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Abstract: We study a dynamic game involving a fossil-fuel exporting cartel and a coalition of importing countries that imposes carbon taxes. We show that there exists a unique Nash equilibrium, where all countries use feedback strategies. We also obtain two Stackelberg equilibria, one where the exporting cartel is the leader, and one where the coalition of importing countries is the leader. Not surprisingly, the world welfare under the Nash equilibrium is lower than that under the social planning, even though both solutions have the same steady state. Comparison of the Stackelberg equilibria with the Nash equilibrium is performed numerically. All our numerical examples reveal that world welfare under the Nash equilibrium is higher than that under the Stackelberg game where the exporting cartel is the leader. The worst outcome for world welfare occurs when the importing coalition is the Stackelberg leader.

Keywords: Exhaustible Resource, Carbon Tax, Nash and Stackerberg Equilibria

JEL Classification Codes: C73, Q34

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1 Introduction

The publication of the Stern Review of the Economics of Climate Change (Stern 2006) has provided impetus to economics analysis of climate change. Much progress has been achieved in climate change studies over the last decade. A large literature has appeared, bringing new insights to the field of climate change research. Some of the pressing economic issues are discussed in Heal (2009) and Haurie et al. (2012), among others.

Issues of climate change are broad and can be analyzed from multiple perspectives. In this paper we adopt a dynamic game approach, because there are strategic considerations that extend into the far future. Hopefully the model will contribute to formulation of climate policy. We focus on international aspect of the exploitation of fossil fuels under the threat of global warming, where carbon taxes are used as policy instruments for mitigating its adverse effects.

We study a dynamic game involving a fossil-fuel exporting cartel and a coalition of fuel importing countries that impose carbon taxes. The fossil fuel is non-renewable resource, and its consumption leads to stock externality in the form of carbon dioxide concentration which is largely responsible for global warming. We will focus on the case where the importing countries form a coalition and agree on their carbon policies. We show that there exists a unique Nash equilibrium in a game by exporting and importing countries, where they use feedback strategies to set fuel price and carbon tax. Further we compare the Nash outcome with the Stackelberg equilibria in which Stackelberg leadership rests with either exporting or importing countries.

Our model borrows some features from Wirl (1995) and Fujiwara and Long (2011). The main differences are that Wirl (1995) derives a Nash equilibrium but does not deal with the Stackelberg leader-follower relationship and Fujiwara and Long (2011) do not consider the externalities of fossil fuel consumption.

After deriving the solutions, we compare welfare levels of participants under Nash equilibrium with the efficient outcome, which is a benchmark scenario where a single world social planner maximizes world welfare. Furthermore, we take two Stackelberg leadership scenarios, one where the importing coalition is the leader and the other with the leadership by exporters. After showing analytical results, we provide numerical comparisons among alternative regimes under a range of possible parameter values.

The paper is organized as follows. Section 2 presents the basic model. In section 3 a benchmark scenario is analyzed by assuming the existence of a world social planer. In section 4 we consider the optimal behavior of the oil cartel facing to an arbitrary carbon-tax rule set by oil importing countries and, in turn, in section 5 the behavior of oil importing countries against an arbitrary price-setting rule of the oil cartel. Section 6 derives the feedback Nash equilibrium. Section 7 compares the Nash equilibrium with the outcome under the social planner, both in terms of welfare and in terms of speed of accumulation of the pollution stock. Section 8 and 9 derive the global Stackelberg equilibrium in linear strategies of the importing and exporting countries as leader, respectively. After pinning down the analytical conditions to solve, numerical examples are presented to shed light on the comparison of welfare under four different regimes.

2 Model

There are three countries, denoted by 1,2, and 3. Countries 1 and 2 import fossil fuels from country 3. The consumption of fossil fuels generates CO_2 emissions, which contributes to greenhouse gas concentration, causing climate change damages. We assume that climate change damages to country 3 are negligible.

For simplicity, assume that country 3 consists of N identical oil producers. (In what follows, "oil" stands for "fossil fuels".) Each producer takes the price path of oil as beyond its control. Its sole objective is to maximize the present value of its stream of revenue. Extraction is assumed to be costless. Each producer j is endowed at time $t = 0$ with a deposit of size R_{j0}. Let $R_0 = \sum_{j=1}^{N} R_{j0}$. Let $q(t) \geq 0$ denote their aggregate extraction at time t. Let $Y(t)$ denote their cumulative extraction. Then

$$\dot{Y}(t) = q(t), Y(0) = Y_0$$

It is required that total cumulative extraction from time zero to time t cannot exceed the available stock at time zero, R_0 :

$$Y(t) - Y_0 \leq R_0 \text{ for all } t \geq 0.$$

The importing country i (where $i = 1, 2$) consists of M_i identical consumers. Let $M = M_1 + M_2$. Each consumer k has a utility function $U(c_k, x_k, g_k)$ where c_k is the consumption of oil, x_k is the consumption of a numeraire good, and g_k is the damage caused by global warming. Assume that $U(c_k, x_k, g_k)$ is of the form

$$U(c_k, x_k) = Ac_k - \frac{1}{2}c_k^2 + x_k - g_k \equiv u(c_k) + x_k - g_k$$

where $u'(c_k) = A - c_k$ is the consumer's marginal utility of oil consumption.

For simplicity, assume that the damage is quadratic in cumulative extraction:

$$g_k(t) = \frac{\gamma}{2}Y(t)^2, \gamma \geq 0.$$

Note that this view (relating damages to cumulative extraction, rather than GHG concentration level) is based on the scientific work of Allen et al. (2009).

At each point in time, each consumer is endowed with $\overline{x}$ units of the numeraire good. It is assumed that $\overline{x}$ is sufficiently large, so that the consumers after paying for the oil they purchase still have some positive amount of the numeraire good to consume.

3 A benchmark scenario: world social planner

As a benchmark, suppose there is a world social planner who wants to maximize the welfare of all consumers and producers. The planner treats all consumers identically. Then, if the aggregate oil extraction at t is $q(t)$, the planner would let each individual consume $c(t) = q(t)/M$ units of oil. Each individual is asked to pay $p(t)$ for each unit of oil consumed. The revenue to the producers is then $p(t)q(t)$. The utility at time t of the representative consumer k is then

$$U(t) = A\frac{q(t)}{M} - \frac{1}{2}\left(\frac{q(t)}{M}\right)^2 + \left[\overline{x} - p(t)\frac{q(t)}{M}\right] - \frac{\gamma}{2}Y(t)^2$$

and the revenue of the collection of producers is $\Pi(t) = p(t)q(t)$. The world's welfare is the weighted sum of producers welfare and consumers' welfare, where ω is the weight given to producers

$$W = \int_0^\infty e^{-rt} \left[\omega\Pi(t) + MU(t)\right] dt. \tag{1}$$

The rate of discount $r > 0$ is exogenously given.

Considering the standard case where $\omega = 1$, i.e., consumers and producers receive the same weight, the social welfare function (1) reduces to

$$W = \int_0^\infty e^{-rt} \left[Aq(t) - \frac{1}{2M}q(t)^2 + M\overline{x} - \frac{M\gamma}{2}Y(t)^2\right] dt. \tag{2}$$

The social planner chooses $q(t)$ to maximize (2) subject to

$$\dot{Y} = q, \tag{3}$$

$$\text{given } Y(0) = Y_0, \ \lim_{t\to\infty} \left[Y(t) - Y_0\right] \leq R_0. \tag{4}$$

Before solving this problem, consider some extreme cases that will provide us some useful intuition.

First, the case where $\gamma = 0$ (i.e. no climate change damages). Then the problem (2) reduces to a standard resource-extraction problem with a quadratic utility function. The marginal benefit of extracting q is

$$A - \frac{1}{M}q.$$

In this case, it is optimal to exhaust the resource at some finite time T. The extraction rate $q(t)$ will fall over time, with $q(T) = 0$. At time T, the price of the resource reaches its "choke price" level A, and extraction stops.

Second, consider the case where γ and Y_0 are so large that at time zero the present value of the stream of marginal damage cost of adding to the cumulative extraction, $\frac{\gamma M Y_0}{r}$, is greater than the marginal utility of consuming oil, A. Then clearly it is optimal not to extract the resource, i.e. $q^*(t) = 0$ for all $t \geq 0$.

Armed with the above intuition, we now consider the case where $0 < \gamma M Y_0 / r < A$.

It is easy to see that in this case, the following result holds:

Proposition 0 *: Assume that* $0 < \gamma M Y_0 / r < A$. *Define* Y_∞ *by*

$$\frac{M\gamma Y_\infty}{r} = A. \tag{5}$$

Then,

(i) it is optimal to extract the resource during some time interval, and

(ii-a) if $(Y_0 + R_0) \geq Y_\infty$*, then exhaustion will not take place, and the remaining resource stock* $R(t)$ *will asymptotically approach a critical level* R_L *defined by*

$$Y_0 + R_L = Y_\infty$$

In this case the steady state pollution is Y_∞ . If $Y_0 = 0$, the social welfare is given by eq (6)

$$\alpha = \frac{M}{2r}\left((A+\beta)^2 + 2\overline{x}\right) \; = \frac{M}{2r}\left(A + \frac{A\mu M}{(r-M\mu)}\right)^2 + \frac{M\overline{x}}{r} \tag{6}$$

where

$$\mu = \frac{r - \sqrt{r^2 + 4\gamma M^2}}{2M} < 0.$$

(ii-b) if $Y_0 + R_0 < Y_\infty$ then extraction should proceed until the remaining resource stock falls to zero (in finite time).

(See Appendix 1 for a proof).

In what follows, we focus on the case where

$$Y_0 + R_0 > \frac{rA}{\gamma M}$$

Then, as shown in the Appendix, the social planner will not exhaust the stock of the resource. The optimal extraction path is positive, with $q(t)$ approaching zero asymptotically, as $t \to \infty$. The optimal consumer price for oil is, as shown in the appendix,

$$p^c(t) = A - c(t) = A - \frac{q(t)}{M} = A - \frac{(\frac{rA}{\gamma M} - Y_0)\gamma M}{(r - \lambda_1)} e^{\lambda_1 t} \tag{7}$$

where

$$\lambda_1 = \frac{1}{2}\left(r - \sqrt{r^2 + 4\gamma M^2}\right) < 0.$$

REMARK 1: In case (i), the resource will never be exhausted. Therefore its scarcity value is zero. This implies that the producer price is zero, while the consumer price is $A - (q/M)$. The difference between the consumer price and the producer price is the carbon tax. We see that the carbon tax rises over time.

REMARK 2: It is easy to introduce a constant extraction cost b, where $A > b > 0$. In this case, we can define $\widetilde{A} = A - b$. Then the eqs (5), (6), and (7) apply, with $\widetilde{A}$ replacing A. The carbon tax per unit is then $p^c - b$. The ad valorem carbon tax is τ where $(1+\tau)b = p^c - b$.

4 Behaviour of the oil cartel facing an arbitrary carbon-tax rule by oil importing countries

In this section, we assume that the coalition of importing countries set a carbon tax rate $\theta(t)$ per barrel of oil at time t. Assume $\theta(t)$ is linked to $Y(t)$ by the following rule

$$\theta = \sigma + \eta Y$$

where $\sigma \geq 0$ and $\eta > 0$ are some constants. Assume $\sigma < A$. Then the tax θ will approach value A when Y approaches the value $\overline{Y}$ defined by

$$\overline{Y} = \frac{A-\sigma}{\eta}. \tag{8}$$

When Y reaches this level, the carbon tax is so high that even if the producer price p is zero, the consumer will not buy oil.

The cartel of oil producers takes the linear Markovian tax rule $\theta = \sigma + \eta Y$ as given.It knows that if it charges a price $p(t) \geq 0$ per barrel at time t, the representative consumer will demand the quantity $c(t)$ such that

$$u'(c) = p(t) + \theta(t) = p(t) + \sigma + \eta Y(t)$$

i.e

$$A - c(t) = p(t) + \sigma + \eta Y(t)$$

i.e. the demand function from each consumer is

$$c(t) = A - p(t) - \sigma - \eta Y(t) \; .$$

Since there are M consumers, the market demand is

$$q(t) = Mc(t) = M\left(A - p(t) - \sigma - \eta Y(t)\right) \equiv q(p, Y).$$

Since extraction cost is zero, the profit of the cartel at time t is

$$\pi(t) = p(t)q(t) = M\left(A - p(t) - \sigma - \eta Y(t)\right) p(t).$$

The cartel seeks to maximize

$$\int_0^\infty e^{-rt}\left\{M\left(A - p(t) - \sigma - \eta Y(t)\right) p(t)\right\} dt$$

subject to

$$\dot{Y}(t) = M\left(A - p(t) - \sigma - \eta Y(t)\right) \tag{9}$$

$$Y(0) = Y_0, Y(t) - Y(0) \leq R_0 \text{ for all } t.$$

Let us solve the cartel's optimal extraction path, and show how it depends on the tax parameters σ and η.

To proceed with the analysis, we make the following assumption

Assumption C:

$$R_0 > \overline{Y} - Y_0 = \frac{A-\sigma}{\eta} - Y_0.$$

This assumption implies that the cartel will never exhaust the stock of oil.

