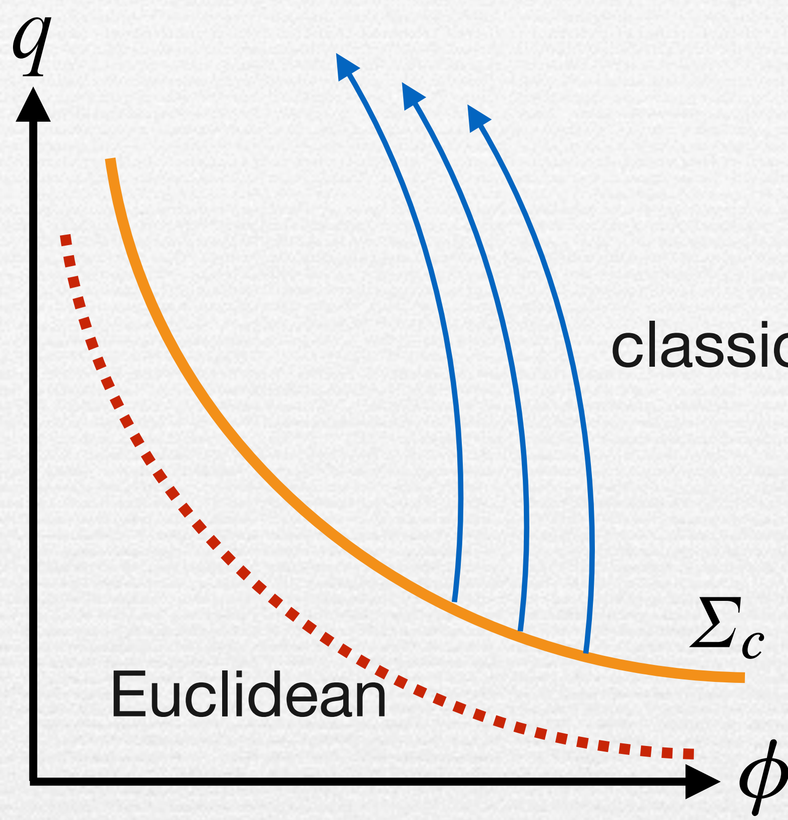
slice at $\phi = 10$



slice at $\phi = 10$



Extraction of probability from wave functions



(0) Obtain wave function with (a,b)

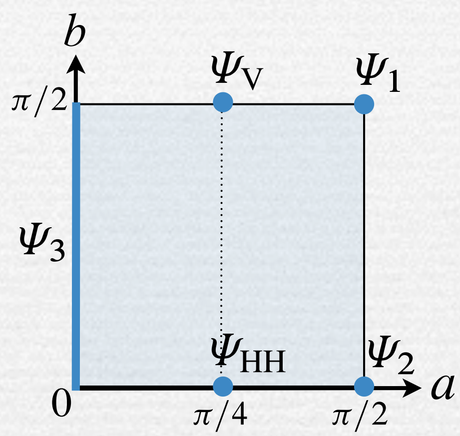
(1) Specify a classicality surface
$$S_0(q,\phi) = \frac{K}{6V(\phi)}\left(2V(\phi)q-1\right)^{3/2} - \frac{\pi}{4}$$

(2) On the classicality surface, we obtain probability measure of inflaton field from the wave function. ➡ initial value of classical equation

