Green's function of the Vector fields on DeSitter Background

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Talk Based On

- *Green's function of the Vector fields on DeSitter Background*, arXiv: 1408.6193.

Motivation: Early Universe

- Big-Bang model explains cosmological evolutions largely, but observations also indicate puzzling issues: Horizon and Flatness problem
- These can be resolved by invoking Inflation: era of exponential expansion in early universe (Guth, Linde, Starobinsky).
- No detailed physical phenomena known to explain inflation
- Several models exists in order to explain inflation and get the required 60 e-folds to resolve issues
- One possible way (the easiest) is to have the inflation driven by vacuum energy density of the space-time (also known as cosmological constant), which has negative pressure
- Einstein's equation then tells accelerated expansion giving rise to maximally symmetric space-time
- Therefore it is widely believed that the early universe had a DeSitter phase

Motivation: Late Times

- Recent cosmological observations indicate that the present universe is undergoing accelerated expansion
- CMB observation indicate that universe is close to flat, and to account for this flatness one needs about 70% of dark energy
- This is further confirmed independently by observations of large scale structures
- One simplest possible way of explaining this is to have acceleration driven by vacuum energy having a negative pressure
- To this date this is the best explanation for the dark energy fitting the variety of observations to a great accuracy
- In these times of accelerated expansion it is expected that the space-time of universe will be of DeSitter type

Motivation: QFT in curved Space-time

- Efforts to combine quantum mechanics and General Relativity led to research area quantum gravity
- At present many approaches to quantum gravity exist, but in all these approaches there are some short comings
- While these problems remains unsettled, its good to study quantum matter fields on curved background, in an effort to understand how the background curvature might effect the known physical phenomenas and QFT of flat space-time
- Such studies also indicate the energy up to which these analysis can be trusted and where it breaks down

Scalar Fields on DeSitter

- First for simplicity scalar fields were studied
- It was found that for massive scalar there is one-parameter family of DeSitter invariant vacuum states, of which the "euclidean" vacuum is special, as the Green's function has Hadamard form at short distances
- For massless scalar there is no DeSitter invariant vacuum state. DeSitter invariance is broken as there is growing behaviour in the propagator in the IR regime
- Also the massless limit of the massive propagator is not well defined
- A problematic massless limit of massive propagator and a problematic IR divergent massless propagator, put the perturbative QFT procedure at stake

Scalar Propagator: Remedies

- Treating scalar theory as a kind of gauge theory, removing the problematic term of the propagator via gauge fixing. The path integral has BRST symmetry and new propagator is well-defined (Folacci 92)
- Subtracting the problematic term by hand (Bros 2010), however this propagator exhibits growing behaviour in IR
- The problematic term is arising due to zero mode, should be removed in order to have smooth massless limit
- IR problem present at tree-level only: incorporating a small local interaction generates a dynamical mass due to quantum corrections, thereby resolving IR problems (Burgess 2010, Serreau 2011). This can be done perturbatively or non-perturbatively by summing infinite set of cactus diagrams
- A non-vanishing mass was also witnessed in the stochastic approach (Starobinsky 1994)
- In these cases it is safe to take massless limit

Scalar Propagator: Remedies

- The problem is important as the massless and IR limit of the massive propagator doesn't commute (which is not so in flat space-time), thereby implying that even a small amount of curvature, as in the present universe might lead to drastic puzzling results
- While the dust on this has not yet been settled, we choose to investigate the issues in the case of vector fields

Vector Fields

- Vector fields are important in standard model of particle physics
- Behaviour of photon field
- Vectors needed to study the nature of force carrier
- They can be massive (electroweak gauge theories) and massless (photon, gluon)
- At high energies all vectors are massless, but acquire mass via Higgs mechanism at low energies.
- Present standard model is set in flat space-time (renormalizable). In curved space-time it is non-renormalizable, as path-integral is not defined.
- Small but nonzero curvature still a problem, as one has to deal with curved space QFT
- Most of cosmological data we get is in form of Photons. Therefore it is important to have a sensible quantum theory describing them in curved space-time
- As they play important role in SM, therefore crucial to investigate their behaviour in the expanding universe