To solve the cartel's optimization problem, we use the Hamilton-Jacobi-Belman (HJB) equation. Let $V_X(Y)$ be the value function of the cartel of oil exporters. Its HJB equation is

$$rV_X(Y) = \max_p \left\{ M\left(A - p - \sigma - \eta Y\right)p + V_X'(Y)M\left(A - p - \sigma - \eta Y\right) \right\}. \tag{10}$$

Maximizing the right-hand side (RHS) of the HJB equation with respect to p yields the FOC

$$-2p + A - \sigma - \eta Y - V_X'(Y) = 0.$$

Therefore the cartel's producer price rule satisfies

$$p = \frac{1}{2}\left(A - \sigma - \eta Y - V_X'(Y)\right) \equiv p(Y). \tag{11}$$

Then the RHS of the HJB equation can be written as

$$M\left[p(Y) + V_X'(Y)\right]\left(A - \sigma - p(Y) - \eta Y\right)$$

$$\begin{aligned} &= M\left(A - \sigma - p(Y) - \eta Y\right)^2 \\ &= \frac{M}{4}\left[A - \sigma - \eta Y + V_X'\right]^2. \end{aligned} \tag{12}$$

Let us conjecture that the value function is quadratic:

$$V_X(Y) = \alpha_X + \beta_X Y + \frac{1}{2}\mu_X Y^2$$

where α_X,β_X and μ_X are to be determined. Then

$$V_X'(Y) = \beta_X + \mu_X Y \tag{13}$$

and eq. (10) becomes

$$r\left(\alpha_X + \beta_X Y + \frac{1}{2}\mu_X Y^2\right) = \frac{M}{4}\left[A - \sigma - \eta Y + \beta_X + \mu_X Y\right]^2 \tag{14}$$

i.e.

$$\begin{aligned} \frac{4r}{M}\left(\alpha_X + \beta_X Y + \frac{1}{2}\mu_X Y^2\right) &= \left[(A - \sigma + \beta_X) + (\mu_X - \eta)Y\right]^2 \\ &= (A - \sigma + \beta_X)^2 + 2(A - \sigma + \beta_X)(\mu_X - \eta)Y + (\mu_X - \eta)^2 Y^2. \end{aligned}$$

This equation must hold for all feasible values of Y. Therefore the coefficient of the Y^2 term on the left-hand side must equal the coefficient of the Y^2 term on the right-hand side:

$$\frac{2r\mu_X}{M} = (\mu_X - \eta)^2. \tag{15}$$

Similarly, the coefficient of the Y term on the left-hand side must equal the coefficient of the Y term on the right-hand side:

$$\frac{4r\beta_X}{M} = 2(A - \sigma + \beta_X)(\mu_X - \eta) \tag{16}$$

and, likewise for the constant term:

$$\frac{4r\alpha_X}{M} = (A - \sigma + \beta_X)^2. \tag{17}$$

The three equations (15), (16), and (17) determine the three coefficients α_X, β_X, μ_X of the quadratic value function $V_X(Y)$. We first determine μ_X from (15):

$$\frac{2r\mu_X}{M} = \mu_X^2 + \eta^2 - 2\eta\mu_X$$

i.e.

$$\mu_X^2 - 2(\eta + \frac{r}{M})\mu_X + \eta^2 = 0.$$

This quadratic equation in μ_X has two positive real roots, μ_{X1} and μ_{X2} where $\mu_{X1} > \mu_{X2} > 0$,

$$\mu_{X1} = \frac{1}{2}\left(2(\eta + \frac{r}{M}) + \sqrt{2^2(\eta + \frac{r}{M})^2 - 4\eta^2}\right) = \eta + \frac{r}{M} + \sqrt{\left(\frac{r}{M}\right)^2 + \frac{2}{M}\eta r}$$

and

$$\mu_{X2} = \frac{1}{2}\left(2(\eta + \frac{r}{M}) - \sqrt{2^2(\eta + \frac{r}{M})^2 - 4\eta^2}\right) = \eta + \frac{r}{M} - \sqrt{\left(\frac{r}{M}\right)^2 + \frac{2}{M}\eta r}.$$

Which root should we select? As usual, we should choose the root such that the differential equation for Y has a solution that converges to a steady state. The differential equation is, from eqs (9), (11) and (13)

$$\begin{aligned} \dot{Y} &= M\,(A - p - \sigma - \eta Y) && (18) \\ &= M(A - \frac{1}{2}\,(A - \sigma - \eta Y - V_X'(Y)) - \sigma - \eta Y) \\ &= \frac{M}{2}\,(A - \sigma + \beta_X - (\eta - \mu_X)\,Y). && (19) \end{aligned}$$

This equation gives a converging solution to a steady state if and only if $(\eta - \mu_X) > 0$. This requires that the smaller root μ_{X2} be chosen.

Therefore

$$\mu_X^* = \mu_{X2} = \eta + \frac{r}{M} - \sqrt{\left(\frac{r}{M}\right)^2 + \frac{2}{M}\eta r}. \tag{20}$$

Notice that

$$\eta - \mu_X^* = \left[\sqrt{\left(\frac{r}{M}\right)^2 + \frac{2}{M}\eta r}\right] - \frac{r}{M} > 0.$$

Having solved for μ_X, we now turn to eq. (16) to solve for β_X

$$\frac{4r\beta_X}{M} = 2(A - \sigma + \beta_X)(\mu_X^* - \eta).$$

Then

$$\beta_X \left[\frac{4r}{M} + 2(\eta - \mu_X^*) \right] = 2(A-\sigma)(\mu_X^* - \eta) < 0.$$

Thus

$$\beta_X^* = -\left[\frac{(A-\sigma)(\eta - \mu_X^*)}{(\eta - \mu_X^*) + \frac{2r}{M}} \right] < 0$$

since $\eta - \mu_X^* > 0$. And thus

$$A - \sigma + \beta_X^* = (A-\sigma)\left[1 + \frac{(\mu_X^* - \eta)}{\frac{2r}{M} + (\eta - \mu_X^*)} \right] = (A-\sigma)\left[\frac{\frac{2r}{M}}{\frac{2r}{M} + (\eta - \mu_X^*)} \right] > 0. \tag{21}$$

Finally, from (17)

we obtain

$$\alpha_X^* = \frac{M}{4r}(A - \sigma + \beta_X^*)^2 > 0.$$

Substituting (21) into (19) we get

$$\dot{Y} = \frac{M}{2}\left\{ (A-\sigma)\left[\frac{\frac{4r}{M}}{\frac{4r}{M} + 2(\eta - \mu_X^*)} \right] - (\eta - \mu_X^*)\, Y \right\}.$$

This equation has a stable steady state $\widehat{Y}$ defined by

$$\widehat{Y} = \frac{(A-\sigma)\left[\frac{\frac{4r}{M}}{\frac{4r}{M} + 2(\eta - \mu_X^*)} \right]}{(\eta - \mu_X^*)}$$

$$= (A-\sigma)\left[\frac{\frac{4r}{M}}{\frac{4r}{M}(\eta - \mu_X^*) + 2(\eta - \mu_X^*)^2} \right].$$

Now we use (15) to simplify $\widehat{Y}$

$$\widehat{Y} = (A-\sigma)\left[\frac{\frac{4r}{M}}{\frac{4r}{M}(\eta - \mu_X^*) + \frac{4r\mu_X^*}{M}} \right] = \frac{A-\sigma}{\eta}.$$

Thus

$$\widehat{Y} = \overline{Y}.$$

The following Proposition summarizes the result of this sub-section.

Proposition 1: *When the oil cartel faces a carbon tax rule of the form $\theta = \sigma + \eta Y$, where $\sigma < A$, and $\eta > 0$, its optimal response is to set the producer price according to the rule*

$$p = \frac{1}{2}\left[(A - \sigma - \beta_X^*) - (\eta + \mu_X^*)Y \right] \text{ for all } Y \leq \overline{Y}$$

where

$$\mu_X^* = \eta + \frac{r}{M} - \sqrt{\left(\frac{r}{M}\right)^2 + \frac{2}{M}\eta r} > 0$$

and

$$\beta_X^* = -\left[\frac{(A-\sigma)(\eta-\mu_X^*)}{(\eta-\mu_X^*)+\frac{2r}{M}}\right] < 0$$

with $\eta+\mu_X^* > 0$ and $A-\sigma-\beta_X^* > 0$. Thus the producer's price will fall over time, and the quantity demanded, q, will also fall over time. As Y approaches $\overline{Y} = (A-\sigma)/\eta$, the producer's price approaches zero, while the consumer's price, $p+\theta$, approaches A.

Proof: It remains to show that $\dot{q}(t) < 0$. Now

$$\begin{aligned} q &= M(A-\sigma-p-\eta Y) \\ &= M(A-\sigma) - M(p+\eta Y). \end{aligned}$$

So

$$\begin{aligned} \dot{q} &= -M(\dot{p}+\eta\dot{Y}) \\ &= -M\left[-\frac{1}{2}(\eta+\mu_X^*)+\eta\right]\dot{Y} \\ &= \frac{1}{2}M\left[\mu_X^*-\eta\right]\dot{Y} \\ &= \frac{1}{2}M\left[\frac{r}{M}-\sqrt{\left(\frac{r}{M}\right)^2+\frac{2}{M}\eta r}\right]\dot{Y} < 0. \end{aligned}$$

5 Behavior of oil importing countries facing an arbitrary price-setting rule of the oil cartel

Now suppose that the oil cartel uses a price-setting rule which relates the price at time t to the state variable $Y(t)$, where $Y(t) \leq Y_0 + R_0$,

$$p = \delta - \lambda Y \tag{22}$$

with $\delta < A$ and $\lambda \gtreqless 0$.

Suppose the governments of the oil importing countries take δ and λ as given, and agree on coordinating their carbon-tax policy to maximize the welfare of the representative consumer.

Let $\theta(t)$ be the carbon tax that consumers must pay to their governments per barrel of oil consumed. Let $c(t)$ be the quantity of oil demanded per person, and $q(t) = Mc(t)$ be the aggregate demand for oil. The aggregate consumer surplus at time t is

$$Aq(t) - \frac{1}{2M}(q(t))^2 - \left[p(t)+\theta(t)\right]q(t).$$

The quantity demanded is

$$q(t) = M\left[A - p(t) - \theta(t)\right] \tag{23}$$

and the carbon-tax revenue is

$$R(t) \equiv \theta(t)q(t). \tag{24}$$

Assume that the carbon-tax revenue is redistributed in a lump-sum fashion to consumers. Let $L(t)$ be the lump-sum transfer to the consumers

Then, the instantatneous welfare flow of the consumers at time t is

$$W(t) = Aq(t) - \frac{1}{2M}(q(t))^2 - [p(t) + \theta(t)]\,q(t) + L(t) + M\overline{x} - M\left[\frac{\gamma}{2}Y(t)^2\right] \tag{25}$$

where $q(t)$ is given by (23) and $p(t) = \delta - \lambda Y(t)$. The coalition of the two governments chooses $\theta(t)$ and $L(t)$ to maximize the integral of the discounted flow of welfare:

$$\max \int_0^{\infty} e^{-rt}W(t)dt$$

subject to the government's budget constraint

$$L(t) = R(t) \tag{26}$$

and the dynamic equation

$$\dot{Y}(t) = M\left[A - p(t) - \theta(t)\right]$$

where $Y(0) = Y_0$ and $Y(t) \leq \widetilde{Y}$.

Using (23), (24), (25) and (26), the instantaneous welfare flow $W(t)$ becomes

$$\begin{aligned} W &= \left[A - p - \frac{1}{2M}q\right]q + M\overline{x} - M\left[\frac{\gamma}{2}Y^2\right] \\ &= \left[A - p - \frac{1}{2}\left(A - p - \theta\right)\right]M\left(A - p - \theta\right) + M\overline{x} - M\left[\frac{\gamma}{2}Y^2\right] \\ &= \frac{M}{2}\left\{\left(A - p + \theta\right)\left(A - p - \theta\right) + 2\overline{x} - \gamma Y^2\right\} \\ &= \frac{M}{2}\left[(A - p)^2 - \theta^2 + 2\overline{x} - \gamma Y^2\right]. \end{aligned} \tag{27}$$

Let $V_I(Y)$ denote the value function for the coalition of the two oil importing countries. The HJB equation is

$$rV_I(Y) = \max_{\theta}\left\{\frac{M}{2}\left[(A - p)^2 - \theta^2 + 2\overline{x} - \gamma Y^2\right] + V_I'(Y)M\left[A - p - \theta\right]\right\} \tag{28}$$

and maximizing the right-hand side of (28) with respect to θ gives the first order condition (FOC)

$$-\theta - V_I'(Y) = 0.$$

Subsititute this FOC into (28) to get

$$
\begin{aligned}
rV_I(Y) &= \frac{M}{2}\left\{(A-p-V_I')\,(A-p+V_I') + 2\overline{x} - \gamma Y^2 + 2V_I'\,(A-p+V_I')\right\} \\
&= \frac{M}{2}\left\{(A-p+V_I')^2 + 2\overline{x} - \gamma Y^2\right\}. \qquad (29)
\end{aligned}
$$

Let us conjecture that

$$V_I(Y) = \alpha_I + \beta_I Y + \frac{\mu_I}{2}Y^2.$$

Then

$$V_I' = \beta_I + \mu_I Y. \qquad (30)$$

Substituting (30) and (22) into (29)

$$r\left(\alpha_I + \beta_I Y + \frac{\mu_I}{2}Y^2\right) =$$

$$\frac{M}{2}\left\{2\overline{x} + (A-\delta+\beta_I)^2 + 2(\mu_I+\lambda)(A-\delta+\beta_I)Y + \left[(\mu_I+\lambda)^2-\gamma\right]Y^2\right\}.$$

