Tensor modes and gauge fields during Inflation

Kei Yamamoto

Kobe University

April 25, 2014

Outline

The background: FLRW

- De facto standard of modern cosmology:

$$ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2$$

- Describes a spatially homogeneous, flat and isotropic universe.

In agreement with;

- Highly isotropic Cosmic Microwave Background Radiation (CMBR).
- Abundance of light elements in the universe (BBN).

Einstein ⇒ Friedman

- The symmetry + Einstein Field Equations ⇒ Energy-momentum tensor of perfect fluid type
- The spacetime is completely characterised by the Hubble scalar:

$$G_{00} = 3H^2 = M_{pl}^{-2}\rho \ , \qquad H = \frac{\dot{a}}{a} \ ,$$
$$G_{11} = G_{22} = G_{33} = -2\dot{H} - 3H^2 = M_{pl}^{-2} p \ .$$

- In particular, $M_{pl}^2 H^2$ is equivalent to the energy density of the universe.

Linearisation, as always...

- The real universe is obviously neither homogeneous nor isotropic.
- Well, almost.
- Linear perturbation will serve to see the stability (and more).

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu} \ .$$

Choosing variables

- Many ways to parametrise the perturbed metric.
- Gauge freedom of the geometric description.
- Einstein's equations form a constraint system.
- Convenience is the deciding factor.
- A typical choice:

$$\delta g_{\mu\nu} = a(t)^2 \begin{pmatrix} 2\phi & \partial_j B - S_j \\ \partial_i B - S_i & -2\psi\delta_{ij} + 2\partial_i\partial_j E + 2\partial_{(i}F_{j)} + h_{ij} \end{pmatrix}$$

Scalar-Vector-Tensor decomposition

- Scalar: ϕ, ψ, B, E (4 variables $-$ 2 algebraic constraints $-$ 2 gauge freedom)
- Vector: S_i, F_i (2×2 variables $-$ 1×2 algebraic constraints $-$ 1×2 gague freedom)

$$\partial_i S_i = \partial_i F_i = 0$$

- Tensor: h_{ij} (1×2 variables, no gauge freedom)

$$\partial_i h_{ij} = 0 \ , \quad h_{ii} = 0$$

The sectors decouple from each other due to the symmetry of the background FLRW spacetime.

Quadratic action for tensor perturbation

$$\mathcal{L} = \frac{M_{pl}^2 a^2}{8}\left(\frac{dh_{ij}}{d\eta}\frac{dh_{ij}}{d\eta} - \partial_k h_{ij}\partial_k h_{ij}\right) , \quad d\eta = a^{-1}dt$$

The conformal time coordinate η is more convenient than the proper time t because:

- Written nicely in the canonical variable:

$$\tilde{h}_{ij} = \frac{M_{pl}a}{2} h_{ij} \ .$$

- The background FLRW is manifestly conforamlly related to Minkowski.

Canonical quantisation

- Action in terms of the canonical variable:

$$\mathcal{L} = \frac{1}{2}\left(\tilde{h}'_{ij}\tilde{h}'_{ij} + (\nabla^2 + \mathcal{H}' + \mathcal{H}^2)\tilde{h}_{ij}\tilde{h}_{ij}\right) \ , \quad \mathcal{H} = \frac{a'}{a} = \dot{a}$$

- Mode decomposition:

$$\tilde{h}_{ij} = \int \frac{d^3k}{(2\pi)^3} \sum_{s=+,\times} \left(h^s_k(\eta)\epsilon^s_{ij}(\mathbf{k})\hat{a}^s_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + (\mathrm{h.c.})\right)$$

$$\epsilon^s_{ii}(\mathbf{k}) = 0 \ , \quad \epsilon^s_{ij}(\mathbf{k})k_j = 0 \ , \quad \epsilon^s_{ij}(\mathbf{k})\epsilon^t_{ij}(\mathbf{k}) = \delta_{st}$$

