Topologically Twisted N=(2,2) SYM Theory on Arbitrary Discretized Riemann Surface

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based on work with T. Misumi and K. Ohta
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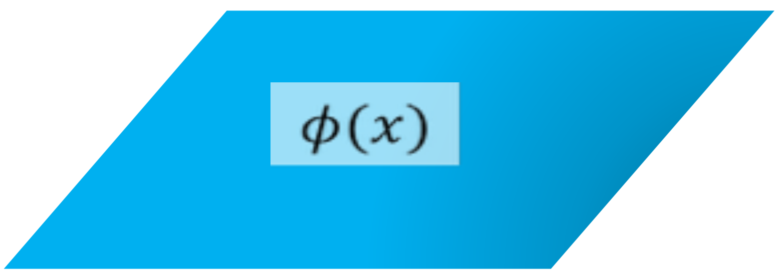
Review of Lattice Theory

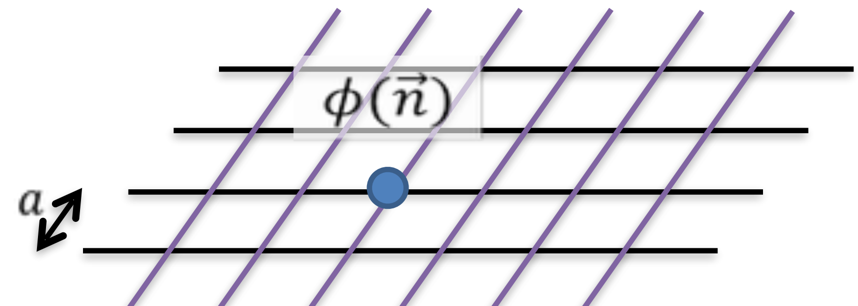
格子理論とは？

時空を格子で近似することで、
理論が有限自由度の統計力学になる。 → 数学的に厳密に

連続時空



格子時空



$$S = \int d^4x \left(\frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{\lambda}{4}\phi(x)^4 \right)$$

積分　微分

$$S = \sum_n \left(\frac{1}{2}\left(\frac{\phi(n+a\mu) - \phi(x)}{a} \right)^2 + \frac{\lambda}{4}\phi(n)^4 \right)$$

和　差分

格子ヤン・ミルズ理論

格子ヤン・ミルズ理論は、ヤン・ミルズ理論の非摂動論的な定式化になっている。

(連続) ヤン・ミルズ理論

基本的な自由度：ゲージ場 $A_\mu(x)$

$$S = \int d^4x Tr\left(\frac{1}{4}F_{\mu\nu}^2\right)$$

$$(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu])$$

ゲージ対称性：

$$A_\mu(x) \to g^{-1}(x)\partial_\mu g(x) + g^{-1}(x)A_\mu(x)g(x)$$

格子ヤン・ミルズ理論

基本的な自由度：リンク変数

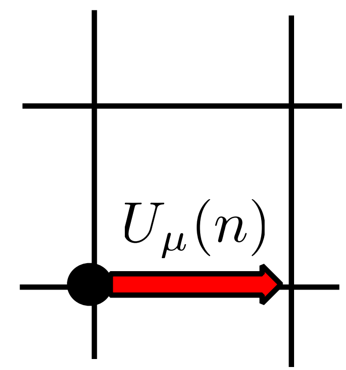
$$U_\mu(n) \sim e^{iaA_\mu(n)} \in U(N)$$

$$S = \sum_n Tr\left(U_\mu(n)U_\nu(n+a\mu)U_\mu^\dagger(n+a\nu)U_\nu^\dagger(n)\right)$$

ゲージ対称性：

$$U_\mu(n) \to g(n)^{-1}U_\mu(n)g(n+a\mu)$$

- （古典的な）連続極限は連続理論に一致する。
- ゲージ対称性が格子上でも保たれている。
- ローレンツ対称性は破れているが、連続極限で回復する。
  （ゲージ対称性とZ_4対称性のため）



格子による場の理論の定式化（一般論）

① a→0とすると連続理論の作用が得られるような、格子上の作用を与える。

② 格子上に出来る限り多くの対称性を残すように工夫する。

③ 連続理論には無いような

古典的にはirrelevant

になり得る。

キーワード

- ➢ 繰り込み群（繰り込み変換）
- ➢ ユニバーサリティ
- ➢ 対称性

(例) $\langle\phi(n)^2\rangle \sim a^{-2} \Longrightarrow a^D\sum_n a\phi(n)^5 \to a^D\sum_n a^{-1}\phi(n)^3$

- 不要なrelevant operatorが生じる時には、予めその項を作用から引いておく必要がある。（fine-tuning。大変面倒）
- 対称性を十分残しておくと、そのようなrelevant operator を事前に禁止することが出来る。すると、取り扱いも易しくなって都合が良い。

超対称性を持つようなゲージ理論

そもそも超対称性って何？

☆　基本的な代数

$$\{Q_\alpha, Q_\beta\} = 2\sigma^\mu_{\alpha\beta} P_\mu$$

← 超対称変換を２回行うと時空の並進になる

☆　bosonとfermionの間の対称性

$$\delta\phi(x) = [\epsilon Q, \phi(x)] \sim \epsilon\psi(x)$$

$$\delta\psi_\alpha(x) = [\epsilon Q, \psi_\alpha(x)] \sim i(\sigma^\mu\epsilon)_\alpha \partial_\mu \phi(x)$$

scalar $\phi(x)$ $\xleftrightarrow{\;Q\;}$ spinor $\psi(x)$

格子上の超対称性？

$$\{Q_\alpha, Q_\beta\} = 2\sigma^\mu_{\alpha\beta} \Delta_\mu$$

← 差分演算子

Leibnitz ruleを満たさない！

２回作用させると差分になるような変換を作るのはほぼ不可能

Lattice上に全ての超対称性を乗せるのはほとんど不可能

cf) Kato-Sakamoto-So (2008)

