Renormalization for harmonic oscillators

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Abstract

We introduce a class of models with a harmonic oscillator coupled to an infinite number of harmonic oscillators. Though the model is free, it requires renormalization. We discuss two models in particular, one mimicking the renormalization of a three dimensional scalar theory, and the other that of a four dimensional scalar theory.

Introduction

1. We often think of the necessity of UV renormalization as a consequence of non-linear interactions in relativistic field theory.

2. We will show that even in the absence of non-linearity UV renormalization becomes necessary when a degree of freedom is coupled to an infinite number of degrees of freedom whose energy goes all the way to infinity.

3. The model was originally introduced by Dirac in his textbook. (Chapter VIII, §52 on **resonance scattering**) It is here transcribed in the language of harmonic oscillators.

4. The Lee model (Phys. Rev. **95**, 1329(1954)) is a particular example.

Ref. H. Sonoda, Phys. Rev. **D89** (2014) 047702 [arXiv: 1311.6936]

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内容紹介

初学者がつまずくところを熟知した著者による，丁寧な解説.
くりこみの勘どころが基礎から理解できる．とにかく，わかりやすい！

The plan of the talk

1. The model of free harmonic oscillators
2. Green function
3. Dispersion relation
4. First example — renormalization of mass
5. Second example — renormalization of mass, coupling, wave function
6. Conclusions

The model

1. The hamiltonian is given by $H = H_0 + H_I$ where

$$\begin{cases} H_0 = \Omega\, a^\dagger a + \sum_n \omega_n a_n^\dagger a_n \\ H_I = -\sum_n g_n \left(a_n^\dagger a + a^\dagger a_n \right) \end{cases}$$

2. Physical interpretations

physics	a	a_n
atomic transition	excited atom	radiations
meson decay	J/ψ	e^+e^- pairs
Cooper's model	?	Cooper's pairs

Green function

1. Define $|0\rangle$ by

$$a\,|0\rangle = a_n\,|0\rangle = 0$$

2. **Retarded Green function**

$$G_R(t) \equiv \theta(t)\,\langle 0|\,a\,\mathrm{e}^{-iHt}\,a^\dagger\,|0\rangle$$

gives the probability amplitude that the state $a^\dagger\,|0\rangle$ (of energy Ω) remains intact after time t.

We define the Fourier transform:

$$G(\omega) \equiv \frac{1}{i}\int_{-\infty}^{\infty} dt\,\mathrm{e}^{i\omega t} G_R(t) = \langle 0|\,a\,\frac{1}{\omega - H + i\epsilon}\,a^\dagger\,|0\rangle$$

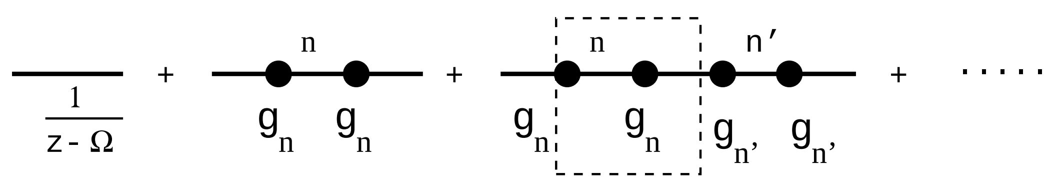
3. Complex valued Green function (resolvent): $\omega + i\epsilon \longrightarrow z$ is analytic in the upper half plane:

$$G(z) \equiv \langle 0|\, a \frac{1}{z - H} a^\dagger \,|0\rangle$$

4. Computing G by summing a geometric series

$$\begin{aligned} G(z) &= \langle 0|\, a \frac{1}{z - H} a^\dagger \,|0\rangle \\ &= \langle 0|\, a \frac{1}{z - H_0} \sum_{k=0}^{\infty} \left(H_I \frac{1}{z - H_0} \right)^k a^\dagger \,|0\rangle \end{aligned}$$

$=$



$$= \frac{1}{z-\Omega} \sum_{l=0}^{\infty} \left(\sum_n g_n^2 \frac{1}{z-\Omega} \frac{1}{z-\omega_n} \right)^l$$

$$= \frac{1}{z - \Omega - \sum_n g_n^2 \frac{1}{z-\omega_n}}$$

5. In the infinite volume limit, g_n^2 gives a continuous function of frequency:

$$g_\omega^2 \equiv \lim_{V\to\infty} g_n^2 \delta(\omega - \omega_n) \quad \text{dimension of frequency}$$

6. Green function in the infinite volume limit

$$\boxed{G(z) = \frac{1}{z - \Omega - \int d\omega \, g_\omega^2 \frac{1}{z-\omega}}}$$