- Commutators:

$$\left[\hat{a}^s_{\mathbf{p}}, \hat{a}^{t\dagger}_{\mathbf{q}}\right] = (2\pi)^3\delta_{st}\delta(\mathbf{p}-\mathbf{q}) \ , \quad \left[\hat{a}^s_{\mathbf{p}}, \hat{a}^t_{\mathbf{q}}\right] = 0 \ , \quad \mathrm{etc.}$$

Slow-roll approximation

- Equation of motion:

$$h_k^{s\prime\prime} + \left(k^2 - \mathcal{H}' - \mathcal{H}^2\right) h_k^s = 0 \ .$$

- Inflationary spacetime is characterised by its deviation from de Sitter:

$$H = H_0 - \epsilon_H H_0^2 t + O(\epsilon_H^2, \eta_H) \ , \quad \epsilon_H, \eta_H \ll 1$$

- Leading order in slow-roll:

$$a = -\frac{1}{H_0 \eta} + O(\epsilon_H) \ , \quad \mathcal{H} = -\frac{1}{\eta} + O(\epsilon_H) \ , \quad -\infty < \eta < 0$$

Vacuum mode function

- A solution (de Sitter mode function):

$$u_k(\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}$$

- General solution is an arbitrary linear combination of u_k and u_k^*.
- Take $h_k^s = u_k$ so that h_k^s is invariant under de Sitter time translation $t \to t + T, \mathbf{x} \to e^{-HT}\mathbf{x}$ (Bunch-Davies vacuum).

$$h_k^s \to \frac{1}{\sqrt{2k}}e^{-ik\eta} \quad \text{as} \quad \eta \to -\infty$$

Power spectrum

- The relative amplitude of tensor perturbation:

$$h_{ij} = 2(M_{pl}a)^{-1}\tilde{h}_{ij} \sim \frac{2H_0\eta}{M_{pl}} h^s_k$$

- Power spectrum:

$$\mathcal{P}_h(k) \propto 2 \times \frac{4H_0^2}{M_{pl}^2}\eta^2 \frac{1}{2k}\left(1 + \frac{1}{k^2\eta^2}\right) \rightarrow \frac{4H_0^2}{M_{pl}^2 k^3} \quad \text{as} \quad \eta \rightarrow 0$$

- As the amplitude freezes out,

$$\left[h_{ij}, \dot{h}_{kl}\right] \rightarrow 0 \quad \text{as} \quad \eta \rightarrow 0 \ .$$

$\Rightarrow$ fluctuations become "classical"

Matter perturbations in FLRW

- Scalar fields: $\varphi = \varphi^{(0)} + \delta\varphi \to$ scalar perturbations
- Perfect fluids (energy): $\rho = \rho^{(0)} + \delta\rho \to$ scalar perturbations
- Perfect fluids (velocity): $\mathbf{v} = \delta\mathbf{v} \to$ scalar + vector perturbations
- Gauge fields: $A_\mu = \delta A_\mu \to$ scalar + vector perturbations

Tensor mode is normally unaffected by matter perturbations

Tensor mode from scalar or vector modes

- Impossible without nonlinear combination.
- For example traceless part of $\delta v_{1i} \delta v_{2j}$.
- nth order perturbative contribution is typically suppressed by a factor of $\left(\frac{H}{M_{pl}}\right)^{2n}$
- Background gauge field may potentially do at linear order $A_\mu^{(0)} \delta A_\nu$

Background $SU(2)$ gauge field

- Consider an $SU(2)$ gauge field coupled to inflaton:

$$\mathcal{L} = -\frac{f(\varphi)^2}{4}\mathrm{tr}\,(F \wedge *F) \ ,$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \ , \quad a = 1, 2, 3$$

- Setting $A^a_0 = 0$ using gauge freedom.
- Isotropic background ansatz:

$$A^a_i = A^{(0)} \delta^a_i$$

- Leads to FLRW inflationary solution for a sufficiently small g.