超対称ゲージ理論を格子上に乗せることは出来ない？

そんなことはない！

アイディア

✓ 超対称性の一部だけ格子上に残す

（extended SUSYの時には、並進を伴わない超対称変換が作れる。）

- Sugino mode
- orbifold lattice

✓ 他の対称性を積極的に使う

- overlap fermionを用いたlattice上のexact chiral symmetry

✓ 格子以外の正則化を使って超対称性を全部残す

- matrix modelへの埋め込み（非可換ゲージ理論）

目次

- 格子理論の復習
- 杉野模型の復習
- 我々の模型の導出と連続極限
- 格子理論から読み取れるトポロジーの情報

Review of
Sugino's Lattice Formulation

Continuum Theory

4D N=1 SYM

$$A_\mu, \quad (\mu = 1, 2)$$
$$A_3, \ A_4$$

dimensional

reduction

2D N=(2,2) SYM

$$A_\mu,$$
$$\Phi, \ \bar{\Phi} \ (\text{scalar fields})$$

$$S = \frac{1}{g_4^2} \int d^4 x Tr \left(\frac{1}{4} F_{MN}^2 + \frac{i}{2} \bar{\Psi} \Gamma^M D_M \Psi \right)$$

$$S = \frac{1}{g^2} \int d^2 x Tr \left(\frac{1}{4} F_{\mu\nu}^2 + \cdots \right)$$

フェルミオンの成分

$$\Psi = (\lambda_1, \lambda_2, \chi, \eta/2)^T$$

+

適当なディラック行列の表示

$$S = \frac{1}{2g^2} \int d^2 x Tr \Big(Y^2 - 2iY F_{12} + |D_\mu \Phi|^2 + \frac{1}{4} [\Phi, \bar{\Phi}]^2$$
$$+ 2i\chi (D_1 \lambda_2 - D_2 \lambda_1) + i\eta D_\mu \lambda_\mu$$
$$- \frac{1}{4} \eta [\Phi, \eta] - \chi [\Phi, \chi] + \lambda_\mu [\bar{\Phi}, \lambda_\mu] \Big)$$

Supersymmetry

4D SUSY transformation

$$\delta A_M = -i\bar{\xi}\Gamma_M\Psi$$

$$\delta\Psi = -\frac{1}{2}F_{MN}\Gamma_{MN}\xi$$

dimensional reduction $\Rightarrow$

2D SUSY transformation

$$\delta A_\mu = -i\bar{\xi}\Gamma_\mu\Psi$$

$$\delta\Phi = -i\bar{\xi}\Gamma_+\Psi$$

etc...

Transformation by $\xi = (0,0,0,-\epsilon)^T$: $\delta = i\epsilon Q$

$$\boxed{\begin{aligned} &QA_\mu = \lambda_\mu, \quad Q\lambda_\mu = iD_\mu\Phi, \\ &Q\bar{\Phi} = \eta, \quad Q\eta = [\Phi,\bar{\Phi}], \\ &Q\chi = Y, \quad QY = [\Phi,\chi], \quad Q\Phi = 0. \end{aligned}}$$

$$Q^2 = \text{gauge transformation by } \Phi$$

that is

Transformation by Q is nilpotent.

Action in Q-exact form

$$S = \frac{1}{2g^2}\int d^2x Tr\Bigg(Y^2 - 2iYF_{12} + |D_\mu\Phi|^2 + \frac{1}{4}[\Phi,\bar{\Phi}]^2$$

$$+ 2i\chi(D_1\lambda_2 - D_2\lambda_1) + i\eta D_\mu\lambda_\mu - \frac{1}{4}\eta[\Phi,\eta] - \chi[\Phi,\chi] + \lambda_\mu[\bar{\Phi},\lambda_\mu]\Bigg)$$

$$= \frac{1}{2g^2}\int d^2x Q Tr\left(\frac{1}{4}\eta[\Phi,\bar{\Phi}] + \chi\left(Y - 2iF_{12}\right) - i\lambda_\mu D_\mu\bar{\Phi}\right)$$

This action is manifestly Q-invariant

Idea

We can make Q-invariant lattice theory
by latticizing this form of action.

Q-transformation on lattice

Hint:

link variable : $U_\mu = \exp(iaA_\mu)$

gauge variable: $QA_\mu = \lambda_\mu$

$$QU_\mu(x) = i\lambda_\mu(x)U_\mu(x), \quad Q\lambda_\mu(x) = iD_\mu\Phi(x) + i\lambda_\mu(x)\lambda_\mu(x),$$
$$Q\bar{\Phi}(x) = \eta(x), \quad Q\eta(x) = [\Phi(x), \bar{\Phi}(x)],$$
$$Q\chi(x) = Y(x), \quad QY(x) = [\Phi(x), \chi(x)], \quad Q\Phi(x) = 0,$$

where

$$D_\mu\varphi(x) \equiv U_\mu(x)\varphi(x+\hat{\mu})U_\mu(x)^{-1} - \varphi(x)$$: covariant difference

Q satisfies

$$Q^2 = \text{gauge transformation by } \Phi$$

Note

All the lattice variables are dimension less, while

$$\Phi, \bar{\Phi} = \mathcal{O}(a), \quad \lambda_\mu, \eta, \chi = \mathcal{O}(a^{3/2}), \quad Y = \mathcal{O}(a^2), \quad Q = \mathcal{O}(a^{1/2})$$

for continuum variables.

Lattice action

$$S_{\text{lat}} = \frac{1}{2g^2}\sum_x QTr\left(\frac{1}{4}\eta[\Phi(x),\bar{\Phi}(x)] + \chi(x)\left(Y(x) - i\mu(x)\right) - i\lambda_\mu(x)D_\mu\bar{\Phi}(x)\right)$$

$$= \frac{1}{2g^2}\sum_x Tr\Bigg(Y(x)^2 - iY(x)\mu(x) + D_\mu\Phi(x)D_\mu\bar{\Phi}(x) + \frac{1}{4}[\Phi(x),\bar{\Phi}(x)]^2$$

$$- \frac{1}{4}\eta(x)[\Phi(x),\eta(x)] - \chi(x)[\Phi(x),\eta(x)]$$

$$- \lambda_\mu(x)\lambda_\mu(x)\left(\bar{\Phi}(x) + U_\mu(x)\bar{\Phi}(x+\hat{\mu})U_\mu(x)^{-1}\right)$$