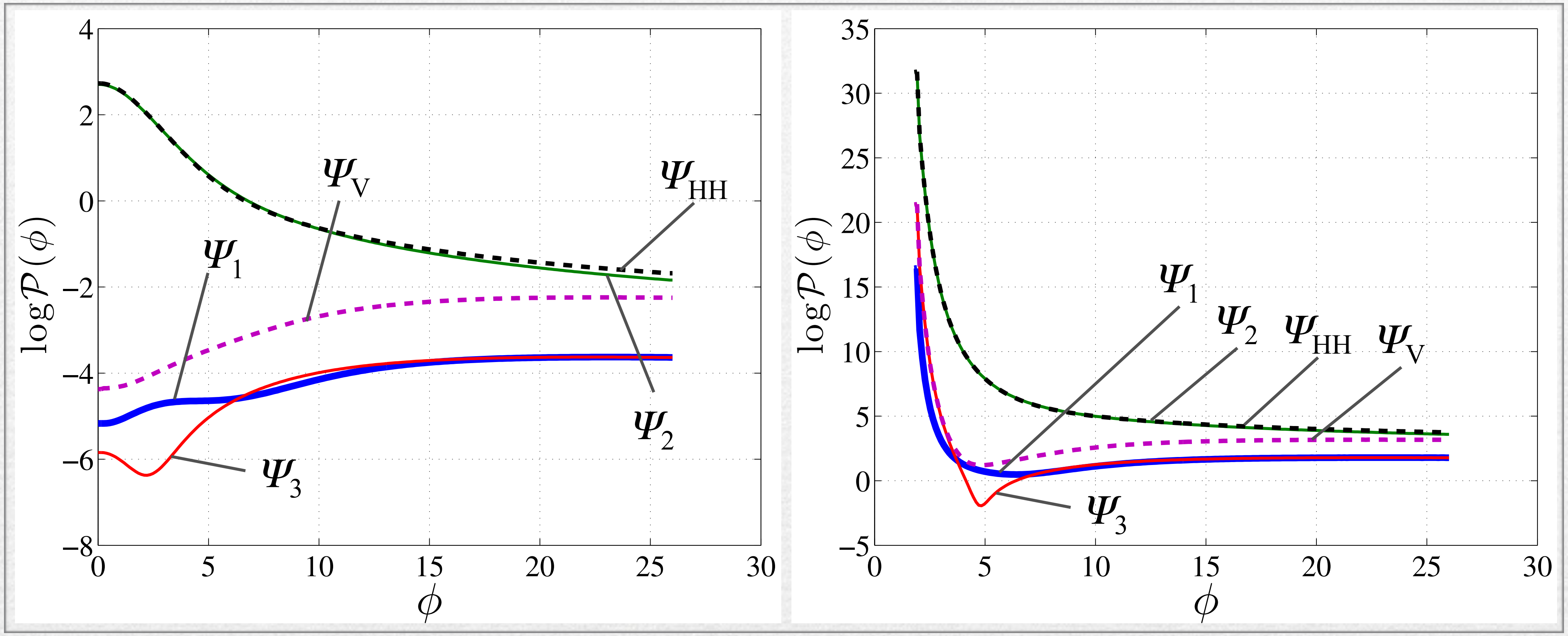
(3) Then, integrate classical eq. motion to obtain e-foldings for scalar field driven inflation and obtain probability for e-foldings.

duration of slow roll inflation

Probability of inflaton field (unnormalized) on the classicality surface for various BCs



$\mu = 0.2$ $\mu = 3$



- The wave function with (HH) prefers small values of potential
- The behavior of the wave function with (V) depends on the value of