7. The positive function g_ω^2 characterizes the model.

8. Assume $g_\omega^2 \neq 0$ only for $\omega \in [\omega_L, \omega_H]$.

$$G(z) = \frac{1}{z - \Omega - \int_{\omega_L}^{\omega_H} d\omega\, g_\omega^2 \frac{1}{z-\omega}}$$

Two cutoffs: ω_L (infrared) & ω_H (ultraviolet)

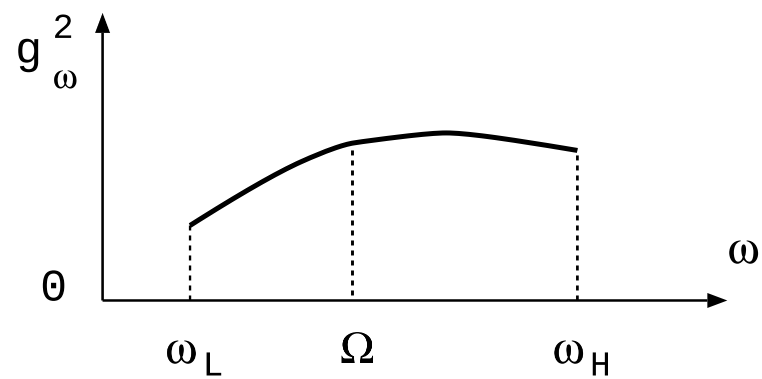
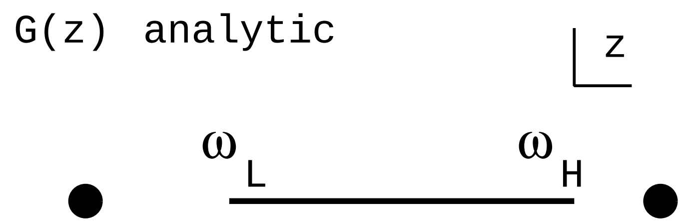


Figure 1: A continuum of states within an energy band

Dispersion relation

1. $G(z)$ is analytic with a cut on the real axis between ω_L and ω_H, and possible isolated singularities (bound states) on the real axis.



2. The imaginary part above the real axis is

$$\Im G(\omega + i\epsilon) = \frac{-\pi g_\omega^2}{b_\omega^2 + \pi^2 g_\omega^4}$$

where

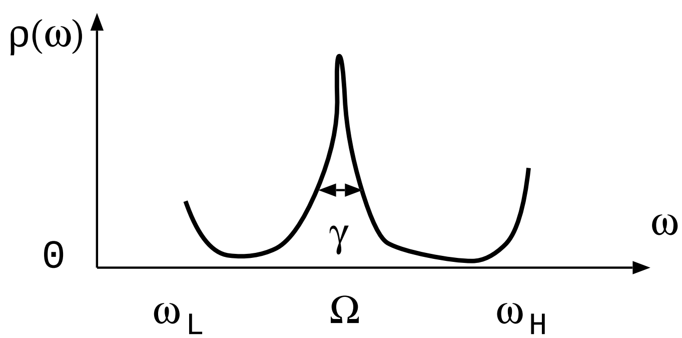
$$b_\omega \equiv \omega - \Omega - \int_{\omega_L}^{\omega_H} d\omega' \, g_{\omega'}^2 \mathbf{P} \frac{1}{\omega - \omega'}$$

3. **dispersion relation**

$$G(z) = \sum_i \frac{r_i}{z - \omega_i} + \int_{\omega_L}^{\omega_H} d\omega \, \frac{1}{z - \omega} \, \rho(\omega)$$

where ρ is a positive **spectral function**

$$\rho(\omega) \equiv -\frac{1}{\pi} \Im G(\omega + i\epsilon) = \frac{g_\omega^2}{b_\omega^2 + \pi^2 g_\omega^4} > 0$$



(a) $\rho(\omega)$ has a peak near Ω.

(b) The width of the peak gives the decay width.

(c) $r_i > 0$ is the probability that $a^\dagger |\Omega\rangle$ is the i-th bound state.