$$+ i\chi(x)Q\mu(x) + i\lambda_\mu(x)D_\mu\eta(x)\Bigg)$$

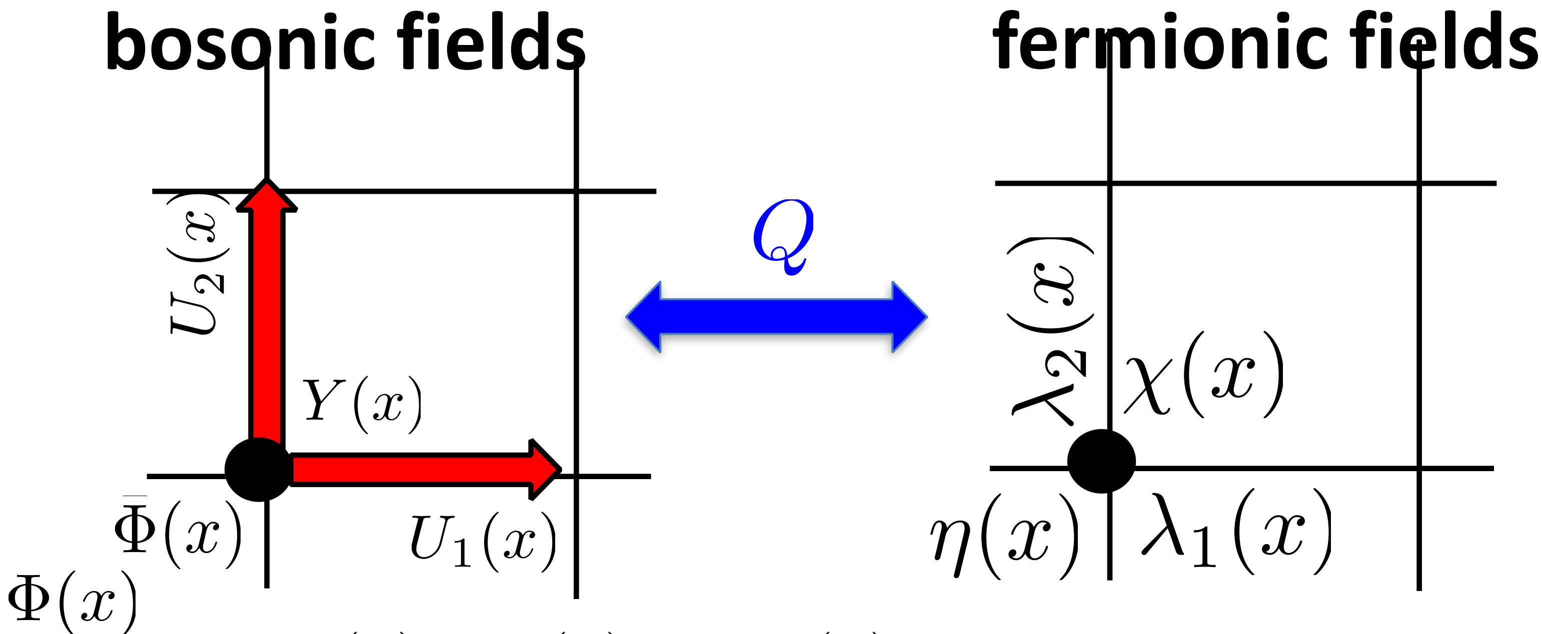
$\mu(x)$ is a Hermitian matrix which satisfies

$$\mu(x) \to 2a^2F_{12}(x)$$

in the continuum limit.

gauge kinetic terms

Geometrical Structure of Sugino Model



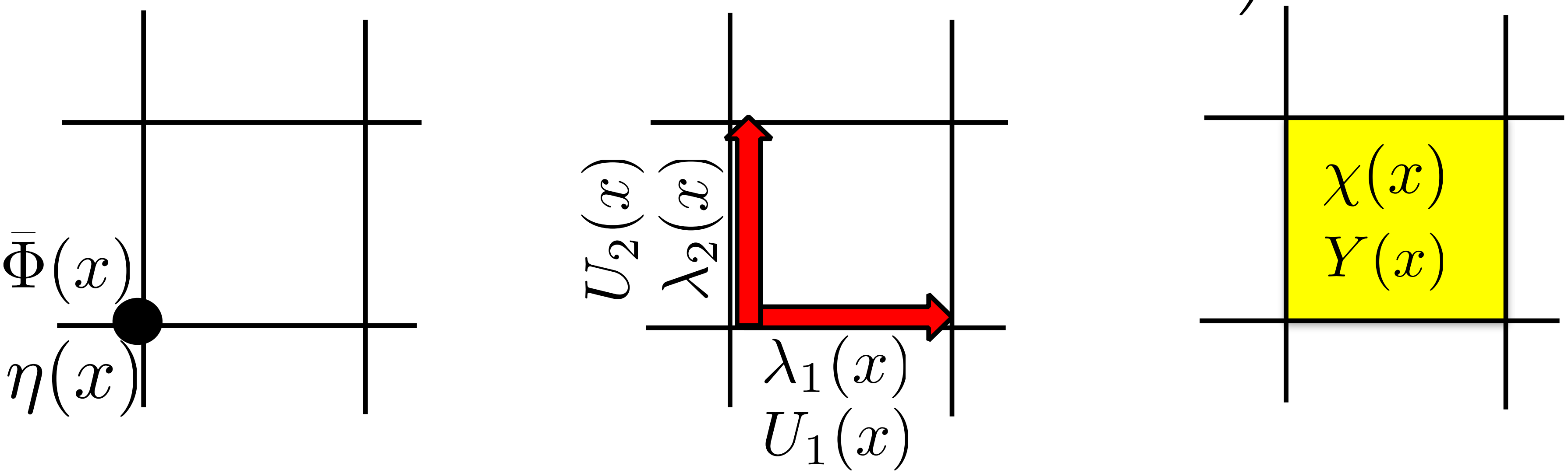
- $Y(x) = \mu(x) \sim F_{12}(x)$ has "plaquette" nature
- $U_\mu(x)$ and $\lambda_\mu(x)$ have "link" nature
- $\Phi(x)$, $\bar{\Phi}(x)$ and $\eta(x)$ have "site" nature

geometric structure of the action

$$S_{\rm lat} = \frac{1}{2g^2}\sum_x {\color{red}Q} Tr\left(\frac{1}{4}\eta[\Phi(x),\bar\Phi(x)] \right. \quad \text{: site}$$