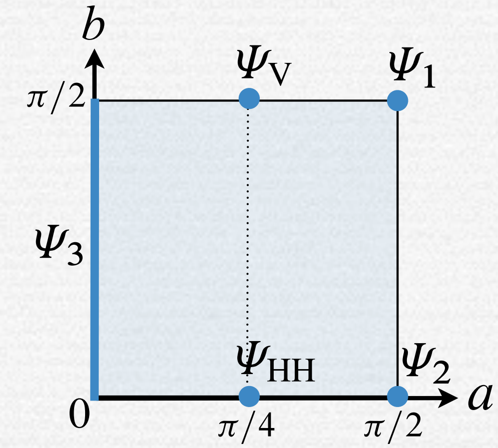
$$\mu \propto m/\Lambda^{1/2}$$

small μ prefers large values of potential slow roll inflation never end

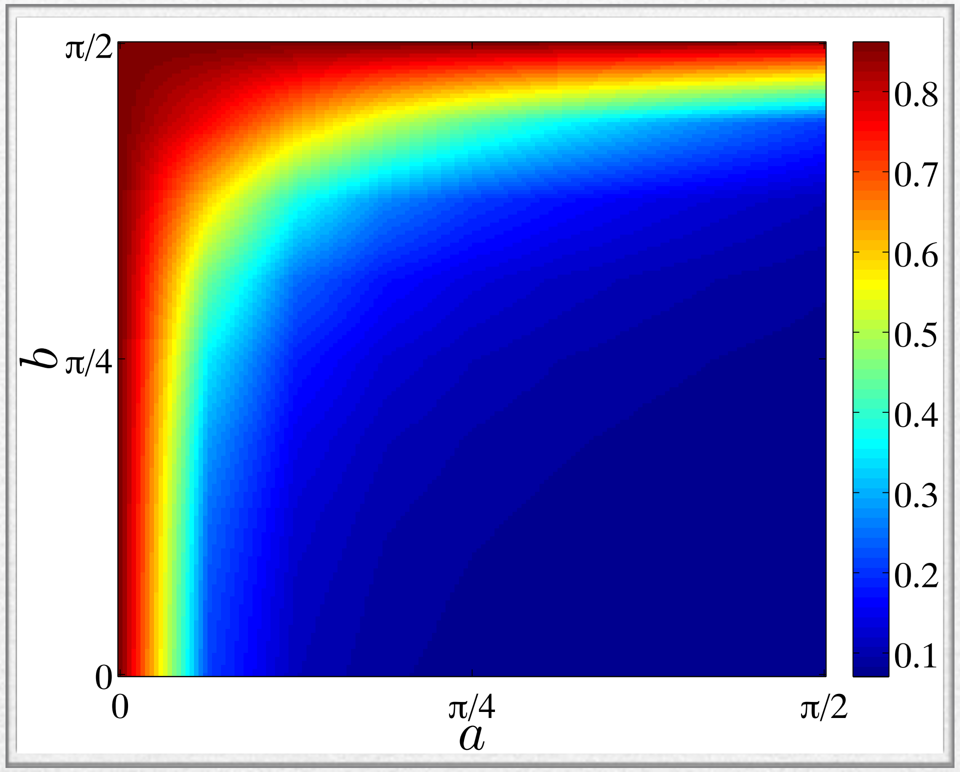
large μ prefers small values of potential slow roll inflation ends at

$$\Phi_f \approx \frac{1}{2\sqrt{3\pi G}}$$

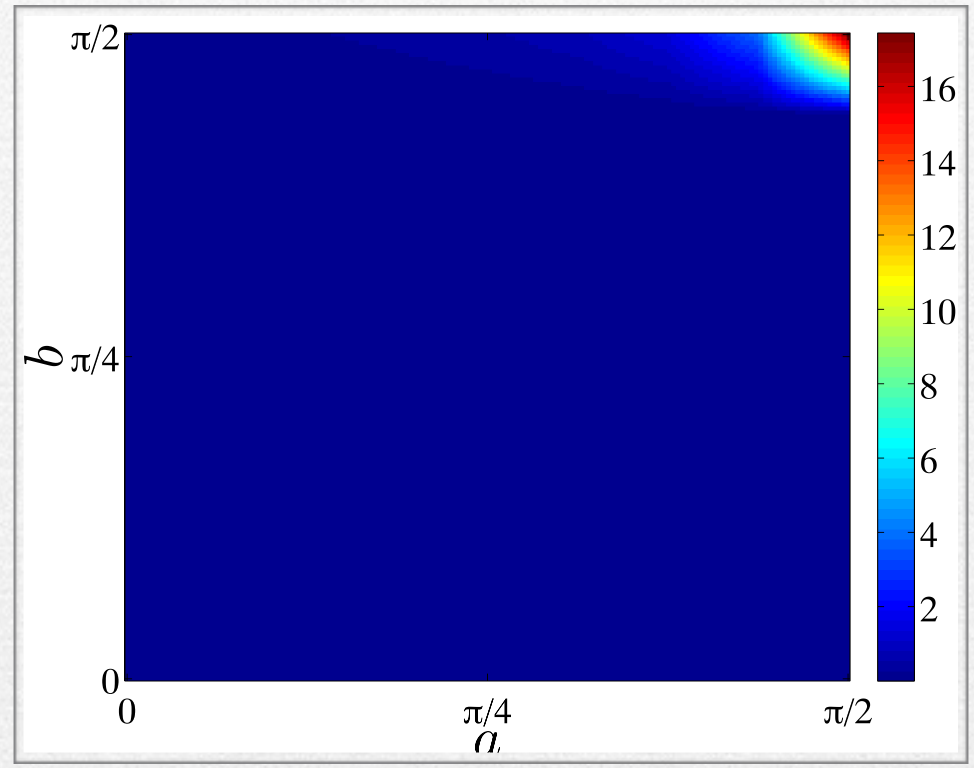
Probability of boundary conditions $P(a,b)$