Perturbation of $SU(2)$ gauge field

- Straightforwardly perturb and split each of the three vector fields into scalar + vector:

$$A_i^a = A^{(0)}\delta_i^a + \partial_i \alpha^a + \alpha_i^a \ , \quad \partial_i \alpha_i^a = 0 \ .$$

- This does not achieve mode decomposition because of the existence of the background gauge field, e.g. α^a, which are scalars above enter equations of vectors by $A^{(0)}\delta_i^a \partial^2 \alpha^a = A^{(0)}\partial^2 \alpha^i$
- The background δ_i^a mixes the gauge indices $a, b, \cdots$ with spatial ones $i, j, \cdots$ in the linear perturbation.

SVT decomposition of gauge field

- The correct decomposition:

$$\delta A^i_j = \alpha\delta^i_j + \theta_{,ij} + \epsilon_{ijk}\left(\tau_{,k} + \lambda_{,k}\right) + \kappa_{(i,j)} + \omega_{ij}$$

- Scalar: α, θ, τ (3 variables, 1 constraint)
- Vector: λ_i, κ_i (2×2 variables, 2×1 constraint)

$$\partial_i\lambda_i = 0\ , \quad \partial_i\kappa_i = 0$$

- Tensor: ω_{ij} (1×2 variables, no constraint)

$$\partial_i\omega_{ij} = 0\ , \quad \omega_{ii} = 0$$

Quadratic action

- Define the energy density of the background gauge field by

$$\rho_g = \frac{3f^2}{2a^2}\left(\frac{dA^{(0)}}{dt}\right)^2$$

- The correction to the free action is given in terms of ρ_g:

$$\mathcal{L} = \frac{M_{pl}^2 a^2}{8}\left(h'_{ij}h'_{ij} - h_{ij,k}h_{ij,k}\right) + \frac{a^4}{6}\rho_g h_{ij}h_{ij} + \frac{f^2}{2}\left(\omega'_{ij}\omega'_{ij} - \omega_{ij,k}\omega_{ij,k}\right) - a^2 f\sqrt{\frac{2\rho_g}{3}}h_{ij}\omega'_{ij}$$

Defining "particles"

- In the end, introduction of canonical variables is not essential (just fixing the normalisation)
- Since the Lagrangian is not diagonal in h_{ij} and ω_{ij}, quanta of gravitons and gauge bosons are not well-separated in general
- Hence Fourier decomposition should be

$$h_{ij} = \frac{2}{M_{pl}a} \sum_{a=1,2,s=+,\times} \int \frac{d^3k}{(2\pi)^3} \epsilon^s_{ij}(\mathbf{k}) \left[h^{s\,a}_k(\eta) \hat{a}^s_{a\mathbf{k}} e^{-\mathbf{k}\cdot\mathbf{x}} + (\text{h.c.}) \right]$$

$$\omega_{ij} = \frac{1}{f} \sum_{a=1,2,s=+,\times} \int \frac{d^3k}{(2\pi)^3} \epsilon^s_{ij}(\mathbf{k}) \left[\omega^{s\,a}_k(\eta) \hat{a}^s_{a\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + (\text{h.c.}) \right]$$

Quantisation ⇔ Determine mode functions

- If the action is quadratic, QFT is all about finding the right mode functions
- Equations up to order $\sqrt{\epsilon}$:

$$h_k^{a\prime\prime} - \frac{2}{\eta} h_k^{a\prime} + k^2 h_k^a = -\frac{4\eta^2}{M_{pl}^2}\sqrt{\frac{2\rho_g}{3}}\omega_k^{a\prime}$$

$$\omega_k^{a\prime\prime} + \frac{4}{\eta}\omega_k^{a\prime} + k^2\omega_k^a = \frac{1}{H_0^2\eta^4}\sqrt{\frac{2\rho_g}{3}} h_k^{a\prime}$$

- Solve with an appropriate initial conditions

Initial conditions

- In the far past, it can be shown that the r.h.s are negligible
- Take Bunch-Davies conditions for each of h^a and ω^a:

$$h^a_k \to -\delta^a_1 \frac{H_0 \eta}{M_{pl}} \sqrt{\frac{2}{k}} \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta}$$

$$\omega^a_k \to \frac{\delta^a_2}{\eta^2 \sqrt{2k}} \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta} \qquad \text{as} \quad \eta \to -\infty$$