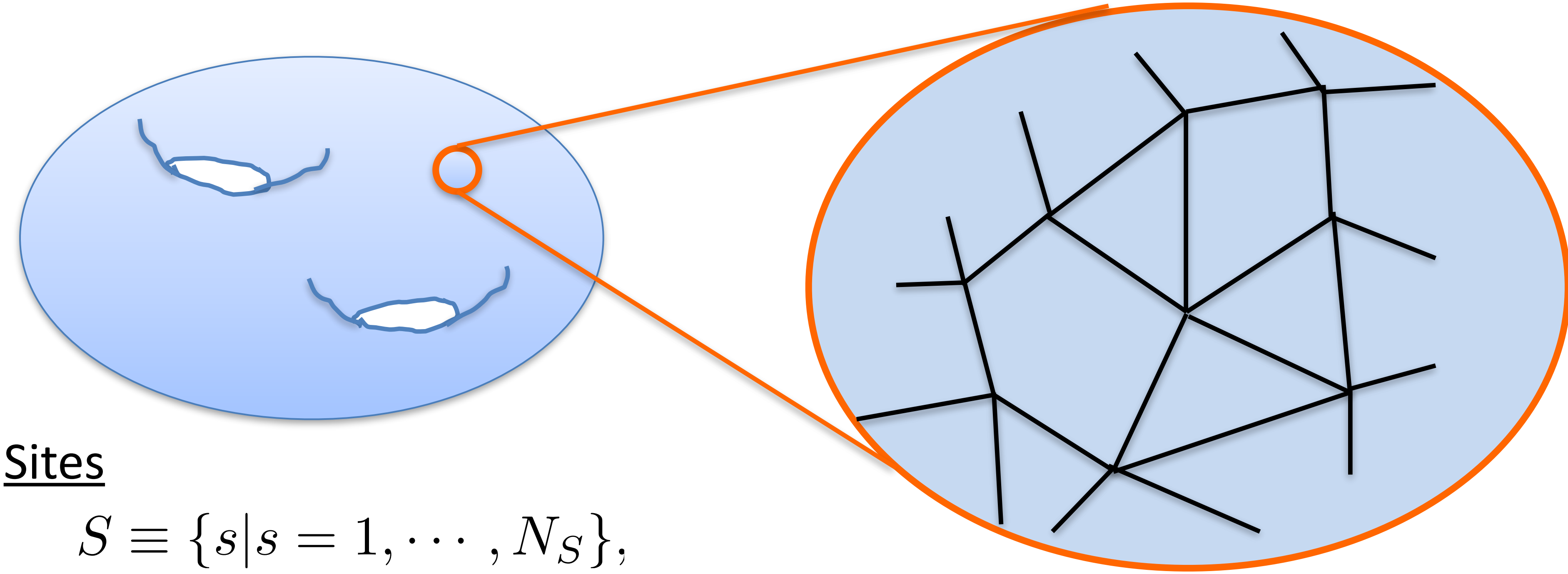
$$- i\lambda_\mu(x) D_\mu \bar\Phi(x) \quad \text{: link}$$

$$\left. + \chi(x)\left(Y(x) - i\mu(x)\right)\right) \quad \text{: face}$$



Can we extend it to a general lattice?

N=(2,2) Topological SYM on Arbitrary Discretized Riemann Surface



Sites

$$S \equiv \{s | s = 1, \cdots, N_S\},$$

Links

$$L \equiv \{\langle st \rangle | s, t \in S\},$$

Faces

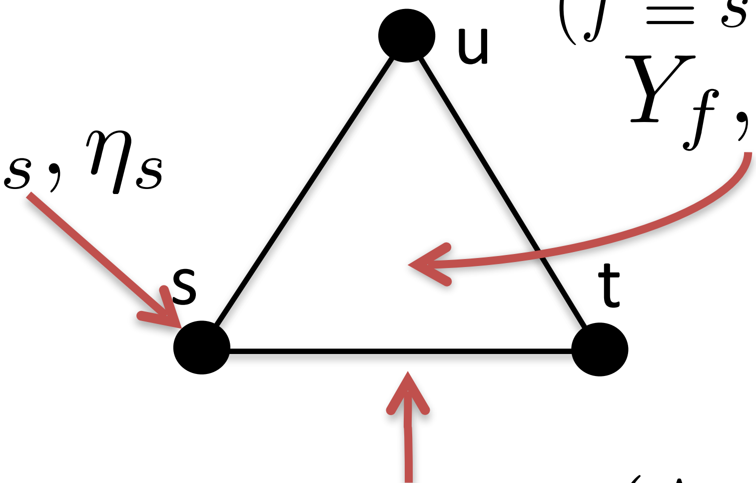
$$F \equiv \{(s_1, \cdots, s_n) | s_1, \cdots, s_n \in S,\ (s_i, s_{i+1}) \in L \text{ or } (s_{i+1}, s_i) \in L\},$$

Fields on the discretized surface

$(f \equiv s : \text{ representative point})$

Y_f, χ_f

$\Phi_s, \bar{\Phi}_s, \eta_s$



$U_{st}, \Lambda_{st} \;\; (\Lambda_{st} \equiv \lambda_{st} U_{st})$

gauge transformation:

$$\Phi_s \to g_s \Phi_s g_s^{-1}, \qquad \bar{\Phi}_s \to g_s \bar{\Phi}_s g_s^{-1}, \qquad \eta_s \to g_s \eta_s g_s^{-1},$$

$$U_{st} \to g_s U_{st} g_t^{-1}, \qquad \Lambda_{st} \to g_s \Lambda_{st} g_t^{-1}, \; \left(\lambda_{st} \to g_s \lambda_{st} g_s^{-1}\right)$$

$$Y_f \to g_f Y_f g_f^{-1}, \qquad \chi_f \to g_f \chi_f g_f^{-1},$$

SUSY transformation

$$Q\Phi_s = 0,$$

$$Q\bar{\Phi}_s = \eta_s, \qquad Q\eta_s = [\Phi_s, \bar{\Phi}_s],$$

$$QU_{st} = i\lambda_{st}U_{st}, \qquad Q\lambda_{st} = i\left(U_{st}\Phi_t U_{st}^{-1} - \Phi_s + \lambda_{st}\lambda_{st}\right),$$

$$QY_f = [\Phi_f, \chi_f], \qquad Q\chi_f = Y_f.$$

Action (proposal)

$$S = S_S + S_L + S_F \equiv Q\left\{\sum_{s\in S}\alpha_s\Xi_s + \sum_{\langle st\rangle\in L}\alpha_{\langle st\rangle}\Xi_{\langle st\rangle} + \sum_{f\in F}\alpha_f\Xi_f\right\}$$

$$\begin{cases} \Xi_s \equiv \dfrac{1}{2g_0^2}\mathrm{Tr}\left\{\dfrac{1}{4}\eta_s[\Phi_s, \bar{\Phi}_s]\right\}, \\ \Xi_{\langle st\rangle} \equiv \dfrac{1}{2g_0^2}\mathrm{Tr}\left\{-i\lambda_{st}\left(U_{st}\bar{\Phi}_t U_{st}^{-1} - \bar{\Phi}_s\right)\right\}, \\ \Xi_f \equiv \dfrac{1}{2g_0^2}\mathrm{Tr}\left\{\chi_f\left(Y_f - i\beta_f\mu(U_f)\right)\right\}, \end{cases} \qquad \left(U_f \equiv \prod_{i=1}^{n} U_{s_i s_{i+1}}\right)$$