$\mu = 0.2$

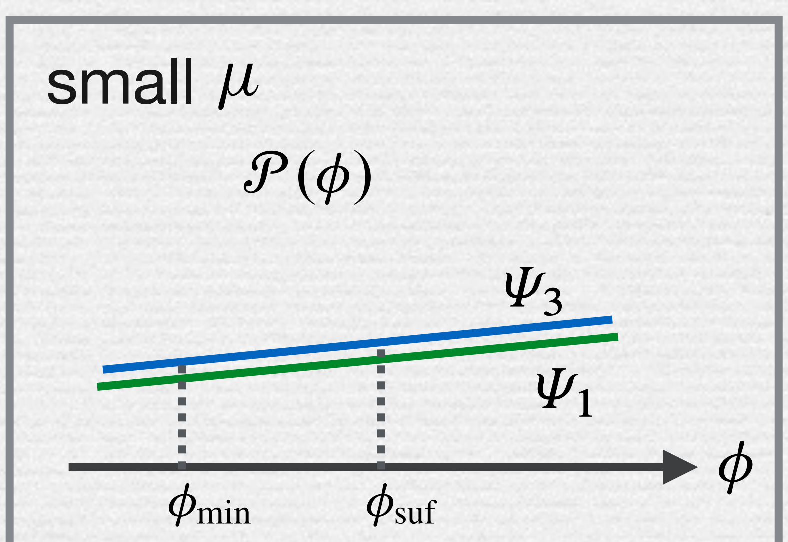


$\mu = 3$

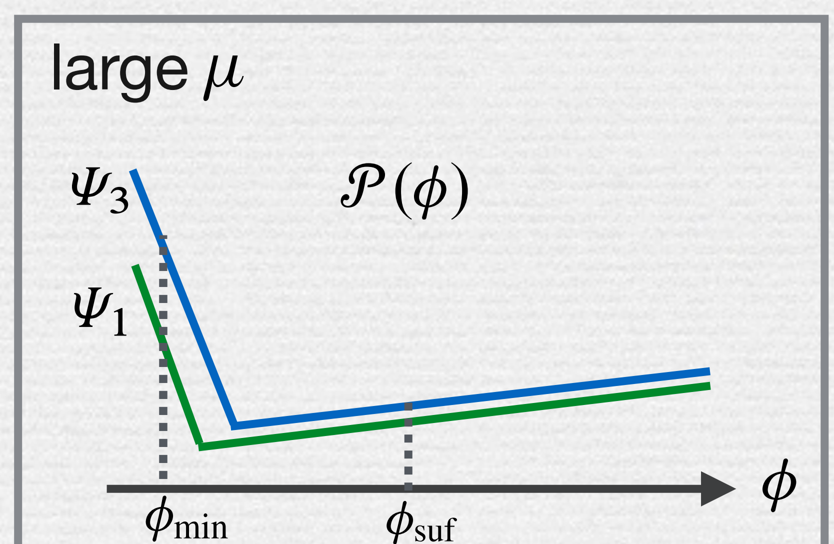


- Ψ_V is superior to Ψ_HH for large e-foldings of inflation
- Location and spread of peak depends on mass parameter μ