It follows, by comparison, that

$$r\mu_I = M\left[(\mu_I+\lambda)^2 - \gamma\right] \qquad (31)$$

$$r\beta_I = M(\mu_I+\lambda)(A-\delta+\beta_I) \qquad (32)$$

$$r\alpha_I = \frac{M}{2}\left[2\overline{x} + (A-\delta+\beta_I)^2\right]. \qquad (33)$$

Equation (31) gives the quadratic equation

$$\mu_I^2 + \left(2\lambda - \frac{r}{M}\right)\mu_I - (\gamma - \lambda^2) = 0.$$

To avoid complex roots and repeated roots, let us assume that the discriminant is positive

$$\Delta \equiv \left(\frac{r}{M} - 2\lambda\right)^2 + 4(\gamma-\lambda^2) > 0.$$

For this to hold, it is *necessary and sufficient* that

$$\gamma > \frac{r}{M}\left(\lambda - \frac{r}{4M}\right)$$

Note: Either of the following conditions is *sufficient* for $\Delta > 0$:

$$\gamma > \frac{r\lambda}{M} \qquad (34)$$

$$\gamma > \lambda^2. \qquad (35)$$

With $\Delta > 0$, we have two roots, $\mu_{I1} > \mu_{I2}$,

$$\mu_{I1} = \frac{1}{2}\left[\left(\frac{r}{M} - 2\lambda\right) + \sqrt{\left(\frac{r}{M} - 2\lambda\right)^2 + 4(\gamma - \lambda^2)}\right]$$

$$\mu_{I2} = \frac{1}{2}\left[\left(\frac{r}{M} - 2\lambda\right) - \sqrt{\left(\frac{r}{M} - 2\lambda\right)^2 + 4(\gamma - \lambda^2)}\right].$$

As before, we should choose the root such that the differential equation for Y has a solution that converges to a steady state. The differential equation is

$$\begin{aligned}\dot{Y} &= M\left[A - p - \theta\right] \\ &= M\left[A - \delta + \lambda Y + V_I'\right] \\ &= M\left[(A - \delta + \beta_I) + (\mu_I + \lambda)Y\right].\end{aligned}$$

This equation gives a converging solution to a steady state if and only if

$$(\mu_I + \lambda) < 0. \tag{36}$$

We must choose μ_I that satisfies the convergence condition (36). Since the bigger root μ_{I1} gives

$$\mu_{I1} + \lambda = \lambda - \lambda + \frac{1}{2}\left[\frac{r}{M} + \sqrt{\left(\frac{r}{M} - 2\lambda\right)^2 + 4(\gamma - \lambda^2)}\right] > 0$$

we reject μ_{I1}. Turning to the smaller root, λ_{I2}, we find that

$$\mu_{I2} + \lambda = \frac{1}{2}\left[\frac{r}{M} - \sqrt{\left(\frac{r}{M} - 2\lambda\right)^2 + 4(\gamma - \lambda^2)}\right] \tag{37}$$

is negative if and only if

$$\left(\frac{r}{M}\right)^2 < \left(\frac{r}{M} - 2\lambda\right)^2 + 4(\gamma - \lambda^2)$$

i.e. iff

$$\gamma > \left(\frac{r}{M}\right)\lambda. \tag{38}$$

In what follows, we assume that condition (38) is satisfied. This condition is satisfied if $\lambda < 0$ or $\lambda > 0$ but sufficiently small).

Under Assumption (34), we select the *smaller root* μ_{I2} and denote it by μ_I^* :

$$\mu_I^* = \frac{1}{2}\left[\frac{r}{M} - 2\lambda - \sqrt{\left(\frac{r}{M} - 2\lambda\right)^2 + 4(\gamma - \lambda^2)}\right].$$

Next, we solve for β_I.

$$\beta_I\left[M(\mu_I^*+\lambda)-r\right]=-M(A-\delta)(\mu_I^*+\lambda)>0.$$

Then $\beta_I<0$ because $(\mu_I^*+\lambda)<\frac{r}{M}$ by eq (37).

$$\beta_I^*=\frac{(A-\delta)(\mu_I^*+\lambda)}{(r/M)-(\mu_I^*+\lambda)}<0$$

i.e.

$$\beta_I^*=\frac{(A-\delta)\left[\frac{r}{M}-\sqrt{\left(\frac{r}{M}-2\lambda\right)^2+4(\gamma-\lambda^2)}\right]}{\frac{r}{M}+\sqrt{\left(\frac{r}{M}-2\lambda\right)^2+4(\gamma-\lambda^2)}}<0 \tag{39}$$

given that condition (38) is satisfied. Finally,

$$\alpha_I^*=\frac{M}{2r}\left[2\overline{x}+(A-\delta+\beta_I^*)^2\right]>0.$$

The steady state is

$$\begin{aligned}
\widetilde{Y} &= \frac{A-\delta+\beta_I}{-(\mu_I+\lambda)}\\
&= \frac{1}{-(\mu_I+\lambda)}\left[\frac{(A-\delta)\left[(r/M)-(\mu_I^*+\lambda)\right]+(A-\delta)(\mu_I^*+\lambda)}{(r/M)-(\mu_I^*+\lambda)}\right]\\
&= \frac{(A-\delta)\,(r/M)}{(\mu_I^*+\lambda)^2-(\mu_I^*+\lambda)(r/M)}\\
&= \frac{(A-\delta)\,(r/M)}{\left[\frac{r\mu_I^*}{M}+\gamma\right]-\frac{r\mu_I^*}{M}-\frac{r\lambda}{M}}=\frac{(A-\delta)\,(r/M)}{\gamma-\frac{r\lambda}{M}}>0.
\end{aligned}$$

The following Proposition summarizes the result of this sub-section:

Proposition 2: *Suppose that the coalition of oil importing countries faces an arbitrary producer's price rule of the form $p=\delta-\lambda Y$, where $\delta<A$, $\lambda\gtrless 0$, and $\gamma>\left(\frac{r}{M}\right)\lambda$.*

Assume that

$$Y_0+R_0\geq\ \widetilde{Y}\equiv\frac{(A-\delta)\,(r/M)}{\gamma-\frac{r\lambda}{M}}.$$

Then, the intertemporal welfare maximizing behaviour of the coalition of importing countries will result in setting the carbon-tax according to the rule

$$\theta=-\beta_I^*-\mu_I^*Y$$

where

$$\mu_I^*=\frac{1}{2}\left[\frac{r}{M}-2\lambda-\sqrt{\left(\frac{r}{M}-2\lambda\right)^2+4(\gamma-\lambda^2)}\right]$$

and

$$\beta_I^* = \frac{(A-\delta)\left[\frac{r}{M} - \sqrt{\left(\frac{r}{M}-2\lambda\right)^2 + 4(\gamma-\lambda^2)}\right]}{\frac{r}{M} + \sqrt{\left(\frac{r}{M}-2\lambda\right)^2 + 4(\gamma-\lambda^2)}} < 0.$$

Thus the consumer's price satisfies

$$p^c = p + \theta = (\delta - \beta_I^*) - (\mu_I^* + \lambda)Y.$$

As Y rises, the consumer's price rises (recall $\mu_I^* + \lambda < 0$). The quantity demanded, q, will fall over time.[1]*. As Y approaches $\widetilde{Y}$, the carbon-tax approaches A, and the consumer's price, $p+\theta$, approaches A.*[2]

Note**:** Since we do not make any assumption about the sign of λ, it is possible that the carbon tax falls as Y rises, provided that $\lambda < 0$, so that the producer's price rises with Y. We will see later that this cannot happen in a Nash equilibrium.

Finally, to prove that q falls over time, we write

$$q = M(A - p - \theta) = M(A - \delta + \lambda - \beta_I^* - \mu_I^* Y).$$

Then

$$\dot{q} = (\lambda + \mu_I^*)\dot{Y} < 0$$

because $\lambda + \mu_I^* < 0$.

6 Nash equilibrium

In the two preceding sub-sections, we looked at the reaction of one player (either the cartel, or the coalition of importing countries) to a given linear Markovian strategy (either a carbon-tax rule, or a producer-price setting rule) of the other player. It is now time to put our pieces together to find the Nash equilibrium of the games between the two players.

Given any linear Markovian tax rule $\theta = \sigma + \eta Y$, we found that the cartel's reaction function (or best reply) is the following pricing rule

$$p = \frac{1}{2}(A - \sigma - \beta_X^*) - \frac{1}{2}(\eta + \mu_X^*)Y$$

where

$$\mu_X^* = \eta + \frac{r}{M} - \sqrt{\left(\frac{r}{M}\right)^2 + \frac{2}{M}\eta r} \equiv \mu_X^*(\eta)$$

and

$$\beta_X^* = -\left[\frac{(A-\sigma)(\eta - \mu_X^*(\eta))}{(\eta - \mu_X^*(\eta)) + \frac{2r}{M}}\right] \equiv \beta_X^*(\sigma, \eta).$$

Conversely, given any linear Markovian producer-price setting rule $p = \delta - \lambda Y$ (where $\lambda \gtrless 0$), we found that the

[1]See the proof below.

[2] This follows from $(A - \delta + \beta_I^*) + (\mu_I^* + \lambda)Y \to 0$ as $Y \to \widetilde{Y}$.

coalition's reaction function (or best reply) is the following carbon-tax rule

$$\theta = -\beta_I^* - \mu_I^* Y$$

where

$$\mu_I^* = \frac{1}{2}\left[\frac{r}{M} - 2\lambda - \sqrt{\left(\frac{r}{M} - 2\lambda\right)^2 + 4(\gamma - \lambda^2)}\right] \equiv \mu_I^*(\lambda)$$

and

$$\beta_I^* = \frac{(A-\delta)\left[\frac{r}{M} - \sqrt{\left(\frac{r}{M} - 2\lambda\right)^2 + 4(\gamma - \lambda^2)}\right]}{\frac{r}{M} + \sqrt{\left(\frac{r}{M} - 2\lambda\right)^2 + 4(\gamma - \lambda^2)}} \equiv \beta_I^*(\delta, \lambda).$$

In a Nash equilibrium, it must hold that, for all Y,

$$\sigma + \eta Y = -\beta_I^* - \mu_I^* Y$$

and

$$\delta - \lambda Y \equiv \frac{1}{2}(A - \sigma - \beta_X^*) - \frac{1}{2}(\eta + \mu_X^*)Y.$$

These two conditions are satisfied if and only if the following four equalitites are met:

$$\sigma = -\beta_I^*(\delta, \lambda) \tag{40}$$

$$\delta = \frac{1}{2}(A - \sigma - \beta_X^*(\sigma, \eta)) \tag{41}$$

$$\eta = -\mu_I^*(\lambda) \tag{42}$$

$$\lambda = \frac{1}{2}(\eta + \mu_X^*(\eta)). \tag{43}$$

Note that the RHS of (40) is positive; and the RHS of (41) is positive if $A - \sigma > 0$. We will verify that in a Nash equilibrium, $A - \sigma > 0$. The four equations (40) to (43) determine the Nash equilibrium tuple $(\sigma, \delta, \eta, \lambda)$.

We are able to show that a solution $(\sigma, \delta, \eta, \lambda)$ exists and is unique.

We find that (see Appendix 2) under Assumption (38), there are two possible values of λ

$$\lambda_1^* = \frac{2}{3}\frac{M\gamma}{r} + \frac{1}{9}\left[\frac{r}{M} - \sqrt{3\gamma + \left(\frac{r}{M}\right)^2}\right] > 0 \tag{44}$$

$$\lambda_2^* = \frac{2}{3}\frac{M\gamma}{r} + \frac{1}{9}\left[\frac{r}{M} + \sqrt{3\gamma + \left(\frac{r}{M}\right)^2}\right] > 0. \tag{45}$$

However, the bigger root is not admissible (see the Appendix for a proof). So in what follows, we define

$$\lambda^* = \lambda_1^* = \frac{2}{3}\frac{M\gamma}{r} + \frac{1}{9}\left[\frac{r}{M} - \sqrt{3\gamma + \left(\frac{r}{M}\right)^2}\right] > 0.$$

After some simple manipulations, we obtain the solution

$$\eta^* = 2\left(\frac{M\gamma}{r} - \lambda^*\right) > 0. \tag{46}$$

Finally, we can solve for σ^* and δ^*. We can show that (see Appendix 2)

$$\delta^* = \frac{Ar\lambda^*}{\gamma M} > 0 \tag{47}$$

$$\sigma^* = 2\delta^* - A > 0.$$

Proposition 3 :

There exists a unique Nash equilibrium. At the equilibrium, the coalition of importing countries imposes a carbon tax rule of the form $\theta(t) = \sigma^* + \eta^* Y(t)$ where $\eta^* > 0$ and $0 < \sigma^* < A$, and the oil cartel sets producers price according to the pricing rule of the form $p(t) = \delta^* - \lambda^* Y(t)$ where $\delta^* > 0$ and $0 < \lambda^* < M\gamma/r$. The quantity demanded will fall over time, and the consumer price will approach A as the stock of pollution Y approaches $\overline{Y}$ where

$$\overline{Y} \equiv \frac{A - \sigma^*}{\eta^*} = \frac{A - \sigma^*}{2((M\gamma/r) - \lambda^*)} = \widetilde{Y} \equiv \frac{A - \delta^*}{(M\gamma/r) - \lambda^*} = Y_\infty \equiv \frac{Ar}{\gamma M} \tag{48}$$

In the Nash equilibrium, the importing countries use the carbon tax strategy

$$\theta = \sigma^* + \eta^* Y$$

while the oil cartel uses the price setting strategy

$$p = \delta^* - \lambda^* Y = \lambda^* \left[\frac{Ar}{\gamma M} - Y\right]$$

Therefore the tax increases and the producer price falls as the stock of pollution increases. The consumer price is

$$p^c = p + \theta = (\sigma^* + \delta^*) + (\eta^* - \lambda^*)Y$$

where $\eta^* - \lambda^* > 0$. As Y increases toward its steady state value $Y_\infty = Ar/(\gamma M)$, the carbon tax tends to A and the cartel's producer price tends to zero.