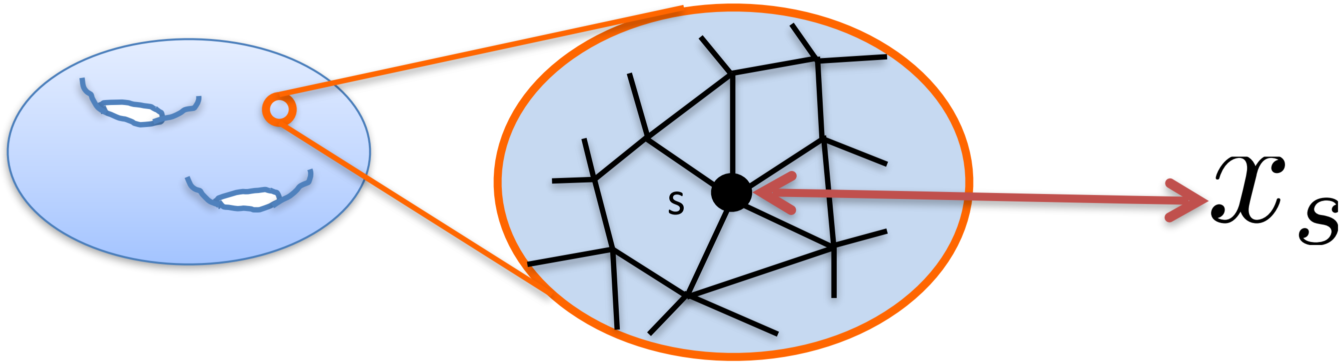
arbitrary parameters (at present)

Comments

- If we consider the usual square lattice as a special case of the discretized Riemann surface with setting α's=β=1, we obtain Sugino's lattice formulation of 2D N=(2,2) SYM on torus.
- The explicit form of μ(U) is determined so that the theory has unique vacuum at U=1:

$$\mu(U_f) = \begin{cases} 2i\Big[\Big(U_f - U_f^{-1}\Big)^{-1}\Big(2 - U_f - U_f^{-1}\Big) \\ \qquad\qquad + \Big(2 - U_f - U_f^{-1}\Big)\Big(U_f - U_f^{-1}\Big)^{-1}\Big] & \text{for } G = U(N), \\ \frac{2i}{M}\Big[\Big(U_f^{M} - U_f^{-M}\Big)\Big(2 - U_f^{M} - U_f^{-M}\Big) \\ \qquad\qquad + \Big(2 - U_f^{M} - U_f^{-M}\Big)\Big(U_f^{M} - U_f^{-M}\Big)\Big] & \text{for } G = SU(N), \end{cases}$$

Continuum Limit



Assumption

- geometry of the continuum Riemann surface is fixed.
- discretization is sufficiently fine to approximate the Riemann surface.
- We can identify the index s of a site with a 2D coordinate x_s.

Lattice spacing

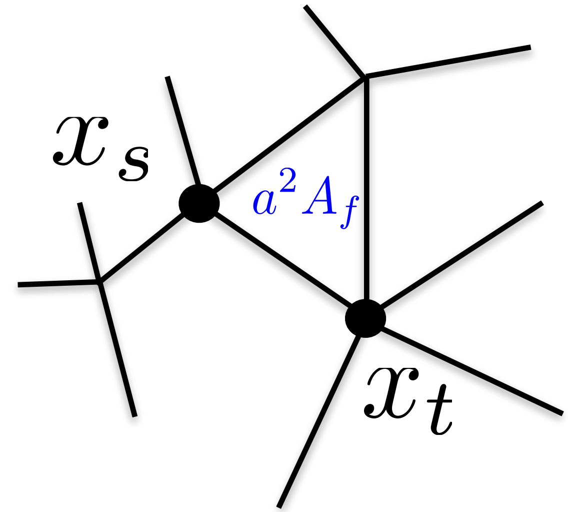
$$a^2 N_F = \int_{\Sigma_g} d^2x \sqrt{g(x)},$$

Continuum limit

$a \to 0,\ N_F \to \infty$ with fixing $a^2 N_F = A$ (area)