  $P(a,b)$ can discriminate value of μ



$P(\Psi_3) = P(\Psi_1)$



$P(\Psi_3) < P(\Psi_1)$

Probability of model parameter $\mu \propto m/\Lambda^{1/2}$

$$P(S|\mu) = \int db P_\mu(S|b) P(b)$$

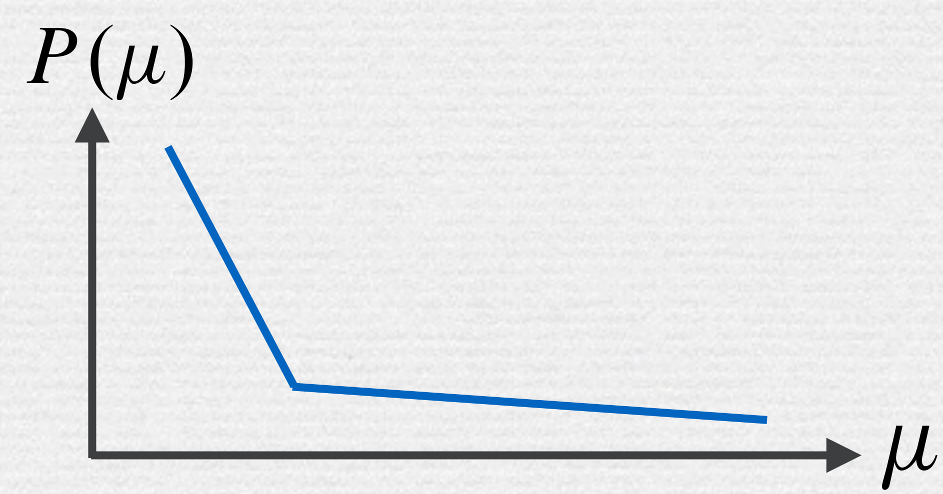
$$P(\mu|S) = \frac{P(S|\mu)P(\mu)}{\int d\mu P(S|\mu)P(\mu)} = \frac{\int db\, P_\mu(S|b)}{\int d\mu db\, P_\mu(S|b)}$$

$P_\mu(S|B_i)$

mass \ wave function	Ψ_1	Ψ_2	Ψ_3	Ψ_{HH}	Ψ_{V}
$\mu = 0.2$	0.604	0.0512	0.627	0.0561	0.621
$\mu = 3$	2.40×10^{-5}	1.60×10^{-10}	1.72×10^{-7}	1.61×10^{-10}	6.74×10^{-7}