The rate of increase in pollution is

$$\dot{Y} = Mc = q = M(A - p - \theta) = M(A - (\sigma^* + \delta^*) - (\eta^* - \lambda^*)Y)$$

Thus, as Y rises, $\dot{Y}$ (the rate of increase in pollution) becomes smaller and smaller:

$$\frac{d\dot{Y}}{dY} = -M(\eta^* - \lambda^*) = \frac{M}{3}\left[\frac{r}{M} - \sqrt{3\gamma + \left(\frac{r}{M}\right)^2}\right] < 0.$$

REMARK 1: The social optimal steady state stock is

$$Y_\infty = \frac{Ar}{M\gamma}$$

Then $Y_\infty = \widetilde{Y}$ is true if and only if

$$\delta^* = \frac{\lambda^* Ar}{\gamma M} \tag{49}$$

For a proof that $\delta^* = \frac{\lambda^* Ar}{\gamma M}$,see the Appendix.

REMARK 2: Under the social planner, the pollution stock also tends to the steady state $Y_\infty = Ar/(\gamma M)$. However, the rate of change in Y is not the same in the two regimes. In fact,

$$\frac{d\dot{Y}}{dY} = -\frac{M}{3}\left[\sqrt{3\gamma + \left(\frac{r}{M}\right)^2} - \frac{r}{M}\right] \text{ for Nash equilibrium}$$

while

$$\frac{d\dot{Y}}{dY} = -\frac{M}{2}\left[\sqrt{4\gamma + \left(\frac{r}{M}\right)^2} - \frac{r}{M}\right] \text{ for social planning.}$$

It is clear that the former takes a smaller negative value than the latter. Thus, compared with the social planner case, the Nash equilibrium results in lower consumption earlier on. This is because cartel conserves the resource stock. This is another confirmation of Solow's claim that the resource monopolist is the conservationist's best friend.

7 Welfare comparison

Since the social planner maximizes world welfare, it is clear that, in terms of world welfare, the Nash equilibrium outcome cannot dominate the outcome under the social planner.

For the sake of illustration, we provide a numerical example. Assume $M = 1, Y_0 = 0, r = 0.05, A = 5$, and $\gamma = 0.02$.

Then the social planner's optimal pollution stock is

$$Y_\infty = \frac{rA}{\gamma M} = \frac{(0.05)5}{0.02} = 12.5$$

and welfare under the social planner is[3]

$$V(0) = \alpha = \frac{M}{2r}\left((A+\beta)^2 + 2\overline{x}\right) \;= \frac{M}{2r}\left(A + \frac{A\xi M}{(r-M\xi)}\right)^2 + \frac{M\overline{x}}{r}$$

where

$$\xi = \frac{r - \sqrt{r^2 + 4\gamma M^2}}{2M}$$

$$\frac{M}{2r}\left(A + \frac{A\mu M}{(r-M\mu)}\right)^2 = 21.\,983\,.$$

In the case of Nash equilibrium

$$\gamma = \lambda_1 = \frac{1}{9}\left[6\frac{M\gamma}{r} + \frac{r}{M} - \sqrt{3\gamma + \left(\frac{r}{M}\right)^2}\right] = 0.24\,4.$$

We keep only λ_1 and call it λ^*. Next, compute η^*

$$\eta^* = 2\left(\frac{M\gamma}{r} - \lambda^*\right) = 0.311.$$

Then

$$\mu_X(\eta^*) = \eta^* + \frac{r}{M} - \sqrt{\left(\frac{r}{M}\right)^2 + \frac{2}{M}\eta^* r} = 0.177$$

$$z(\eta^*) = \frac{\eta^* - \mu_X^*}{\eta^* - \mu_X^* + \frac{2r}{M}} = 0.571$$

$$\delta^* = \frac{A(1-z(\eta^*))(1+z(\eta^*))}{2 - z(\eta^*)(1+z(\eta^*))} = 3.\,055$$

$$\sigma^* = \frac{Az(\eta^*)(1-z(\eta^*))}{2 - z(\eta^*)(1+z(\eta^*))} = 1.\,111$$

$$2\delta^* - A = 1.\,111$$

$$\overline{Y} = \frac{A - \sigma^*}{\eta^*} = 12.\,499$$

$$Y_\infty = \frac{A - \delta^*}{(M\gamma/r) - \lambda^*} = 12.499.$$

Therefore, we confirm that $\overline{Y} = Y_\infty$.

Notice that the steady state pollution stock in the Nash equilibrium is the same as under the social planner. However, the rates at which the pollution stock grows toward the steady state are different under the two regimes.

Concerning welfare in the Nash equilibrium, for simplicity, we set $Y_0 = 0$. The welfare of the importing coalition,

[3]Since the term $\frac{M\overline{x}}{r}$ is a constant, we can omit it in all welfare expressions.

as seen from time $t = 0$, is

$$V_I(0) = \alpha_I^* = \frac{M}{2r}\left[2\overline{x} + (A - \delta^* + \beta_I^*)^2\right] = \frac{M\overline{x}}{r} + \frac{M}{2r}(A - \delta^* + \beta_I^*)^2 .$$

And the welfare of the cartel of oil exporter is

$$V_X(0) = \alpha_X^* = \frac{M}{4r}(A - \sigma^* + \beta_X^*)^2$$

$$\begin{aligned} &= \frac{M}{4r}(A - \sigma^* - \frac{(A-\sigma^*)(\eta - \mu_X^*)}{(\eta - \mu_X^*) + \frac{2r}{M}})^2 \\ &= \frac{M}{4r}\left((A-\sigma^*)(1 - z(\eta^*)\right)^2 . \end{aligned}$$

In the Nash equilibrium, $\lambda = \lambda_1 = 0.244\,44$. So the welfare of the coalition of importers is $\frac{M\overline{x}}{r}$ plus

$$\frac{1}{2r}(A - \delta^* + \beta_I^*)^2 = \frac{1}{2r}(A - \delta^* - \sigma^*)^2 = 6.\,944 .$$

And the welfare of the cartel of exporters is

$$\alpha_X^* = 13.\,888 .$$

The sum of their welfare levels is

$$\frac{M\overline{x}}{r} + 6.\,944 + 13.\,888 = \frac{M\overline{x}}{r} + 20.\,832 .$$

Recall that the welfare under the social planner, which is $\frac{M\overline{x}}{r} + 21.\,983$ .This implies that welfare in the social planner regime is greater than that in the Nash equilibrium.

8 Equilibrium when the importing coalition is the leader

In Section 4, we have shown how the cartel determines its pricing strategy facing a given tax rule $\theta = \sigma + \eta Y$ by the importing coalition, i.e., given the parameters $\sigma < A$ and $\eta > 0$. Suppose the importing coalition knows this "reaction function" of the cartel. Then it seems tempting for the coalition to choose the "best" parameters σ and η to maximize its welfare.

Let us formulate this problem. We have found that given (σ, η) the cartel's best reply takes the form

$$p = \frac{1}{2}(A - \sigma - \beta_X^*(\sigma,\eta)) - \frac{1}{2}(\eta + \mu_X^*(\eta))Y.$$

For simplicity, define

$$G(\eta) = \left(\frac{r}{M}\right)^2 + \frac{2r\eta}{M}$$

and define the cartel's reaction functions

$$\delta^R(\sigma,\eta) \equiv \frac{1}{2}(A-\sigma-\beta_X^*(\sigma,\eta)) = \frac{(A-\sigma)\left[\frac{r}{M}+2\eta-\sqrt{G(\eta)}\right]}{2\eta} \tag{50}$$

and

$$\lambda^R(\eta) \equiv \frac{1}{2}(\eta+\mu_X^*(\eta)) = \frac{1}{2}\left(\frac{r}{M}+2\eta-\sqrt{G(\eta)}\right) \tag{51}$$

then the cartel's price setting reaction is

$$p^R = \delta^R(\sigma,\eta) - \lambda^R(\eta)Y.$$

The consumer price is

$$p = p^R + \theta.$$

Then the transition equation becomes

$$\dot{Y} = M\left[A - p^R - \theta\right] = M\left\{A - \left(\sigma+\delta^R\right)\right\} - M(\eta-\lambda^R)Y.$$

Then $Y(t)$ converges to the steady state $\overline{Y} = (A-\sigma)/\eta$ and

$$Y(t) = \overline{Y} + (Y_0 - \overline{Y})\exp\left[\frac{(r-M\sqrt{G(\eta)})t}{2}\right]. \tag{52}$$

From (27), the instantaneous welfare of the importing country is

$$W = M\overline{x} + M\left[\frac{(A-p^R)^2-\theta^2}{2} - \frac{\gamma Y^2}{2}\right]$$

which can be expressed as

$$W = \kappa Y^2 + \rho Y + \psi + M\overline{x}$$

where

$$\kappa(\eta) \equiv -\frac{M}{2}\left[(\eta-\lambda^R(\eta))(\eta+\lambda^R(\eta))+\gamma\right]$$

$$\rho(\sigma,\eta) \equiv M(-\sigma\eta + (A-\delta^R(\sigma,\eta))\lambda^R(\eta))$$

$$\psi(\sigma,\eta) \equiv \frac{M}{2}(A-\sigma-\delta^R(\sigma,\eta))(A+\sigma-\delta^R(\sigma,\eta)).$$

It follows that, after substituting for $Y(t)$ using (52), instantaneous welfare at t is

$$W(t) = \kappa(Y_0-\overline{Y})^2 e^{(r-MG^{1/2})t} + (2\kappa\overline{Y}+\rho)(Y_0-\overline{Y})e^{(r-MG^{1/2})t/2} + \kappa\overline{Y}^2 + \rho\overline{Y} + \psi + M\overline{x}.$$

Thus

$$\int_0^\infty e^{-rt}W(t)dt = \frac{\kappa}{MG^{1/2}}(Y_0-\overline{Y})^2 + \frac{2(2\kappa\overline{Y}+\rho)}{r+MG^{1/2}}\left(Y_0-\overline{Y}\right) + \frac{\kappa\overline{Y}^2+\rho\overline{Y}+\psi}{r} + \frac{M\overline{x}}{r}$$

which can be simplified as

$$\int_0^\infty e^{-rt}W(t)dt = \frac{\kappa}{MG^{1/2}}(Y_0 - \overline{Y})^2 + \frac{2(2\kappa\overline{Y} + \rho)}{r + MG^{1/2}}\left(Y_0 - \overline{Y}\right) + \frac{M\overline{x}}{r} - \frac{M\gamma\overline{Y}^2}{2r}. \tag{53}$$

The task of the importing coalition, acting as leader, is to choose η and σ to maximize the right-hand side of eq. (53). Note that $\overline{Y} = \overline{Y}(\sigma, \eta)$. The first order conditions that determine the optimal pair (σ, η) would involve the term Y_0. Suppose that at some future time τ if the leader can replan, by choosing η and σ again to maximize the integral of instantaneous welfare flow starting from time τ, where Y_τ is the current pollution stock:

$$V_\tau(Y_\tau) = \int_\tau^\infty e^{-r(t-\tau)}W(t)dt = \frac{\kappa}{MG^{1/2}}(Y_\tau - \overline{Y})^2 + \frac{2(2\kappa\overline{Y} + \rho)}{r + MG^{1/2}}\left(Y_\tau - \overline{Y}\right) + \frac{M\overline{x}}{r} - \frac{M\gamma\overline{Y}^2}{2r}. \tag{54}$$

Then the new first order conditions that determine the optimal pair (σ, η) would involve the term Y_τ, which is different from Y_0. This observation leads us to conclude that the optimal policy of the leader is time-inconsistent. This time inconsistency in dynamic games with Stackelberg leadership is a well-known result, see e.g. Kemp and Long (1980).

To resolve the problem of time inconsistency, several authors have imposed time-consistent conditions that would constrain the choice set available to the Stackelberg leader, see for example Karp (1984), Fujiwara and Long (2011). In what follows we use the approach advocated by Fujiwara and Long (2011). They propose that the leader's choice of the parameters of the tax function should lead to the socially optimal steady state. The rationale for this requirement is that if a policy leades to a steady state that is not efficient, there will be incentive for the leader to deviate from it to achieve gains. In terms of our model, this requirement is

$$\overline{Y} = \frac{A - \sigma}{\eta} = \frac{Ar}{\gamma M} = Y_\infty$$

i.e.

$$\sigma = \frac{A(M\gamma - \eta r)}{M\gamma}. \tag{55}$$

This requirement allows us to simplify the coefficients of the terms $\left(Y_\tau - \overline{Y}\right)$ and $(Y_\tau - \overline{Y})^2$ as follows

$$\frac{2(2\kappa\overline{Y} + \rho)}{r + MG^{1/2}} = -A$$

$$\frac{\kappa}{MG^{1/2}} = \frac{r(3\eta + r/M) - (2\eta + r/M)MG^{1/2} - 2\gamma M}{4MG^{1/2}}.$$

Therefore the right-hand side of eq (54) becomes

$$V_\tau(Y_\tau) = \left[\frac{r(3\eta + r/M) - (2\eta + r/M)MG^{1/2} - 2\gamma M}{4MG^{1/2}}\right]\left(Y_\tau - \frac{Ar}{\gamma M}\right)^2 +$$

$$-A\left(Y_\tau - \frac{Ar}{\gamma M}\right) + \frac{M\overline{x}}{r} - \frac{M\gamma}{2r}\left(\frac{Ar}{\gamma M}\right)^2. \tag{56}$$

It follows that the time-consistent leader's optimization problem amounts to choosing η to maximize the term inside the square brackets. Set $M = 1$ for simplicitly. The first order condition for this optimization problem is

$$3rG - [r(3\eta + r) - 2\gamma]\, r - 2G^{3/2} = 0. \tag{57}$$

We can show (see the Appendix) that the above FOC has a unique positive root η. Unfortunately, it is not possible to express the leader's optimal choice of η as an explicit function of the parameter values r and γ. We must therefore resort to numerical computations.