Area of a face

$$a^2 A_f = \int_{\sigma_f} d^2x \sqrt{g(x)},$$



Measure in the continuum limit

$$a^2 \sum_{f \in F} A_f \to \int_{\Sigma_g} d^2x \sqrt{g(x)},$$

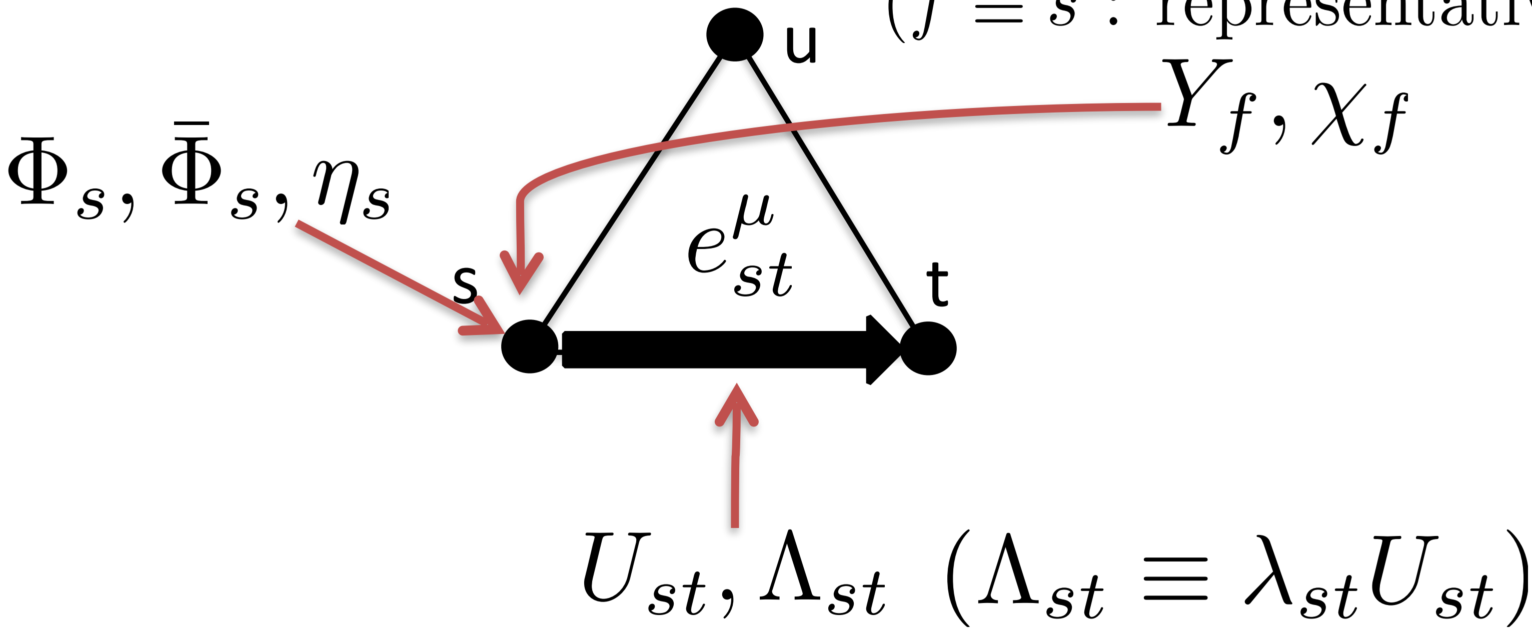
Covariant vector corresponding to a link

$$e^{\mu}_{st} \equiv \frac{1}{a} \left(x^{\mu}_t - x^{\mu}_s \right),$$

Correspondence between lattice and continuum fields

$$\Phi_s = a\Phi(x_s), \quad \bar{\Phi}_s = a\bar{\Phi}(x_s), \quad \eta_s = a^{\frac{3}{2}}\eta(x_s),$$
$$U_{st} = e^{iae^{\mu}_{st}A_{\mu}(x_s+\frac{a}{2}e^{\mu}_{st})},$$
$$\lambda_{st} = a^{\frac{3}{2}} e^{\frac{i}{2}ae^{\mu}_{st}A_{\mu}(x_s+\frac{a}{2}e^{\mu}_{st})} e^{\nu}_{st}\lambda_{\nu}(x_s+\frac{a}{2}e_{st}) e^{-\frac{i}{2}ae^{\mu}_{st}A_{\mu}(x_s+\frac{a}{2}e^{\mu}_{st})},$$
$$Y_f = a^2 Y(x_f), \quad \chi_f = a^{\frac{3}{2}}\chi(x_f).$$
$$Q = a^{1/2}\hat{Q}, \quad g_0^2 = a^2 g_{2d}^2.$$

$(f \equiv s:$ representative point$)$



Action in the classical continuum limit

$$S_S = \frac{\hat{Q}}{2g_{2d}^2}\sum_{f\in F} a^2 A_f \left(\sum_{s\in S_f}\frac{\alpha_s^f}{A_f}\mathrm{Tr}\left(\frac{1}{4}\eta(x_s)[\Phi(x_s),\bar{\Phi}(x_s)]\right)\right),$$

$$S_L = \frac{\hat{Q}}{2g_{2d}^2}\sum_{f\in F} a^2 A_f \left(\sum_{\langle st\rangle\in L_f}\frac{\alpha^f{}_{\langle st\rangle}}{A_f}e^\mu_{st}e^\nu_{st}\mathrm{Tr}\left\{-i\lambda_\mu(x_s)\mathcal{D}_\nu\bar{\Phi}(x_s)+\mathcal{O}(a)\right\}\right),$$

$$S_F = \frac{\hat{Q}}{2g_{2d}^2}\sum_{f\in F} a^2 A_f \left(\frac{\alpha_f}{A_f}\mathrm{Tr}\left\{\chi(x_f)\left(Y(x_f)-i\beta_f A_f\frac{\epsilon^{\mu\nu}}{\sqrt{g(x_f)}}F_{\mu\nu}+\mathcal{O}(a)\right)\right\}\right),$$

continuum limit

with

$$\alpha_s = \sum_{f\in F_s}\frac{A_f}{|S_f|},\quad \alpha_f = A_f,\quad \beta_f = \frac{1}{A_f}.$$

$$\sum_{\langle st\rangle\in L_f}\alpha^f_{\langle st\rangle}e^\mu_{st}e^\nu_{st} = A_f g^{\mu\nu}(x_f).$$

$$S = \frac{\hat{Q}}{2g_{2d}^2}\int_{\Sigma_g} d^2x\sqrt{g}\mathrm{Tr}\left[\frac{1}{4}\eta\left[\Phi,\bar{\Phi}\right]-ig^{\mu\nu}\lambda_\mu D_\nu\bar{\Phi}+\chi\left(Y-2if\right)\right],$$

Topologically twisted 2D N=(2,2) SYM theory

Radiative corrections

Theory 1 $\bar{\Phi}(x) = (\Phi(x))^\dagger$

- The action is invariant under the U(1) transformation,

$$\Phi \to e^{2i\alpha}\Phi, \quad \bar{\Phi} \to e^{-2i\alpha}\bar{\Phi}, \quad A_\mu \to A_\mu,$$
$$\eta \to e^{-i\alpha}\eta, \quad \lambda_\mu \to e^{i\alpha}\lambda_\mu, \quad \chi \to e^{-i\alpha}\chi.$$

- Because of the Q-symmetry and the U(1) symmetry, there is no relevant operator which breaks other symmetries in the continuum limit.

Theory 2 $\Phi(x), \ \bar{\Phi}(x)$: independent hermitian matrices

- There is no additional U(1) symmetry.
- We need to add the local counter terms,

$$S_C = \begin{cases} \sum_{s\in S} \mathrm{Tr}\left(c_1\Phi_s^2 + c_2\Phi_s\right) & \text{for } G = U(N), \\ \sum_{s\in S} \mathrm{Tr}\left(c_1\Phi_s^2\right) & \text{for } G = SU(N), \end{cases}$$

to the action.