$$P(\mu = 0.2|S) = 0.999987$$

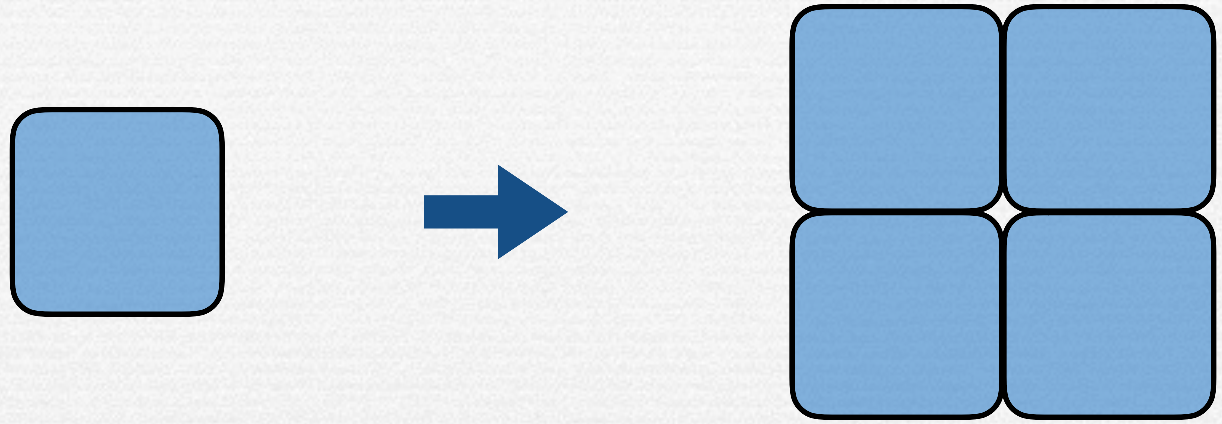
$$P(\mu = 3|S) = 0.000013$$



For a fixed value of mass, large cosmological constant is preferred

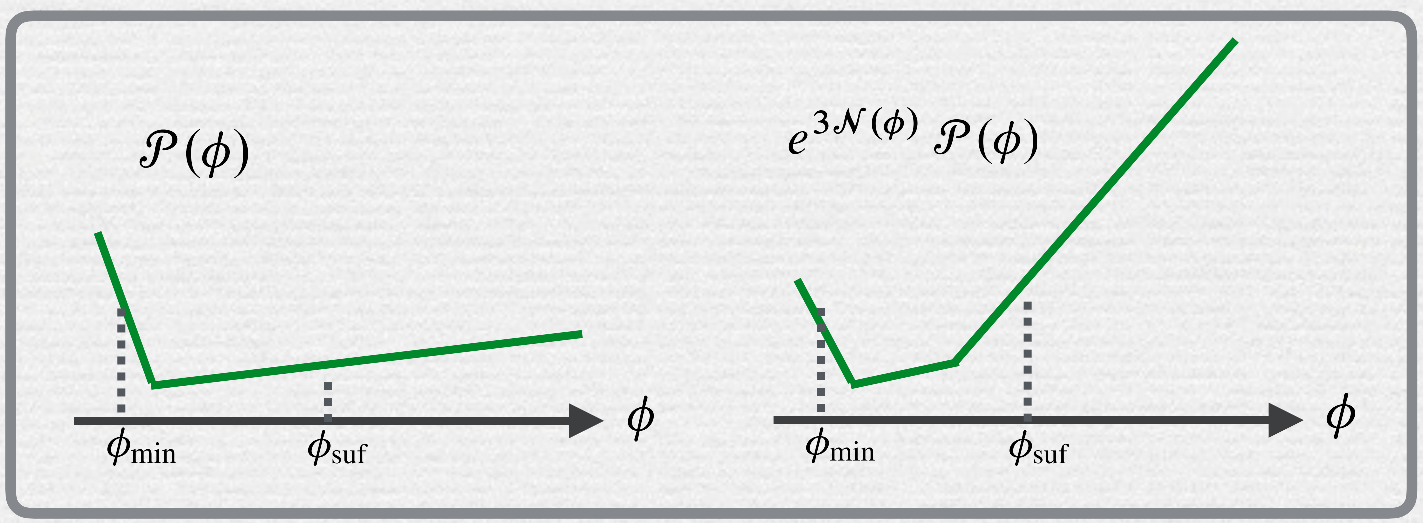
Volume weighting

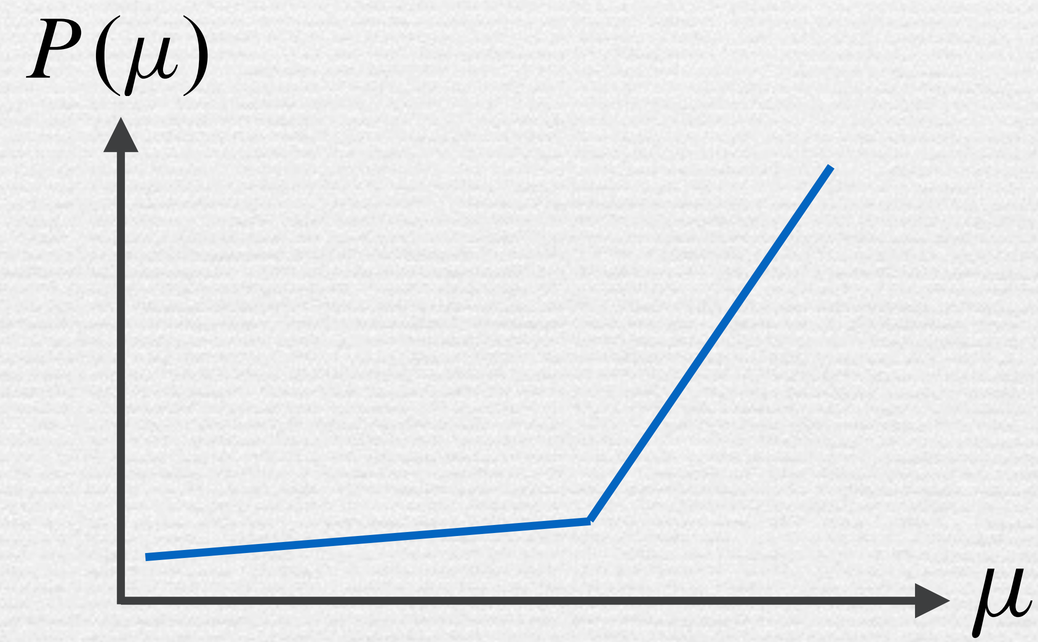
physical volume



$e^{3\mathcal{N}(\phi)}$

$\mathcal{P}(\phi)$ $\quad$ $e^{3\mathcal{N}(\phi)}\,\mathcal{P}(\phi)$





probability for large field value is enhanced

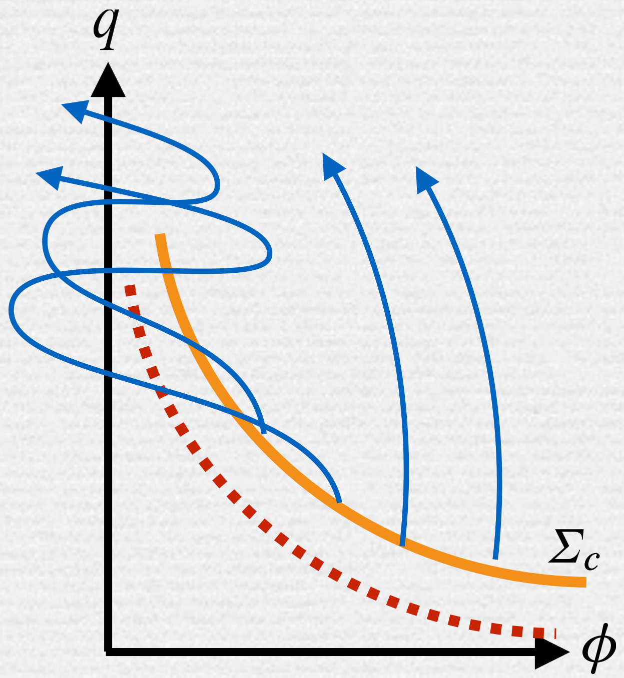
$P_\mu(S|B_i)$ with large mass is enhanced

small cosmological constant is expected

Summary

Summary

- One purpose of quantum cosmology is to predict inflationary universe.
- Introducing parametrization in BC space of the wave function, we evaluated probability of BC under the condition of sufficient e-foldings of inflation. The probability sharply depends of the value of parameters in the model.
- prediction on model parameters
- treatment of oscillatory phase   large μ case



superposition of WKB wave functions

technically difficult to decompose
each WKB components

Baysian update?

$$\boxed{P(B_i|S)} = \frac{\boxed{P(B_i)}P(S|B_i)}{\sum_k \boxed{P(B_k)}P(S|B_k)}$$

$P(B_i|S)$: posterior probability

$P(B_k)$: prior probability

宇宙の生成実験　　　　multiverse?