Given the numerical values of r and γ, we must solve for η. The solution proceeds as follows. Define the new variable $s = 2\eta + r$. First, we must find the unique positive root of the following cubic equation in s

$$4s^3 - \frac{9r}{4}s^2 - 3r\left(\frac{r}{2} + \frac{2\gamma}{r}\right)s - r\left(\frac{r}{2} + \frac{2\gamma}{r}\right)^2 = 0.$$

Next, we find η from $s = 2\eta + r$, and compute

$$G^{1/2} = \sqrt{\left(\frac{r}{M}\right)^2 + \frac{2r\eta}{M}}.$$

After that, we find the welfare $V_\tau(Y_\tau)$.

Numerical example

Assume $M = 1, r = 0.05$, $\gamma = 0.02$, $A = 5$. Solving the cubic equation in s, we obtain the real root $s = 0.265$. Then

$$\eta = 0.107.$$

Turning to $G(\eta)^{1/2} = \sqrt{\left(\frac{r}{M}\right)^2 + \frac{2r\eta}{M}}$,

$$G(\eta)^{1/2} = 0.115.$$

And

$$\sigma = \frac{A(M\gamma - \eta r)}{M\gamma} = 3.653.$$

Then, assuming $Y_0 = 0$,the leader's payoff is $13.644 + \frac{\overline{x}}{r}$. This is an improvement over the Nash equilibrium welfare (which was $6.944 + \frac{\overline{x}}{r}$)

What about the follower's welfare?

Recall that in section 4, for any arbitrary tax function $\theta = \sigma + \eta$, the payoff of the cartel, when $Y_0 = 0$, is

$$\alpha_{X^*} = \frac{M}{4r}\left(A - \sigma + \beta_X^*\right)^2$$

where

$$\beta_X^* = -\left[\frac{(A-\sigma)(\eta - \mu_X^*)}{(\eta - \mu_X^*) + \frac{2r}{M}}\right]$$

and

$$\eta - \mu_X^* = \sqrt{\left(\frac{r}{M}\right)^2 + \frac{2}{M}\eta r} - \frac{r}{M} = \sqrt{G(\eta)} - \frac{r}{M} = 0.065.$$

Then $\beta_X^* = -0.531$ and $\alpha_X^* = 3.321$. Therefor the payoff for the follower (the cartel) is much smaller than under the Nash equilibrium (which was 13.88).

World welfare under the leadership of the importing coalition is $16.966 + \frac{\overline{x}}{r}$,which is smaller than under the Nash equilibrium, which is in turn smaller than under the social optimum.

9 Leadership by exporters

Now let us turn to the other case, where the cartel of exporters know the reaction functions of the importers to its pricing policy $p = \delta - \lambda Y$ where $\delta < A$ and $\lambda \gtreqless 0$. Then it seems tempting for the cartel to choose the "best" parameters δ and λ to maximize its integral of discounted profits.

From our analysis of the previous section we know that the importers respond to (δ, λ) by setting

$$\theta = -\beta_I - \mu_I Y$$

where (assuming $\lambda r/M < \gamma$),

$$\mu_I = \frac{1}{2}\left[\frac{r}{M} - 2\lambda - \sqrt{\left(\frac{r}{M}\right)^2 + 4\left(\gamma - \frac{\lambda r}{M}\right)}\right]$$

and

$$\mu_I + \lambda = \frac{1}{2}\left[\frac{r}{M} - \sqrt{\left(\frac{r}{M}\right)^2 + 4\left(\gamma - \frac{\lambda r}{M}\right)}\right] < 0$$

And

$$\beta_I = \frac{(A-\delta)\frac{1}{2}\left[\frac{r}{M} - \sqrt{\left(\frac{r}{M}\right)^2 + 4\left(\gamma - \frac{\lambda r}{M}\right)}\right]}{\frac{r}{M} - \frac{1}{2}\left[\frac{r}{M} - \sqrt{\left(\frac{r}{M}\right)^2 + 4\left(\gamma - \frac{\lambda r}{M}\right)}\right]}$$

$$= \frac{(A-\delta)\left[\frac{r}{M} - \sqrt{\left(\frac{r}{M}\right)^2 + 4\left(\gamma - \frac{\lambda r}{M}\right)}\right]}{\frac{r}{M} + \sqrt{\left(\frac{r}{M}\right)^2 + 4\left(\gamma - \frac{\lambda r}{M}\right)}} < 0.$$

It is convenient to define the reaction functions

$$\sigma^R(\delta, \lambda) = -\beta_I = \frac{(A-\delta)\left[\sqrt{\left(\frac{r}{M}\right)^2 + 4\left(\gamma - \frac{\lambda r}{M}\right)} - \frac{r}{M}\right]}{\frac{r}{M} + \sqrt{\left(\frac{r}{M}\right)^2 + 4\left(\gamma - \frac{\lambda r}{M}\right)}} > 0$$

and

$$\eta^R(\lambda) = -\mu_I = \frac{1}{2}\left[-\frac{r}{M} + 2\lambda + \sqrt{\left(\frac{r}{M}\right)^2 + 4\left(\gamma - \frac{\lambda r}{M}\right)}\right].$$

Then the importers' tax setting reaction function is

$$\theta^R = \sigma^R(\delta, \lambda) + \eta^R(\lambda)Y$$

and the consumer price is

$$p^c = p + \theta^R = \delta - \lambda Y + \sigma^R(\delta, \lambda) + \eta^R(\lambda)Y.$$

The transition equation becomes

$$\dot{Y} = M\left[A - p - \theta^R\right] = M\left\{A - (\sigma^R(\delta, \lambda) + \delta\right\} - M(\eta^R(\lambda) - \lambda)Y$$

where

$$\eta^R(\lambda) - \lambda = \frac{1}{2}\left[-\frac{r}{M} + \sqrt{\left(\frac{r}{M}\right)^2 + 4\left(\gamma - \frac{\lambda r}{M}\right)}\right] > 0.$$

Thus the path $Y(t)$ converges to a steady state,

$$\widetilde{Y} = \frac{(A - \delta)(r/M)}{\gamma - \frac{\lambda r}{M}}$$

and

$$\begin{aligned} Y(t) &= \widetilde{Y} + (Y_0 - \widetilde{Y})\exp\left[\frac{M}{2}\left[\frac{r}{M} - \sqrt{\left(\frac{r}{M}\right)^2 + 4\left(\gamma - \frac{\lambda r}{M}\right)}\right]t\right] \\ &= \widetilde{Y} + (Y_0 - \widetilde{Y})\exp\left[\frac{1}{2}\left[r - M\sqrt{\left(\frac{r}{M}\right)^2 + 4\left(\gamma - \frac{\lambda r}{M}\right)}\right]t\right].. \end{aligned} \tag{58}$$

Define

$$F(\lambda) = \left(\frac{r}{M}\right)^2 + 4\left(\gamma - \frac{\lambda r}{M}\right) > 0 \text{ for } \lambda < M\gamma/r.$$

Then

$$Y(t) = \widetilde{Y} + (Y_0 - \widetilde{Y})\exp\left[\left(\frac{r - M\sqrt{F}}{2}\right)t\right].$$

The profit of the cartel is

$$\pi = pq = (\delta - \lambda Y)\left[M\left\{A - (\sigma^R(\delta, \lambda) + \delta\right\} - M(\eta^R(\lambda) - \lambda)Y\right].$$

Substituting eq (58) into the above expression for π, we get

$$\begin{aligned}\frac{\pi}{M} &= \frac{\left(M\sqrt{F(\lambda)}-r\right)\lambda}{2M}\left[Y_0-\widetilde{Y}\right]^2 e^{(r-M\sqrt{F})t} \\ &\quad + \frac{(r-M\sqrt{F})}{2(\gamma M-r\lambda)}\left[\delta\gamma - Ar\lambda/M\right]\left[Y_0-\widetilde{Y}\right] e^{\frac{1}{2}(r-M\sqrt{F})t}.\end{aligned}$$

In what follows, we set $M=1$ for simplicity. Then the integral of discounted profit flow is

$$\begin{aligned}\int_0^\infty e^{-rt}\pi(t)dt &= \frac{\left(\sqrt{F(\lambda)}-r\right)\lambda}{2\sqrt{F(\lambda)}}\left[Y_0-\frac{r(A-\delta)}{\gamma-\lambda r}\right]^2 \\ &\quad + \frac{(r-\sqrt{F(\lambda)})\left[\delta\gamma - rA\lambda\right]}{\left(r+\sqrt{F(\lambda)}\right)(\gamma-r\lambda)}\left[Y_0-\frac{r(A-\delta)}{\gamma-\lambda r}\right].\end{aligned} \tag{59}$$

The task of the oil cartel, acting as leader, is to choose λ and δ to maximize the right-hand side of eq. (59). The first order conditions that determine the optimal pair (λ,δ) would involve the term Y_0. Suppose that at some future time τ if the leader can replan, by choosing λ and δ again to maximize the integral of instantaneous profit flow starting from time τ, where Y_τ is the current pollution stock:

$$\begin{aligned}J_\tau(Y_\tau) &= \int_\tau^\infty e^{-r(t-\tau)}\pi(t)dt = \frac{\left(\sqrt{F(\lambda)}-r\right)\lambda}{2\sqrt{F(\lambda)}}\left[Y_\tau-\frac{r(A-\delta)}{\gamma-\lambda r}\right]^2 \\ &\quad + \frac{(r-\sqrt{F(\lambda)})\left[\delta\gamma - rA\lambda\right]}{\left(r+\sqrt{F(\lambda)}\right)(\gamma-r\lambda)}\left[Y_\tau-\frac{r(A-\delta)}{\gamma-\lambda r}\right].\end{aligned} \tag{60}$$

Then the new first order conditions that determine the optimal pair (λ,δ) would involve the term Y_τ, which is different from Y_0. This observation leads us to conclude that the optimal policy of the leader is also time-inconsistent.

To resolve the problem of time inconsistency, we again use the approach advocated by Fujiwara and Long (2011). They propose that the leader's choice of the parameters of the tax function should lead to the socially optimal steady state. The rationale for this requirement is that if a policy leades to a steady state that is not efficient, there will be incentive for the leader to deviate from it to achieve gains. In terms of our model, this requirement is

$$\widetilde{Y} = \frac{(A-\delta)(r/M)}{\gamma-\frac{\lambda r}{M}} = \frac{Ar}{\gamma M} = Y_\infty.$$

This condition implies

$$\delta = \frac{A\lambda r}{\gamma M}. \tag{61}$$

Substituting the condition (61) into the objective function (60) we get

$$J_\tau(Y_\tau) = \frac{\left(\sqrt{F(\lambda)} - r/M\right)\lambda}{2\sqrt{F(\lambda)}}\left[Y_\tau - \frac{Ar}{\gamma M}\right]^2 \tag{62}$$

Then the choice of λ that maximizes (62) is independent of Y_τ. The FOC for this maximization problem is

$$rF(\lambda) + 2\lambda r^2 - F(\lambda)^{\frac{3}{2}} = 0.$$

We can show that the above FOC has a unique positive root λ (see the Appendix). Unfortunately, it is not possible to express the leader's optimal choice of λ as an explicit function of the parameter values r and γ. We must therefore resort to numerical computations.

Numerically, given the numerical values of r and γ, we must solve for λ. The solution proceeds as follows. Define the new variable z by

$$z = r + \frac{4\gamma}{r} - 4\lambda.$$

First, we must find the unique positive root of the following cubic equation in z

$$4z^3 - rz^2 - 2r\left(\frac{4\gamma}{r} + r\right)z = r\left(\frac{4\gamma}{r} + r\right)^2.$$

By an argument similar to that made in the preceding section, we can show that there is a unique $z > 0$ that satisfies the above cubic equation.

Then we solve for λ and compute $F(\lambda)^{1/2}$. Finally, we compute $J_\tau(Y_\tau)$.

Numerical example

Assume $M = 1, r = 0.05$, $\gamma = 0.02$, $A = 5$. Solving the cubic equation,

$$4z^3 - rz^2 - 2r\left(\frac{4\gamma}{r} + r\right)z - r\left(\frac{4\gamma}{r} + r\right)^2 = 0$$

we obtain the unique real root 0.370.

Then we find $\lambda = 0.319$, and $\delta = 3.997$. Next,

$$\sqrt{F(\lambda)} = 0.136\,19.$$

Let $Y_0 = 0$. Then the payoff of the cartel (as the leader) is

$$J_0(Y_0) = 15.810.$$

This is greater than its Nash payoff (which was 13.88).