Review of localization

~ preparation ~

General Setup

$$S = S(x^i, \psi^i) \text{ : function of } \begin{cases} x^i & : \text{ bosonic variables} \\ \psi^i & : \text{ fermionic variables} \end{cases}$$

invariant under the "supersymmetry",

$$Qx^i = \psi^i, \quad Q\psi^i = {}^{\exists}X^i(x)$$

that is,

$$QS = 0$$

Proposition

Consider the integral,

$$Z_t = \int dx d\psi e^{-S+t\mathcal{K}}.$$

If Z_t is t-invariant, $Z_{t=0} = Z_t$ and the saddle point approximation at $t = \infty$ is exact.

Localization

Once t-invariance is guaranteed, we can estimate the integral as a summation over saddle points:

$$\begin{aligned} Z &\equiv Z_{t=0} = Z_{t=\infty} \\ &= \sum_{x_0:\mathrm{S.P.}} e^{-H(x_0)} \times [\text{gaussian integral of } \mathcal{K} \text{ around } x_0] \end{aligned}$$

Claim

In many cases, the expression,

$$Z_t \equiv \int dx^i d\psi^i e^{-S+tQ\Xi}$$

is t-invariant for a function $\Xi(x,\psi)$ satisfying $Q^2\Xi(x,\psi)=0$.

(We need some conditions (later).)

Hamiltonian

$$S = H(x) - \omega(x,\psi)$$

$$Q\Xi = K(x) - \Omega(x,\psi)$$

"Kamiltonian"

"Proof" (Karki-Niemi 1994)

Under the transformation,

$$x'^i = x^i + \delta t\, \Xi\, \psi^i = x^i + \delta t\, \Xi\, (Qx^i),$$

$$\psi'^i = \psi^i + \delta t\, \Xi\, X^i = \psi^i + \delta t\, \Xi\, (Q\psi^i),$$

1) the "action" is invariant: $\delta(S - Q\Xi) = 0$

2) the measure changes as $dx'^i d\psi'^i = (1 + \delta t\, (Q\Xi))\, dx^i d\psi^i = e^{\delta t\, Q\Xi}$

Thus we see

$$Z_t = \int dx'^i d\psi'^i e^{-S(x',\psi')+tQ\Xi(x',t')}$$

$$= \int dx^i d\psi^i e^{-S(x,\psi)+(t+\delta t)Q\Xi(x,\psi)} = Z_{t+\delta t}$$

Trivial example: gauss integral

$$S = \frac{1}{2}(x^2 + y^2) + \psi_x \psi_y \Rightarrow Z = \int dx dy d\psi_x d\psi_y e^{-S} = 2\pi$$

SUSY transformation

$$Qx = \psi_x, \quad Q\psi_x = -y,$$
$$Qy = \psi_y, \quad Q\psi_y = x,$$

“Kamiltonian”

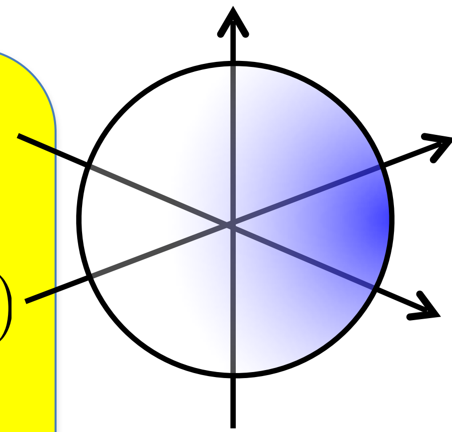
$$\Xi \equiv x\psi_y - y\psi_x$$
$$\Rightarrow S = \frac{1}{2} Q\Xi$$

deformed partition function

$$\begin{aligned} Z_t &\equiv \int dx dy d\psi_x d\psi_y e^{-tQ\Xi} \\ &= \int dx dy d\psi_x d\psi_y e^{-\frac{t}{2}(x^2+y^2) - t\psi_x\psi_y} \\ &= 2\pi \end{aligned}$$

Slightly nontrivial example: height function

$$Z = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sin\theta e^{-\beta H}, \quad (H \equiv -r\cos\theta)$$
$$= \frac{1}{\beta}\int d\theta d\phi d\psi^{\theta} d\psi^{\phi} e^{-\beta S}, \quad (S \equiv -r\cos\theta + \sin\theta\psi^{\theta}\psi^{\phi})$$
$$= \frac{2\pi}{\beta r}\left(e^{\beta r} - e^{-\beta r}\right)$$



SUSY transformation

$$Q\theta = \psi^{\theta}, \quad Q\psi^{\theta} = 0$$
$$Q\phi = \psi^{\phi}, \quad Q\psi^{\phi} = r.$$

"Kamiltonian"

$$\Xi \equiv r\sin^2\theta\psi^{\phi}$$
$$Q\Xi = r\sin^2\theta + r\sin 2\theta\psi^{\theta}\psi^{\phi}$$

$$Z_t = \frac{1}{\beta}\int d\theta d\phi d\psi_{\theta} d\psi_{\phi} e^{-\beta S - tQ\Xi}$$

saddle point at $t \to \infty$:

$$\theta = 0,\ \pi$$

$$(1)\ \theta \sim 0$$

$$[\text{action at } \theta = 0] = \int d\phi e^{-\beta S(\theta=0)} = 2\pi e^{\beta r}$$

$$[\text{1-loop around } \theta \sim 0] = \frac{1}{r}$$

$$(2)\ \theta \sim \pi$$

$$[\text{action at } \theta = \pi] = \int d\phi e^{-\beta S(\theta=0)} = 2\pi e^{-\beta r}$$

$$[\text{1-loop around } \theta \sim \pi] = \frac{1}{r}$$

$$Z_{t\to\infty} = \frac{1}{\beta}\left(\frac{2\pi}{r}e^{\beta r} - \frac{2\pi}{r}e^{-\beta r}\right)$$

Non-trivial example: Harish-Chandra-Itzykson-Zuber integral

$$Z \equiv \int dU e^{-\beta H}$$
$$= \int dU e^{-\beta \mathrm{tr}\left(AUBU^\dagger\right)}$$

$$U \in U(N)$$
$$A = \mathrm{diag}\left(a_1, \cdots, a_N\right)$$
$$B = \mathrm{diag}\left(b_1, \cdots, b_N\right)$$