4. **The sum rule**: the asymptotic behavior $G(z) \stackrel{|z|\to\infty}{\longrightarrow} \frac{1}{z}$ implies

$$\boxed{\sum_i r_i + \int_{\omega_L}^{\omega_H} d\omega\, \rho(\omega) = 1}$$

Normalization of the state $a^\dagger |0\rangle$:

$$\langle 0| a a^\dagger |0\rangle = 1$$

First example

1. Constant $g_\omega^2 = g^2 \quad (\omega_L < \omega < \omega_H)$ gives

$$G(z)^{-1} = z - \Omega - g^2 \int_{\omega_L}^{\omega_H} d\omega \frac{1}{z-\omega} = z - \Omega - g^2 \ln \frac{z - \omega_L}{z - \omega_H}$$

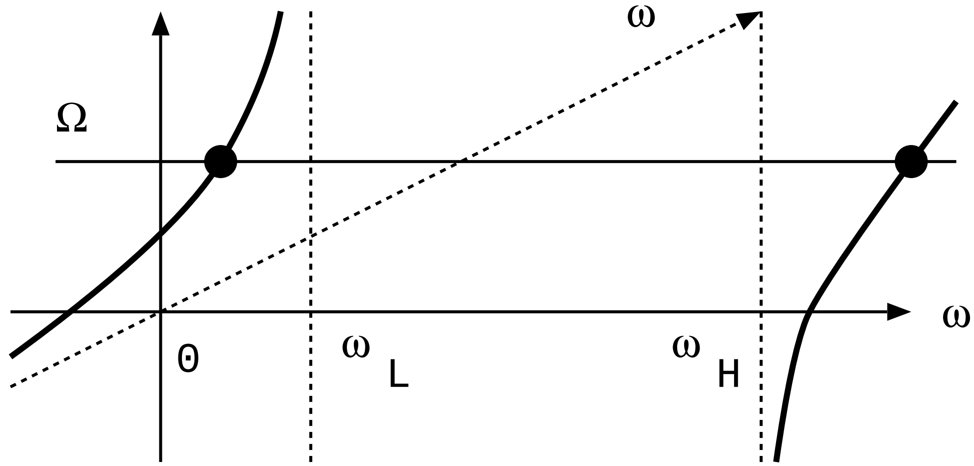


Figure 2: Plot of $\omega - g^2 \ln(\omega - \omega_L)/(\omega - \omega_H)$ for $\omega_L < \Omega < \omega_H$

2. Two isolated states, one below ω_L (attractive), another above ω_H (repulsive) arise.

3. Second order perturbation theory gives

$$\Delta\omega_n = \frac{g_n^2}{\omega_n - \Omega} = \begin{cases} \text{negative} & (\omega_n < \Omega) \\ \text{positive} & (\omega_n > \Omega) \end{cases}$$

4. $\omega_H \to \infty$ limit

$$\begin{aligned} G(z)^{-1} &= z - \Omega - g^2 \ln \frac{\omega_L - z}{\omega_H - z} \\ &= z - \Omega - g^2 \ln \frac{\mu}{\omega_H - z} - g^2 \ln \frac{\omega_L - z}{\mu} \\ &\xrightarrow{\omega_H \to \infty} z - \Omega_r - g^2 \ln \frac{\omega_L - z}{\mu} \end{aligned}$$

where

$$\boxed{\Omega_r \equiv \lim_{\omega_H \to \infty} \left(\Omega + g^2 \ln \frac{\mu}{\omega_H} \right)}$$

is the renormalized frequency (mass).

$$\boxed{G_r(z)^{-1} = z - \Omega_r - g^2 \ln \frac{\omega_L - z}{\mu}}$$

5. Renormalization group equation

$$\left(\mu \frac{\partial}{\partial \mu} + g^2 \frac{\partial}{\partial \Omega_r} \right) G_r(z) = 0$$

6. G_r has only one pole at $\omega = \omega_b < \omega_L$:

$$\omega_b - \Omega_r - g^2 \ln \frac{\omega_L - \omega_b}{\mu} = 0 \Longrightarrow \omega_b = \omega_L - g^2 W_0 \left(\frac{\mu}{g^2} \, \mathrm{e}^{-\frac{\Omega_r}{g^2}} \right)$$

where W_0 is the Lambert W function defined by $W_0(x)\mathrm{e}^{W_0(x)} = x$.

7. The bound state frequency ω_b is an RG invariant:

$$\left(\mu\frac{\partial}{\partial\mu} + g^2\frac{\partial}{\partial\Omega_r}\right)\omega_b = 0$$

(μ, Ω_r) and $(\mu\mathrm{e}^{\Delta t}, \Omega_r + g^2\Delta t)$ give the same physics.