The resulting payoff to the importing countries is

$$\alpha_I = \frac{1}{2r}\left(2\overline{x} + (A - \delta + \beta_I)^2\right)$$

where

$$\beta_I = \frac{(A-\delta)\left(\frac{r}{M} - \sqrt{\left(\frac{r}{M}\right)^2 + 4\left(\gamma - \frac{r\lambda}{M}\right)}\right)}{\frac{r}{M} + \sqrt{\left(\frac{r}{M}\right)^2 + 4\left(\gamma - \frac{r\lambda}{M}\right)}}.$$

Computing with the assumed numerical values, we obtain $\beta_I = -0.464$, and $\alpha_I = 2.\,901$,which is lower than its Nash equilibrium payoff.

Thus the world welfare under the exporter's leadership is 15.810+2.901+$\frac{\overline{X}}{r}$ which is greater than that under importer's leadership, $16.966 + \frac{\overline{X}}{r}$.

To check the robustness of our results, we have computed a number of numerical examples to make the welfare comparisons for three entities (world welfare, importer and exporter welfare) across four regimes (social planner, Nash, leadership by exporters, and leadership by importers). Calculations are conducted with different values of r, keeping other parameters fixed at $M = 1, A = 5, \gamma = 0.02$. The rate of discount varies from 0.001 to 0.10. The results are summerized here by the following Table 1. For all the different values of r, we find that world welfare is highest under social planning, which dominates world welfare under Nash, which is in turn superior to world welfare when the exporters are the Stackelberg leader. World welfare is always lowest when the coalition of importers is the Stackelberg leader. The last inequality is interesting. This direction of inequality seems hold in general (or at least over all numerical examples of ours.) This could be model specific since the exporters manage to control the resource and to act as the conservationist's friend and by that reason, atmostpheric carbon accumulates at a slower rate.

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Appendix

A.1 Appendix 1: Proof of Proposition 0

Suppose the social planner chooses a time T (which may be finite or infinite) at which extraction ceases, and the terminal stock $R(T) = \widetilde{R} \leq R_0$. Then the utility flow from time T on is

$$0 + M\overline{x} - \frac{M\gamma}{2} Y(t)^2 = M\overline{x} - \frac{M\gamma}{2}\left[Y_0 + R_0 - \widetilde{R}\right]^2 \text{ for all } t \geq T.$$

The present value of the integral of this flow is

$$\int_T^\infty e^{-rt}\left\{M\overline{x} - \frac{M\gamma}{2}\left[Y_0 + R_0 - \widetilde{R}\right]^2\right\} dt = e^{-rT}\frac{1}{r}\left\{M\overline{x} - \frac{M\gamma}{2}\left[Y_0 + R_0 - \widetilde{R}\right]^2\right\} \equiv e^{-rT} S\left(\widetilde{R}\right).$$

The optimal T and $\widetilde{R}$ and the optimal extraction path $q(t)$ must solve the problem

$$\max_{T,\widetilde{R},q} \int_0^T e^{-rt}\left[Aq(t) - \frac{1}{2M}q(t)^2 + M\overline{x} - \frac{M\gamma}{2}Y(t)^2\right] dt + e^{-rT} S\left(\widetilde{R}\right)$$

subject to

$$\dot{Y}(t) = q(t) \text{ for } t \in [0, T]$$

$$Y(0) = Y_0$$

$$Y(T) = Y_0 + R_0 - \widetilde{R}$$

$$\widetilde{R} \leq R_0.$$

Let $\mu(t)$ be the co-state variable. The Hamiltonian is

$$H(q, \mu, Y, t) = e^{-rt}\left[Aq(t) - \frac{1}{2M}q(t)^2 + M\overline{x} - \frac{M\gamma}{2}Y(t)^2\right] + \mu(t)q(t).$$

The necessary conditions include

$$\frac{\partial H}{\partial q} = e^{-rt}\left[A - \frac{q(t)}{M}\right] + \mu(t) = 0 \text{ for } t \leq T$$

$$\dot{\mu}(t) = -\frac{\partial H}{\partial Y} = e^{-rt} M\gamma Y(t) \text{ for } t \leq T$$

$$\dot{Y}(t) = q(t).$$

The transversality condition with respect to T is $\lim_{t \to T}\left\{H + \frac{\partial}{\partial T}\left[e^{-rT} S\left(\widetilde{R}\right)\right]\right\} = 0$, which is equivalent to

$$\lim_{t\to T} e^{-rT}\left[Aq(T) - \frac{1}{2M}q(T)^2 + M\overline{x} - \frac{M\gamma}{2}Y(T)^2\right] - \lim_{t\to T} re^{-rT} S\left(\widetilde{R}\right) = 0$$

which is satisfied if

$$\lim_{t \to T} e^{-rT} \left[Aq(T) - \frac{1}{2M} q(T)^2 \right] = 0.$$

(This is satisfied if $q(T) = 0$ at some finite T, or if $\lim_{t \to \infty} q(t) = 0$.)

The transversality condition with respect to $\widetilde{R}$ is

$$\lim_{t \to T} \left\{ \mu(T) + \frac{\partial}{\partial \widetilde{R}} \left[e^{-rT} S\left(\widetilde{R}\right) \right] \right\} \geq 0,\ R_0 - \widetilde{R} \geq 0$$

$$\lim_{t \to T} \left\{ \mu(T) + \frac{\partial}{\partial \widetilde{R}} \left[e^{-rT} S\left(\widetilde{R}\right) \right] \right\} \left(R_0 - \widetilde{R} \right) = 0.$$

If we define the current-value shadow price of Y by $\psi(t)$

$$\psi(t) = e^{rt}\mu(t) \iff \mu(t) = e^{-rt}\psi(t)$$

then the above conditions become

$$\left[A - \frac{q(t)}{M} \right] + \psi(t) = 0 \text{ for } t \leq T \tag{A.1}$$

$$\dot{\psi} = r\psi + M\gamma Y(t) \tag{A.2}$$

$$\dot{Y} = q \tag{A.3}$$

$$\lim_{t \to T} e^{-rT} q(T) = 0 \tag{A.4}$$

$$\lim_{t \to T} e^{-rT} \left[\psi(T) + \frac{M\gamma}{r} \left[Y_0 + R_0 - \widetilde{R} \right] \right] \geq 0,\ R_0 - \widetilde{R} \geq 0 \tag{A.5}$$

$$\lim_{t \to T} e^{-rT} \left[\psi(T) + \frac{M\gamma}{r} \left[Y_0 + R_0 - \widetilde{R} \right] \right] \left(R_0 - \widetilde{R} \right) = 0. \tag{A.6}$$

Differentiate (A.1) with respect to t

$$\frac{1}{M}\dot{q} = \dot{\psi}.$$

Substituting into (A.2)

$$\frac{1}{M}\dot{q} = -r\left[A - \frac{q(t)}{M} \right] + M\gamma Y.$$

Consider the system of differential equations

$$\dot{q} = -rMA + rq + \gamma M^2 Y$$

$$\dot{Y} = q.$$

This system has a unique steady state

$$(\widehat{Y}, \widehat{q}) = \left(\frac{rA}{\gamma M}, 0 \right). \tag{A.7}$$

This steady state has the saddlepoint property. It takes infinite time to reach the steady state. Define

$$z = Y - \widehat{Y}.$$

Then we have the homogeneous system

$$\dot{q} = rq + \gamma M^2 z$$

$$\dot{z} = q$$

i.e.

$$\begin{bmatrix} \dot{q} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} q \\ z \end{bmatrix} = \begin{bmatrix} r & \gamma M^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q \\ z \end{bmatrix}. \tag{A.8}$$

The characteristic equation of this marix is

$$(r - \lambda)(0 - \lambda) - \gamma M^2 = 0 \iff \lambda_2 - r\lambda - \gamma M^2 = 0.$$

Let λ_1 and λ_2 be the charateristic roots. Then

$$\lambda_1 = \frac{1}{2}\left(r - \sqrt{r^2 + 4\gamma M^2}\right) < 0 \tag{A.9}$$

$$\lambda_2 = \frac{1}{2}\left(r + \sqrt{r^2 + 4\gamma M^2}\right) > 0. \tag{A.10}$$

The general solution is

$$\begin{bmatrix} q(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 1 \\ v_1 \end{bmatrix} h_1 e^{\lambda_1 t} + \begin{bmatrix} 1 \\ v_2 \end{bmatrix} h_2 e^{\lambda_2 t} \tag{A.11}$$

where

$$v_1 = -(a_{12})^{-1}(a_{11} - \lambda_1) \text{ and } v_2 = -(a_{12})^{-1}(a_{11} - \lambda_2)$$

and h_1 and h_2 are determined by the boundary conditions.

$$v_1 = -\frac{r - \lambda_1}{\gamma M^2} < 0.$$

It what follows, we assume $Y_0 < \widehat{Y}$.

Now, consider two cases.

Case 1: $R_0 > \widehat{Y} - Y_0$

Case 2: $R_0 < \widehat{Y} - Y_0$

Case 1: Clearly, there exists a unique positive value $\widetilde{R} < R_0$ such that $Y_0 + R_0 - \widetilde{R} = \widehat{Y}$. Then the social planner's optimal program is to take the stable branch of the saddlepoint and approach the steady state $(\widehat{Y}, \widehat{q})$ asymptotically. As $t \to T$, $q(t) \to 0$ and $R(t) \to \widetilde{R}$.

For the stable branch of the saddle-point, we set $h_2 = 0$.

Then

$$q - \widehat{q} = (q_0 - \widehat{q})e^{\lambda_1 t} = h_1 e^{\lambda_1 t}$$

$$z = Y - \widehat{Y} = (Y_0 - \widehat{Y})e^{\lambda_1 t} = z_0 e^{\lambda_1 t} = v_1 h_1 e^{\lambda_1 t}$$

where Y_0 is known but q_0 has to be determined. We can determine h_1 from

$$z(0) = v_1 h_1$$

i.e.

$$h_1 = \frac{z_0}{v_1} = \frac{(Y_0 - \widehat{Y})\gamma M^2}{-(r - \lambda_1)} > 0 \text{ for } Y_0 < \widehat{Y}.$$

Hence the optimal path of extraction expressed as a function of time is

$$q(t) - \widehat{q} = \frac{(\frac{rA}{\gamma M} - Y_0)\gamma M^2}{(r - \lambda_1)} e^{\lambda_1 t} \text{ for } 0 \leq t < \infty. \tag{A.12}$$

Thus, under $0 < \frac{\gamma M Y_0}{r} < A$ and $Y_0 + R_0 > \widehat{Y}$, the consumer's price is

$$p^c(t) = A - c_i(t) = A - \frac{q(t)}{M} = A - \frac{(\frac{rA}{\gamma M} - Y_0)\gamma M}{(r - \lambda_1)} e^{\lambda_1 t}. \tag{A.13}$$

The optimal consumer price is rising, and asymptotically approaches the choke price A as $t \to \infty$.

Remark 1: Instead of expressing the optimal extraction q as a function of time, it is sometimes more convenient to express the optimal control in the feedback form, $q = q^{FB}(Y)$. This is done as follows. Since $\dot{z} = \dot{Y} = q$, we have, using (A.8)

$$\ddot{Y}(t) = \dot{q}(t) = rq(t) + \gamma M^2 z(t) = r\dot{Y} + \gamma M^2 \left[Y(t) - \widehat{Y} \right]$$

which again gives the characteristic equation $\lambda_2 - r\lambda - \gamma M^2 = 0$, and hence, with $\lambda_1 < 0$ and $\lambda_2 > 0$,

$$z(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}.$$

To ensure convergence to the steady state, we set $A_2 = 0$. Then

$$z(t) = A_1 e^{\lambda_1 t} = z_0 e^{\lambda_1 t}.$$

Differentiate with respect to t

$$\dot{z}(t) = \lambda_1 z_0 e^{\lambda_1 t} = \lambda_1 z(t).$$

Thus the optimal control in feedback form is

$$q = \lambda_1 z = \frac{1}{2}\left(r - \sqrt{r^2 + 4\gamma M^2}\right) z = \frac{1}{2}\left(-r + \sqrt{r^2 + 4\gamma M^2}\right)\left(\widehat{Y} - Y\right) \text{ for } Y < \widehat{Y}. \tag{A.14}$$

This is consistent with eq (A.12) since $-\lambda_1 = \frac{\gamma M^2}{r-\lambda_1}$.

With $q = -\lambda_1(\widehat{Y} - Y)$, the optimal consumer's price in feedback form is

$$p = A - \frac{q}{M} = A + \frac{\lambda_1(\widehat{Y} - Y)}{M}.$$

This indicates that the price rises as $Y(t)$ approaches $\widehat{Y}$.

Remark 2: The value function approach.

Instead of solving (2) using the maximum principle, we can solve it using the Hamilton-Jacobi-Belman (HJB) equation, by seeking a value function $V(Y)$ that satisfies the HJB eq.

$$rV(Y) = \max_q \left[Aq - \frac{q^2}{2M} + M\overline{x} - \frac{M\gamma Y^2}{2} + V'(Y)q\right]. \tag{A.15}$$

Maximizing the right-hand side of eq. (A.15) we get

$$A - \frac{q}{M} = -V'(Y) \tag{A.16}$$

as the necessary condition. In the case where $Y_0 + R_0 > \widehat{Y}$, we can show that a quadratic value function would satisfy the HJB equation. To prove this, let us try the quadratic value function

$$V(Y) = \alpha + \beta Y + \frac{\xi}{2} Y^2 \tag{A.17}$$

where α, β and ξ are to be determined.