Deformation

$$Z_\omega \equiv \int dU d\psi e^{-\beta H + \omega}$$

$$\begin{pmatrix} H = \mathrm{tr}\left(AUBU^\dagger\right) \\ \omega \equiv -\frac{1}{2}\mathrm{tr}\left(\psi[X,\psi]\right) \end{pmatrix}$$

where

$$X = UBU^\dagger$$

SUSY

$$QU = i\psi U$$
$$Q\psi = i\beta A + i\psi\psi$$

relation

$$Z_\omega = \Delta(b) Z$$
$$\Delta(b) \equiv \prod_{i<j} (b_i - b_j)$$

"Kamiltonian"

$$\Xi \equiv i\beta \mathrm{tr}\left([\psi, X][A, X]\right) \qquad Q\Xi = \beta^2 K - \beta\Omega$$

$$\begin{cases} K = -\mathrm{tr}[A, X]^2 \\ \Omega = -\mathrm{tr}\left([\psi, X][A, [\psi, X]]\right) \end{cases}$$

Saddle points

$$K = 0 \Leftrightarrow X = UBU^\dagger \ \text{diagonal} \Leftrightarrow U = \Gamma_\sigma, \quad (\sigma \in S_N)$$

Contribution from a saddle point σ

[action at S.P.] $\displaystyle\prod_{i=1}^{N} e^{\beta a_i b_{\sigma(i)}}$

[1-loop around S.P.] $\displaystyle\underbrace{(-1)^{|\sigma|}}_{\text{measure}}\underbrace{(2\pi)^N}_{\text{U(1) modes}} \times \underbrace{\left(\frac{\pi}{2t\beta^2}\right)^{N(N-1)/2} \frac{1}{\Delta(a)^2\Delta(b)^2}}_{\text{boson}} \times \underbrace{(2t\beta)^{N(N-1)/2}\Delta(a)\Delta(b)^2}_{\text{fermion}}$

Summation over all saddle points

$$Z = (2\pi)^N \left(\frac{\pi}{\beta}\right)^{N(N-1)/2} \frac{\det_{i,j}(e^{a_i b_j})}{\Delta(a)\Delta(b)}$$ **: HCIZ integral !**

ちょっとこわいお話

Hight function

$$Z = \frac{1}{\beta}\int d\theta d\phi d\psi^\theta d\psi^\phi e^{-\beta\left(-r\cos\theta + \sin\theta\psi^\theta\psi^\phi\right)}$$

$$= \frac{2\pi}{\beta r}\left(e^{\beta r} - e^{-\beta r}\right)$$

$$Q\theta = \psi^\theta, \quad Q\psi^\theta = 0$$
$$Q\phi = \psi^\phi, \quad Q\psi^\phi = r.$$

Q-exactなのにβ依存性がある・・・

何故か？

$x'^i = x^i + \delta t\,\Xi(x)\,\psi^i$: 積分の境界をずらす可能性がある

パラメータに依存するようなKamiltonianを使うとlocalizationは失敗する。

“Damiltonian”

パラメータ依存性は本当はちゃんと見る必要がある。

Topological Aspect of the Discretized Theory

$$Z = \int d\bar{\Phi}_s d\eta_s dY_f d\chi_f \cdots e^{-S(\Phi_s, \bar{\Phi}_s, \cdots)}$$

$$= \int d\bar{\Phi}_s \cdots e^{-Q\left(g_s \sum_{s\in S} \Xi_s + g_l \sum_{\langle st\rangle \in L} \Xi_{\langle st\rangle} + g_{f_1} \sum_{f\in F} \Xi_{f_1} + g_{f_2} \sum_{f\in F} \Xi_{f_2}\right)}$$

$$\Xi_s \equiv \mathrm{Tr}\left\{\frac{1}{4}\eta_s[\Phi_s, \bar{\Phi}_s]\right\}, \qquad \Xi_{\langle st\rangle} \equiv \mathrm{Tr}\left\{-i\lambda_{st}\left(U_{st}\bar{\Phi}_t U_{st}^{-1} - \bar{\Phi}_s\right)\right\},$$

$$\Xi_{f_1} \equiv \mathrm{Tr}\left\{\chi_f Y_f\right\}, \qquad \Xi_{f_2} \equiv \mathrm{Tr}\left\{-i\chi_f \mu(U_f)\right\},$$

From scaling argument, $Z = Z(g_s/g_l^2, g_{f_1}/g_{f_2}^2)$.

Z is independent of g_s and g_{f_1}.

We can take $g_\circ \to \infty$ without changing Z.

Integral is localized to the saddle points.

Fixed points

non-trivial fixed point equations:
$$\Phi_s U_{st} - U_{st}\Phi_t = 0$$
$$\mu(U_f) = 0$$

$$U_{st} = \Gamma_{\sigma_{st}} \quad (\sigma_{st} \in S_N)$$
$$\Phi_t = \Gamma^{\dagger}_{\sigma_{st}} \Phi_s \Gamma_{\sigma_{st}}$$
$$\oplus$$
$$U_f \Big|_{U=\Gamma} = 1$$

The eigenvalues of Φ_s are independent of s.

1-loop contribution

ghosts $c_s, \bar{c}_s$

face variables χ_f

$$\frac{(\Delta(\phi)^2)^{\#(\text{sites})} \times (\Delta(\phi))^{\#(\text{faces})}}{(\Delta(\phi))^{\#(\text{links})} \times (\Delta(\phi))^{\#(\text{sites})}} \qquad \left(\Delta(\phi) \equiv \prod_{i<j} (\phi_i - \phi_j) \right)$$

link variablea U_{st}

site variables $\bar{\Phi}_s$

$$= \prod_{i<j} (\phi_i - \phi_j)^{\chi}$$

Euler characteristic !

Partition function

$$Z = \mathcal{N} \int d\phi_i \prod_{i<j} (\phi_i - \phi_j)^{\chi}$$

- *N* is a constant.
- essentially the same with the partition function of topologically twisted 2D N=(2,2) SYM.
- It depends only on the topology of the network.
- Independent of the detail of the discretization.

Summary and Discussion

- We discretized topologically twisted 2D N=(2,2) SYM on an arbitrary discretized Riemann surface.
- The partition function of the discretized theory depends only on the topology of the generalized lattice and is essentially the same with that of the continuum theory.
- Simulation of gauge theory on an arbitrary Riemann surface?
- Dynamical triangulation with keeping Q-symmetry?