8. Dispersion relation for the continuum limit

$$G_r(z) = \frac{r_b}{z-\omega_b} + \int_{\omega_L}^{\infty} d\omega\, \frac{\rho(\omega)}{z-\omega}$$

where

$$\rho(\omega) = \frac{g^2}{\left(\omega - \Omega_r - g^2 \ln\frac{\omega-\omega_L}{\mu}\right)^2 + \pi^2 g^4}$$

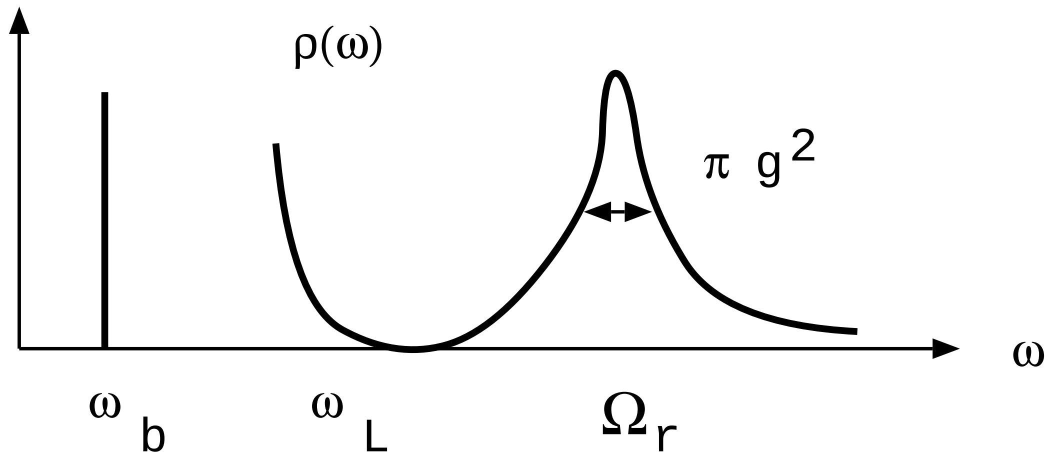


Figure 3: The narrow ($\sim \mu \mathrm{e}^{-\frac{\Omega_r - \omega_L}{g^2}}$) peak at the threshold ω_L is an artifact due to the discontinuity of g_ω^2 at $\omega = \omega_L$.

9. The example is similar to the superrenormalizable ϕ^4 theory in $D = 3$ which requires renormalization of only the squared mass.

Second example

1. Consider $g_\omega^2 = \omega\,\bar{g}^2$ where $\bar{g}^2$ is a dimensionless constant.

$$G(z)^{-1} = z - \Omega + \bar{g}^2 \left(\omega_H - \omega_L - z \ln \frac{z - \omega_L}{z - \omega_H} \right)$$

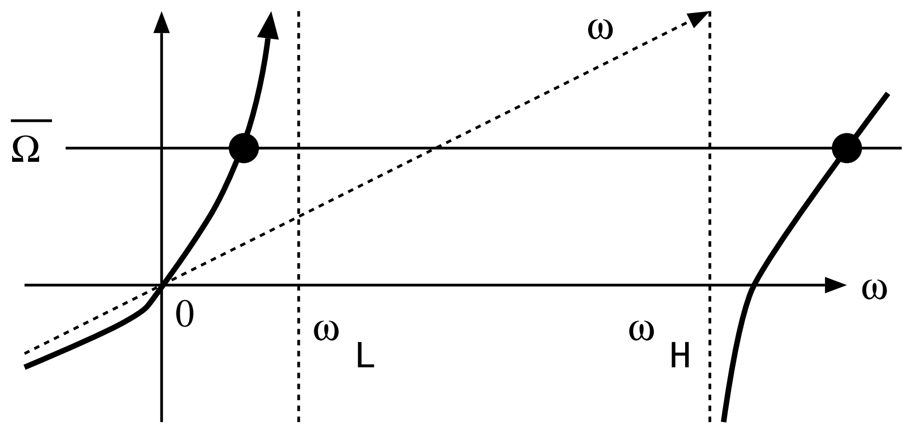


Figure 4: Plot of $\omega \left(1 - \bar{g}^2 \ln \frac{\omega - \omega_L}{\omega - \omega_H} \right)$, where $\bar{\Omega} \equiv \Omega - \bar{g}^2(\omega_H - \omega_L) > 0$

2. Relation to the Lee model (Phys. Rev. **95** (1954) 1329; stationary nucleons of mass difference $\Delta M = \Omega$ interacting with a neutral meson of mass m) $\quad V \longleftrightarrow N + \phi$

$$H_I = g \int d^3x \, \left(\bar{V} N \phi^- + \bar{N} V \phi^+ \right) \Longrightarrow g_\omega^2 = \frac{g^2}{4\pi^2} \sqrt{\omega^2 - m^2} \quad \stackrel{\omega \gg m}{\longrightarrow} \omega \, \frac{g^2}{4\pi^2}$$