Ultraviolet behaviour

- In the limit $\eta \to 0$, the terms proportional to k^2 can be dropped
- Solve as:

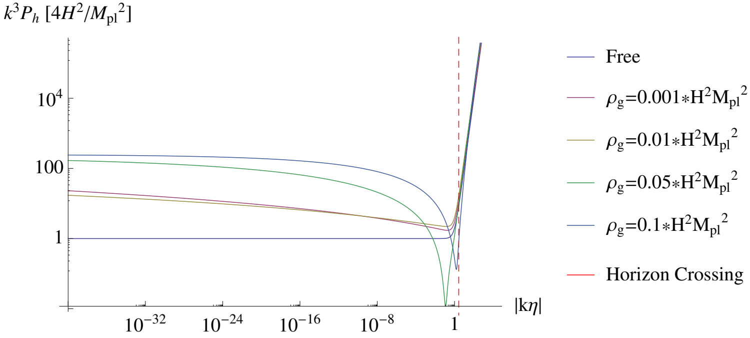
$$\begin{aligned} h_k^a &= \text{const} + O(\eta) \\ \omega_k^a &= \frac{\text{const}}{\eta^3} + O(\eta^{-2}) \end{aligned} \quad \text{as} \quad \eta \to 0$$

- The gravitational power spectrum becomes constant in far future:

$$\langle h_k^2 \rangle \to \frac{\text{const}}{k^3}$$

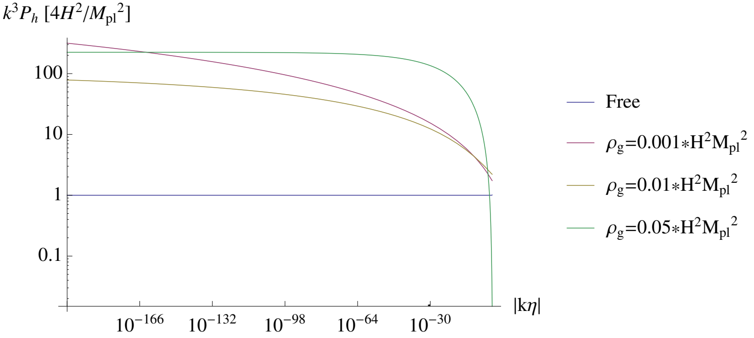
Numerical solution

- Tensor power spectrum for different values of the background gauge energy density



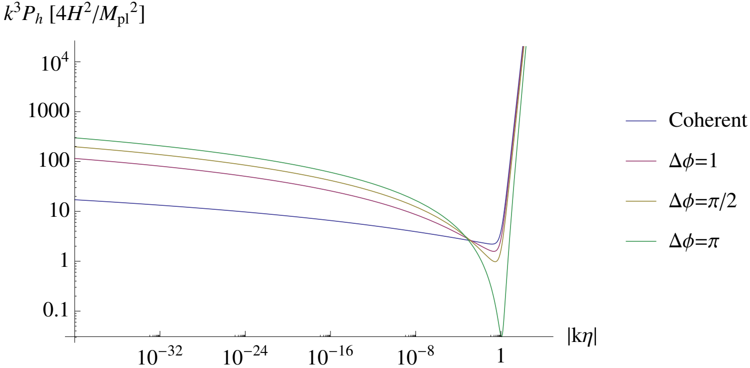
Numerical solution

- Tensor power spectrum for different values of the background gauge energy density



Numerical solution

- Dependence of power spectrum on phase difference $\Delta\phi$ between h_k and ω_k for $\rho_g = 0.01 M_{pl}^2 H_0^2$



Numerical solution

- Tensor-to-scalar ratio for different values of the slow-roll parameter ϵ_H for $\rho_g = 0.001 M_{pl}^2 H_0^2$