Then

$$V'(Y) = \beta + \xi Y. \tag{A.18}$$

Substituting this into (A.16) we get

$$A - \frac{q}{M} = -\beta - \xi Y \tag{A.19}$$

Eq (A.19) gives the linear feedback control rule

$$q = M\left(A + \beta + \xi Y\right). \tag{A.20}$$

Then the differential equation for Y is

$$\dot{Y} = M\left(A + \beta + \xi Y\right). \tag{A.21}$$

The solution path for the differential eq. (A.21) approaches a steady state iff $\xi < 0$.

The steady state is

$$Y_\infty = \frac{A + \beta}{-\xi}$$

where β and ξ are determined below.

Substitute (A.18) and (A.20) into the right-hand side of eq. (A.15)

$$AM\left(A+\beta+\xi Y\right)-\frac{M}{2}\left(A+\beta+\xi Y\right)^{2}+M\overline{x}-\frac{M\gamma Y^{2}}{2}+\left(\beta+\xi Y\right)M\left(A+\beta+\xi Y\right)$$

i.e.

$$M\left\{A\left(A+\beta+\xi Y\right)-\frac{1}{2}\left(A+\beta+\xi Y\right)^{2}+\overline{x}-\frac{\gamma Y^{2}}{2}+\left(\beta+\xi Y\right)\left(A+\beta+\xi Y\right)\right\}.$$

So the RHS of the HJB equation is

$$\frac{Y^{2}M}{2}\left(\xi^{2}-\gamma\right)+\xi MY\left(A+\beta\right)+M\left[\frac{1}{2}\left(A+\beta^{2}\right)+\overline{x}\right].$$

The LHS of the HJB equation is

$$r\alpha+r\beta Y+\frac{r\xi}{2}Y^{2}.$$

Since the LHS must equal the RHS for all feasible values of Y, we must equate the coefficients of Y^2

$$\frac{r\xi}{2}=\frac{M}{2}\left(\xi^{2}-\gamma\right)$$

i.e.

$$M\xi^{2}-r\xi-\gamma M=0$$

$$\xi=\frac{r\pm\sqrt{r^{2}+4\gamma M^{2}}}{2M}. \tag{A.22}$$

We take the negative root to ensure that the solution path for the differential eq. (A.21) approaches a steady state. Then

$$\xi=\frac{r-\sqrt{r^{2}+4\gamma M^{2}}}{2M}<0 \text{ since } \gamma>0.$$

Next, equating the coefficients of Y

$$r\beta=\xi M\left(A+\beta\right).$$

Thus

$$\beta=\frac{A\xi M}{\left(r-M\xi\right)}<0. \tag{A.23}$$

Then

$$V'(Y)=\beta+\xi Y<0$$

and

$$q=M\left(A+\beta+\xi Y\right)=MA+\frac{A\xi M^{2}}{\left(r-M\xi\right)}+M\xi Y.$$

Compare with

$$q=\frac{1}{2}\left(-r+\sqrt{r^{2}+4\gamma M^{2}}\right)\left(\widehat{Y}-Y\right).$$

The two equations are the same because $\mu=\frac{1}{2M}\left(r-\sqrt{r^{2}+4\gamma M^{2}}\right)$.

The steady state is

$$Y_\infty = \frac{A+\beta}{-\xi} = A\left[\frac{r}{-r\xi + M\xi^2}\right] = \frac{Ar}{M\gamma}.$$

Finally, we can solve for α

$$\alpha = \frac{M}{2r}\left((A+\beta)^2 + 2\overline{x}\right) \;= \frac{M}{2r}\left(A + \frac{A\xi M}{(r-M\xi)}\right)^2 + \frac{M\overline{x}}{r}. \tag{A.24}$$

Case 2: Clearly, if $Y_0 + R_0 < \widehat{Y} \equiv \frac{rA}{\gamma M}$, then there does not exist a value $\widetilde{R} \leq R_0$ such that $Y_0 + \widetilde{R} = \widehat{Y}$. Thus it is not feasible to reach $\widehat{Y}$, given that R_0 is so small. The optimal solution in this case is to exhaust the resource stock in finite (at some time $T^* < \infty$), and $R^*(T^*) = 0$, $Y^*(T^*) = Y_0 + R_0 < \widehat{Y}$. The optimal path is not a saddle path, but lies below the saddle path, with $q^*(T^*) = 0$. Then

$$z^*(T^*) = Y^*(T^*) - \widehat{Y} = (Y_0 + R_0) - \widehat{Y}$$

and

$$q^*(T^*) = 0.$$

Since $z(0) = Y_0 - \widehat{Y}$, we get from (A.11) the following 4 equations to solve for the 4 unknowns h_1, h_2, T^* and $q^*(0)$: $\begin{bmatrix} q(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 1 \\ v_1 \end{bmatrix} h_1 e^{\lambda_1 t} + \begin{bmatrix} 1 \\ v_2 \end{bmatrix} h_2 e^{\lambda_2 t}$

$$0 = h_1 e^{\lambda_1 T^*} + h_2 e^{\lambda_2 T^*}$$

$$(Y_0 + R_0) - \widehat{Y} = v_1 h_1 e^{\lambda_1 T^*} + v_2 h_2 e^{\lambda_2 T^*}$$

$$q^*(0) = h_1 + h_2$$

$$Y_0 - \widehat{Y} = v_1 h_1 + v_2 h_2.$$

A.2 Finding the Nash equilibrium values

Equations (42) and (43) can be used to solve for λ and η. After that, we can solve for δ and σ, using eq.s (40) and (41).

From (43),

$$2\lambda = \eta + \mu_X^*(\eta) = 2\eta + \frac{r}{M} - \sqrt{\left(\frac{r}{M}\right)^2 + \frac{2}{M}\eta r}$$

i.e.

$$\sqrt{\left(\frac{r}{M}\right)^2 + \frac{2}{M}\eta r} = 2(\eta - \lambda) + \frac{r}{M}.$$

Squaring both sides

$$\left(\frac{r}{M}\right)^2 + \frac{2}{M}\eta r = 4(\eta - \lambda)^2 + \left(\frac{r}{M}\right)^2 + 4(\eta - \lambda)\frac{r}{M}.$$

Thus

$$(2\lambda - \eta) = \frac{2M}{r}(\lambda - \eta)^2 \geq 0. \tag{A.25}$$

This equation requires that $2\lambda \geq \eta$.

Consider next eq. (42)

$$\eta = -\frac{1}{2}\left[\frac{r}{M} - 2\lambda - \sqrt{\left(\frac{r}{M} - 2\lambda\right)^2 + 4(\gamma - \lambda^2)}\right] \tag{A.26}$$

i.e.

$$2(\eta - \lambda) + \frac{r}{M} = \sqrt{\left(\frac{r}{M} - 2\lambda\right)^2 + 4(\gamma - \lambda^2)}\ .$$

Squaring both sides

$$\begin{aligned} \left(\frac{r}{M}\right)^2 + 4(\eta-\lambda)^2 + 4(\eta-\lambda)\frac{r}{M} &= \left(\frac{r}{M} - 2\lambda\right)^2 + 4(\gamma - \lambda^2) \\ &= \left(\frac{r}{M}\right)^2 + 4\lambda^2 - 4\lambda\frac{r}{M} + 4\gamma - 4\lambda^2. \end{aligned}$$

So

$$(\eta - \lambda)^2 + \frac{r\eta}{M} = \gamma > 0.$$

Thus

$$\frac{M}{r}\gamma - \eta = \frac{M}{r}(\lambda - \eta)^2. \tag{A.27}$$

From eq. (A.27) and (A.25) we get

$$2\lambda - \eta = \frac{2M}{r}\gamma - 2\eta.$$

So

$$2\lambda = \frac{2M}{r}\gamma - \eta$$

or

$$\eta = \frac{2M}{r}\gamma - 2\lambda > 0 \text{ by condition (38).} \tag{A.28}$$

Substitute (A.28) into (A.25)

$$2\lambda - \left(\frac{2M}{r}\gamma - 2\lambda\right) = \frac{2M}{r}\left[\lambda - \left(\frac{2M}{r}\gamma - 2\lambda\right)\right]^2$$

or

$$2\lambda - \frac{M}{r}\gamma = \frac{M}{r}\left[\lambda - \left(\frac{2M}{r}\gamma - 2\lambda\right)\right]^2$$

or

$$2\lambda - \frac{M}{r}\gamma = \frac{M}{r}\left[3\lambda - \frac{2M}{r}\gamma\right]^2$$

or

$$\begin{aligned}\frac{2r}{M}\lambda-\gamma &= \left[3\lambda-\frac{2M}{r}\gamma\right]^2\\ &= 9\lambda^2+4\left(\frac{M\gamma}{r}\right)^2-12\frac{M\gamma}{r}\lambda\end{aligned}$$

$$9\lambda^2-\lambda\left(12\frac{M\gamma}{r}+\frac{2r}{M}\right)+4\left(\frac{M\gamma}{r}\right)^2+\gamma=0. \tag{A.29}$$

Solving this quadratic eq. for λ. The discriminant is

$$\Delta\equiv\left(12\frac{M\gamma}{r}+\frac{2r}{M}\right)^2-36\left[4\left(\frac{M\gamma}{r}\right)^2+\gamma\right]=12\gamma+4\left(\frac{r}{M}\right)^2>0.$$

So there are two positive real roots, λ_1 and λ_2

$$\lambda_1=\frac{1}{9}\left[6\frac{M\gamma}{r}+\frac{r}{M}-\sqrt{3\gamma+\left(\frac{r}{M}\right)^2}\right]>0.$$

Note: $\lambda_1>0$ because

$$6\frac{M\gamma}{r}+\frac{r}{M}>\sqrt{3\gamma+\left(\frac{r}{M}\right)^2}$$

$$\lambda_2=\frac{1}{9}\left[6\frac{M\gamma}{r}+\frac{r}{M}+\sqrt{3\gamma+\left(\frac{r}{M}\right)^2}\right]>0.$$

Can we use both λ_1 and λ_2?

We now show that λ_2 (the bigger root) must be ruled out because it cannot satisfy simultaneously the conditions (A.28) and (A.26). To see this, suppose λ satisfy both (A.28) and (A.26). Then

$$-\frac{1}{2}\left[\frac{r}{M}-2\lambda-\sqrt{\left(\frac{r}{M}-2\lambda\right)^2+4(\gamma-\lambda^2)}\right]=\eta=2\left(\frac{M\gamma}{r}-\lambda\right)$$

i.e.

$$-\frac{1}{2}\left[\frac{r}{M}-2\lambda-\sqrt{\left(\frac{r}{M}\right)^2+4(\gamma-\frac{r\lambda}{M})}\right]=\eta=2\left(\frac{M\gamma}{r}-\lambda\right).$$

Then

$$6\lambda-\frac{4M\gamma}{r}=\frac{r}{M}-\sqrt{\left(\frac{r}{M}\right)^2+4(\gamma-\frac{r\lambda}{M})}. \tag{A.30}$$

Recall the restriction that (see eq. (38)):

$$\lambda<\frac{M\gamma}{r}.$$

Therefore the right hand side of (A.30) is negative.

This in turn implies that

$$6\lambda - \frac{4M\gamma}{r} < 0$$

i.e

$$\lambda < \frac{2}{3}\left(\frac{M\gamma}{r}\right). \tag{A.31}$$

So the root λ_2 would be admissible only if

$$\frac{1}{9}\left[6\frac{M\gamma}{r} + \frac{r}{M} + \sqrt{3\gamma + \left(\frac{r}{M}\right)^2}\right] < \left(\frac{2}{3}\right)\frac{M\gamma}{r}$$

i.e only if

$$\frac{r}{M} + \sqrt{3\gamma + \left(\frac{r}{M}\right)^2} < 0.$$

But this inequality cannot be satisfied given that $\gamma > 0$.

Therefore the only admissible root is λ_1, i.e., the Nash equilibrium is unique.

Having found λ_i^*, the associated η_i^* is

$$\eta_i^* = -\frac{1}{2}\left[\frac{r}{M} - 2\lambda_i^* - \sqrt{\left(\frac{r}{M} - 2\lambda_i^*\right)^2 + 4\gamma - 4(\lambda_i^*)^2}\right].$$

Note that $\eta_i^* > 0$ iff

$$\gamma > (\lambda_i^*)^2.$$

Since we require that $\eta_i^* > 0$ (to ensure that $\overline{Y} > 0$), we will impose the restriction $\gamma > (\lambda_i^*)^2$ for a Nash equilibrium. Making use of (A.28) we get

$$\eta_i^* = 2\left(\frac{M\gamma}{r} - \lambda_i^*\right) > 0.$$

Before calculating δ^* and σ^* let us state an important lemma:

LEMMA 1

$$\mu_I^*(\lambda_i^*) + \lambda_i^* = \frac{1}{2}(\mu_X^*(\eta_i^*) - \eta_i^*) < 0$$

Proof

$$\mu_I + \lambda^* = \frac{1}{2}\left[\frac{r}{M} - \sqrt{\left(\frac{r}{M} - 2\lambda^*\right)^2 + 4(\gamma - \lambda^{*2})}\right]$$

while

$$\frac{1}{2}(\mu_X^*(\eta_i^*) - \eta_i^*) = \frac{1}{2}\left[\frac{r}{M} - \sqrt{\left(\frac{r}{M}\right)^2 + \frac{2r\eta^*}{M}}\right]$$

The two RHS expressions are equal because

$$\eta_i^* = 2\left(\frac{M\gamma}{r} - \lambda_i^*\right).$$

QED.