3. For $\omega_H \to \infty$, we need three types of renormalization:

   (a) wave function

$$Z = 1 + \bar{g}^2 \ln \frac{\omega_H}{\mu}$$

   (b) mass

$$\Omega_r = \frac{\bar{\Omega}}{Z}$$

   (c) coupling

$$\bar{g}_r^2 = \frac{\bar{g}^2}{Z}$$

4. Renormalized Green function

$$G_r(z) \equiv \lim_{\omega_H \to \infty} Z \cdot G(z) = \frac{1}{z - \Omega_r - \bar{g}_r^2 z \ln \frac{\omega_L - z}{\mu}}$$

has a ghost pole.

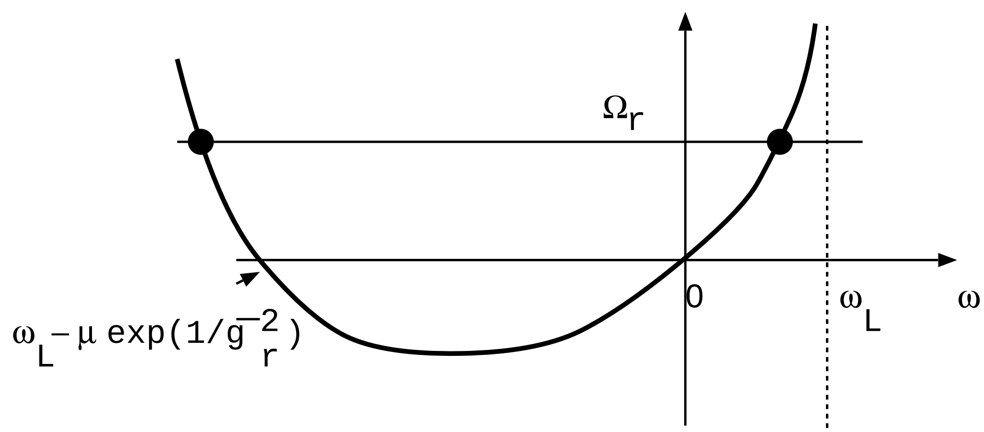


Figure 5: A ghost pole ω_t is found below $\omega_L - \mu\, \mathrm{e}^{\frac{1}{\bar{g}_r^2}}$

The ghost has a negative norm:

$$G_r(z) \stackrel{\omega\to\omega_t}{\longrightarrow} \frac{z_t}{\omega - \omega_t} \quad (z_t < 0)$$

5. $G_r(z)$ is unphysical! $\Longrightarrow$ The limit $\omega_H \to \infty$ does not exist.

6. **Triviality**

$$\bar{g}_r^2 = \frac{\bar{g}^2}{1 + \bar{g}^2 \ln \frac{\omega_H}{\mu}} = \frac{1}{\frac{1}{\bar{g}^2} + \ln \frac{\omega_H}{\mu}} \stackrel{\omega_H\to\infty}{\longrightarrow} 0$$

(a) Inequality

$$\frac{1}{\bar{g}_r^2} \le \frac{1}{\ln \frac{\omega_H}{\mu}} \Longleftrightarrow \omega_H \le \mu\, \mathrm{e}^{\frac{1}{\bar{g}_r^2}}$$

(b) Landau pole $\bar{g}^2 = \infty$ at $\omega_H = \mu\, \mathrm{e}^{\frac{1}{\bar{g}_r^2}}$.

7. We can take ω_H large but only finite. The same as in
   (a) The Lee model
   (b) ϕ^4 theory in $D = 4$
   (c) QED
   (d) the Standard Model

8. Comments
   (a) We can make the Lee model "equivalent" to our model by

$$\frac{g^2}{4\pi^2} \to \frac{g_\omega^2}{\vec{p}^{\,2}} = \frac{g_\omega^2}{\omega^2 - m^2}$$

   (b) Our model becomes similar to the large N limit of the O(N) linear σ model in D dimensions if we choose

$$g_\omega^2 \propto \omega^{D-3} \qquad 2 < D \leq 4$$

Conclusions

1. We have seen that such simple models as non-interacting harmonic oscillators can provide us with non-trivial examples of UV renormalization.

2. Renormalization is necessary when an infinite number of degrees of freedom with energy going toward infinity is mixed with a finite number of degrees of freedom.

3. Example 2 gives us a nice example of the physics of “triviality.”

4. The model reproduces only the mathematical prescription for renormalization. **Scaling picture is missing!**

5. Possible generalization to mimic 1-loop renormalization of multiple parameters?