From eq. $\delta = \frac{1}{2}(A - \sigma - \beta_X^*(\sigma, \eta))$, we get

$$2\delta - A + \sigma - (A - \sigma)\left[\frac{\eta - \mu_X^*}{\eta - \mu_X^* + \frac{2r}{M}}\right] = 0. \tag{A.32}$$

From eq. $\sigma = -\beta_I^*,$ we get

$$\sigma - (A - \delta)\frac{2(\mu_I^* + \lambda)}{2(\mu_I^* + \lambda) - (2r/M)} = 0.$$

Using Lemma 1, the above equation becomes

$$\sigma - (A - \delta)\left[\frac{\eta - \mu_X^*}{\eta - \mu_X^* + \frac{2r}{M}}\right] = 0. \tag{A.33}$$

Define

$$Q = z(\eta^*) = \left[\frac{\eta - \mu_X^*}{\eta - \mu_X^* + \frac{2r}{M}}\right]$$

Note that $0 < Q < 1$. Then the system of eq.s (A.32) and (A.33) becomes

$$\begin{bmatrix} 2 & 1+Q \\ Q & 1 \end{bmatrix}\begin{bmatrix} \delta \\ \sigma \end{bmatrix} = \begin{bmatrix} A(1+Q) \\ AQ \end{bmatrix}$$

Then

$$\delta^* = \frac{A(1+Q)(1-Q)}{2 - Q(1+Q)} \tag{A.34}$$

$$\sigma^* = \frac{AQ(1-Q)}{2 - Q(1+Q)} \tag{A.35}$$

This implies that

$$2\delta^* - A = \sigma^*.$$

A.3 Proof of linear relationship between δ^* and λ^*

We must prove that for the smaller root, λ_1, the following equation holds: $\delta^* = \frac{\lambda^* Ar}{\gamma M}$.

We must show

$$\frac{A(1+Q)(1-Q)}{2 - Q(1+Q)} = \frac{\lambda^* Ar}{\gamma M}$$

i.e

$$(1 - Q^2)\left(1 - \frac{\lambda^* r}{\gamma M}\right) = \frac{\lambda^* r}{\gamma M}(1 - Q)$$

i.e

$$(1 + Q)\left(1 - \frac{\lambda^* r}{\gamma M}\right) = \frac{\lambda^* r}{\gamma M}.$$

We must show

$$(1+Q)(\gamma - \lambda\frac{r}{M}) = \lambda\frac{r}{M}.$$

Now, let us compute $1+Q$. Recall that

$$Q = \left[\frac{\eta - \mu_X^*}{\eta - \mu_X^* + \frac{2r}{M}}\right] = \frac{-2(\mu_I^*(\lambda_i^*) + \lambda_i^*)}{-2(\mu_I^*(\lambda_i^*) + \lambda_i^*) + \frac{2r}{M}}$$

where

$$\begin{aligned}
-2(\mu_I^*(\lambda_i^*) + \lambda_i^*) &= -\frac{r}{M} + \sqrt{\left(\frac{r}{M} - 2\lambda^*\right)^2 + 4(\gamma - \lambda^{*2})} \\
&= -\frac{r}{M} + \sqrt{\left(\frac{r}{M}\right)^2 + 4(\gamma - \frac{\lambda r}{M})}.
\end{aligned}$$

Thus

$$Q = \frac{\sqrt{\left(\frac{r}{M}\right)^2 + 4(\gamma - \frac{\lambda r}{M})} - \frac{r}{M}}{\sqrt{\left(\frac{r}{M}\right)^2 + 4(\gamma - \frac{\lambda r}{M})} + \frac{r}{M}}$$

So we must show

$$2\left[(\gamma - \lambda\frac{r}{M})\right]\sqrt{\left(\frac{r}{M}\right)^2 + 4(\gamma - \frac{\lambda r}{M})} = \lambda\frac{r}{M}\left[\sqrt{\left(\frac{r}{M}\right)^2 + 4(\gamma - \frac{\lambda r}{M})} + \frac{r}{M}\right]$$

i.e

$$\left[2\gamma - 3\lambda\frac{r}{M}\right]\sqrt{\left(\frac{r}{M}\right)^2 + 4(\gamma - \frac{\lambda r}{M})} = \frac{r^2\lambda}{M^2}. \tag{A.36}$$

Square both sides of (A.36)

$$\left(2\gamma - 3\lambda\frac{r}{M}\right)^2\left(\left(\frac{r}{M}\right)^2 + 4(\gamma - \frac{\lambda r}{M})\right) = \left(\frac{r}{M}\right)^2\left(\frac{r\lambda}{M}\right)^2.$$

Then we have

$$\left[4\gamma^2 + 9\lambda^2\left(\frac{r}{M}\right)^2 - 12\lambda\gamma\frac{r}{M}\right]\left(\left(\frac{r}{M}\right)^2 + 4(\gamma - \frac{\lambda r}{M})\right) = \left(\frac{r}{M}\right)^2\left(\frac{r\lambda}{M}\right)^2.$$

Dividing both sides by $(r/M)^2$, we get

$$\left[4\gamma^2\frac{M^2}{r^2} + 9\lambda^2 - 12\lambda\gamma\frac{M}{r}\right]\left(\left(\frac{r}{M}\right)^2 + 4(\gamma - \frac{\lambda r}{M})\right) = \left(\frac{r\lambda}{M}\right)^2 \tag{A.37}$$

But $9\lambda^2 - \lambda\left(12\frac{M\gamma}{r} + \frac{2r}{M}\right) + 4\left(\frac{M\gamma}{r}\right)^2 + \gamma = 0$ (eq. (A.29) above). Therefore

$$4\gamma^2\frac{M^2}{r^2} + 9\lambda^2 - 12\lambda\gamma\frac{M}{r} = \frac{2r\lambda}{M} - \gamma$$

Then eq. (A.37) becomes

$$\left(\frac{2r\lambda}{M}-\gamma\right)\left(\left(\frac{r}{M}\right)^2+4(\gamma-\frac{\lambda r}{M})\right)=\left(\frac{r\lambda}{M}\right)^2$$

i.e.

$$-\gamma\left(\frac{r}{M}\right)^2-4\gamma^2+12\gamma\lambda\frac{r}{M}+2\lambda\left(\frac{r}{M}\right)^3=9\left(\frac{r\lambda}{M}\right)^2.$$

Divide by $(r/M)^2$

$$-\gamma-4\left(\frac{\gamma M}{r}\right)^2+12\lambda\frac{M\gamma}{r}+\frac{2r\lambda}{M}=9\lambda^2 \tag{A.38}$$

which is true, by (A.29).

A.4 Proof that the FOC of the importing coalition acting as leader has a unique positive root

We have the FOC

$$3rG-[r(3\eta+r)-2\gamma]\,r=2G^{3/2}.$$

Define

$$s=2\eta+r.$$

Then $G=rs$ and the FOC becomes

$$r^{1/2}\left(\frac{3s}{2}+\frac{r}{2}+\frac{2\gamma}{r}\right)=2s^{3/2}.$$

Squaring both sides, we get the equation

$$4s^3-\frac{9r}{4}s^2-3r\left(\frac{r}{2}+\frac{2\gamma}{r}\right)s-r\left(\frac{r}{2}+\frac{2\gamma}{r}\right)^2=0.$$

Define

$$g(s)=4s^3-\frac{9r}{4}s^2-3r\left(\frac{r}{2}+\frac{2\gamma}{r}\right)s=s\left[4s^2-\frac{9r}{4}s-3r\left(\frac{r}{2}+\frac{2\gamma}{r}\right)\right].$$

Then we look for a value of s such that

$$g(s)=r\left(\frac{r}{2}+\frac{2\gamma}{r}\right)^2.$$

Graphing the LHS, we see it is a curve that goes through the origin, and cuts the horizontal axis at a positive s_1 and a negative s_2, where s_1 and s_2 solve

$$\left[4s^2-\frac{9r}{4}s-3r\left(\frac{r}{2}+\frac{2\gamma}{r}\right)\right]=0.$$

Furthermore, $g(s)\to\infty$ as $s\to\infty$. So $g(s)$ intersects the RHS (which is a horiontal line) at a unique value $s^*>s_1>0$.

A.5 Proof that the FOC of the oil cartel acting as leader has a unique positive root

We have the FOC

$$rF(\lambda) + 2\lambda r^2 - F(\lambda)^{\frac{3}{2}} = 0.$$

Define the new variable z by

$$z = r + \frac{4\gamma}{r} - 4\lambda.$$

Then the FOC becomes

$$r^{1/2}\left[\frac{z+r}{2} + \frac{2\gamma}{r}\right] z^{1/2} = z^2.$$

Squaring both sides, we obtain the following cubic equation in z

$$4z^3 - rz^2 - 2r\left(\frac{4\gamma}{r} + r\right)z = r\left(\frac{4\gamma}{r} + r\right)^2.$$

Next, find λ using the formula

$$\lambda = \frac{1}{4}\left[r + \frac{4\gamma}{r} - z\right].$$

Next, compute

$$F^{1/2} = \sqrt{\left(\frac{r}{M}\right)^2 + 4\left(\gamma - \frac{\lambda r}{M}\right)}.$$

References

[1] Allen, Myles R., D. J. Frame, C. Huntingford, C.D. Jones, J. A. Lowe, M. Meinhausen and N. Meinhausen (2009), Warming caused by cumulative carbon emissions towards the trillionth tonne, *Nature*, Volume 458(7242), pp. 1163-1166, 30 April 2009, doi:10.1038/nature08019.

[2] Fujiwara, Kenji. and Ngo Van Long (2011), “Welfare Implications of Leadership in a Resource Market under Bilateral Monopoly, *Dynamic Games and Applications*, Volume 1, December 2011.

[3] Haurie, M. B.C. Tavoni, C.van der Zwaan, (2012), “Modeling Uncertainy and the Climate Change:Recommendations for Robust Energy Policy, *Environ Model Assess* 17:1-5.

[4] Heal,G. (2009), “The economics of climate change: A post-Stern perspective,” *Climate Change* 96, 275-297.

[5] Karp, L. (1984), Optimality and Consistency in a Differential Game with Non-Renewable Resources, *Journal of Economic Dynamics and Control* 8: 73-97.

[6] Kemp M. C. and N. V. Long, 1980, ”Optimal Tariffs on Exhaustible Resources,” in Kemp and Long (Jeds), *Exhaustible Resources, Optimality, and Trade*, North Holland, pp 183-186.

[7] Stern, N., (2006), *The Stern Review of the Economics of Climate Change*, The British Government.

[8] Stern, N. at al., (2006), *The economics of Climate Change: The Stern Review*, Cambridge University Press.

[9] Wirl, F (1995), The Exploitation of Fossil fuels under the Threat of Global Warming and Carbon Taxes: A Dynamic Game Approach, *Environmental and Resource Economics* 39: 1125-1136.

Table 1 Welfare Comparison by Time Discount Rate

Example. 0 (r=0.05)	World welfare	importer welfare	exporter welfare
Social planning	$\frac{M\bar{x}}{r}+21.983275$		
Nash equilibrium	$\frac{M\bar{x}}{r}+20.832849$	$\frac{M\bar{x}}{r}+6.9442906$	3.8384
Exporter Stackelberg	$\frac{M\bar{x}}{r}+18.712$	$\frac{M\bar{x}}{r}+2.9019$	15.810
Importer Stackelberg	$\frac{M\bar{x}}{r}+16.99611$	$\frac{M\bar{x}}{r}+13.644777$	3.3213284
Example. 1 (r=0.01)			
Social planning	$\frac{M\bar{x}}{r}+5.8234$		
Nash equilibrium	$\frac{M\bar{x}}{r}+5.7572$	$\frac{M\bar{x}}{r}+1.9188$	3.8384
Exporter Stackelberg	$\frac{M\bar{x}}{r}+5.3768$	$\frac{M\bar{x}}{r}+0.33423$	5.0426
Importer Stackelberg	$\frac{M\bar{x}}{r}+5.19396$	$\frac{M\bar{x}}{r}+4.7857$	0.40826
Example.2 (r=0.1)			
Social planning	$\frac{M\bar{x}}{r}+31.25$		
Nash equilibrium	$\frac{M\bar{x}}{r}+28.215$	$\frac{M\bar{x}}{r}+9.4045$	18.810
Exporter Stackelberg	$\frac{M\bar{x}}{r}+25.4781$	$\frac{M\bar{x}}{r}+5.4541$	20.024
Importer Stackelberg	$\frac{M\bar{x}}{r}+21.949$	$\frac{M\bar{x}}{r}+16.061$	5.8878
Example.3 (r=0.001)			
Social planning	$\frac{M\bar{x}}{r}+0.62062$		

Nash equilibrium	$\frac{M\bar{x}}{r}+0.61742$	$\frac{M\bar{x}}{r}+0.20605$	0.41137
Exporter Stackelberg	$\frac{M\bar{x}}{r}+0.60658$	$\frac{M\bar{x}}{r}+0.0086861$	0.59786
Importer Stackelberg	$\frac{M\bar{x}}{r}+0.60199$	$\frac{M\bar{x}}{r}+0.5911$	0.01089