Vectors Fields: Past Studies

- First study on Electromagnetic fields on maximally symmetric spaces (Katz, Peters)
- More extensive study of Green's function was conducted in (Allen and Jacobson 1986). But many issues like gauge dependence, massless limit and infrared divergences have been left out.
- Recently gauge dependences and IR pathologies have been studied (Youssef 2010). IR divergences are purely gauge artefacts and disappear in Landau gauge.
- Massive vector fields have recently been studied more accurately (Higuchi 2014)
- We use path-integrals methods to find the equation satisfied by two-point correlation function, and find the solution to it by applying appropriate boundary conditions

Bitensors

- In performing the computation of Green's function, we use the language of bi-scalars and bi-tensors.
- In curved space-time, we use the geodetic interval between two points as the bi-scalar. This is $\sigma(x, x')$ and satisfies $\sigma_\mu \sigma^\mu = 2\sigma$, where $\sigma_\mu = \nabla_\mu \sigma(x, x')$.
- We use the parallel displacement bi-vector $g_{\mu\nu'}$. Which satisfies $\sigma^\tau \nabla_\tau g_{\mu\nu'} = 0$.
- Moreover we have following simple identities

$$
\begin{aligned}
&g_{\mu\nu'}\sigma^{\nu'} = -\sigma_\mu\,, \quad g_{\mu\nu'}\sigma^{\mu} = -\sigma_{\nu'} \\
&g_{\mu\rho'}g_{\nu}{}^{\rho'} = g_{\mu\nu}\,, \quad g_{\rho\mu'}g^{\rho}{}_{\nu'} = g_{\mu'\nu'} \\
&(\det\ g^{\nu\rho'}) = \frac{1}{(\det\ g_{\nu\rho'})}\,, \\
&\det g_{\mu\nu'} = \sqrt{g}\sqrt{g'}\,.
\end{aligned}
\tag{1}
$$

Maximally Symmetric Space-time

- On a maximally symmetric space-time any bi-tensors can be written as a linear combination of bi-tensors constructed using σ_μ, $\sigma_{\mu'}$ and $g_{\mu\nu'}$, with coefficient being function of bi-scalar $\sigma(x, x')$ (Katz, Peters)
- Due to this

$$\sigma_{\mu\nu} = A(\sigma)\left[g_{\mu\nu} - \frac{1}{2\sigma}\sigma_\mu\sigma_\nu\right] + \frac{1}{2\sigma}\sigma_\mu\sigma_\nu \,,$$

$$\sigma_{\mu\nu'} = C(\sigma)\left[g_{\mu\nu'} + \frac{1}{2\sigma}\sigma_\mu\sigma_{\nu'}\right] + \frac{1}{2\sigma}\sigma_\mu\sigma_{\nu'}$$

$$g_{\alpha\beta';\mu} = -\frac{A+C}{2\sigma}(g_{\mu\alpha}\sigma_{\beta'} + g_{\mu\beta'}\sigma_\alpha)$$

- For a Euclidean DeSitter it is found that

$$A = \sqrt{\frac{2\sigma R}{d(d-1)}}\cot\sqrt{\frac{2\sigma R}{d(d-1)}}$$

$$C = -\sqrt{\frac{2\sigma R}{d(d-1)}}\csc\sqrt{\frac{2\sigma R}{d(d-1)}}$$

Massive Vector

$$Z[J] = \int \mathcal{D}A_\mu \exp\left[-\int \mathrm{d}^d x \sqrt{g} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \,, \right) - \int \mathrm{d}^d x \sqrt{g} J_\mu A^\mu \right]$$

$$W[J] = \frac{1}{2} \int \mathrm{d}^d x \sqrt{g} J_\mu \left(\Delta_F^{-1}\right)^{\mu\nu} J_\nu + \frac{1}{2} \mathrm{Tr} \ln \Delta_F^{\mu\nu}$$

$$\Delta_F^{\mu\nu} = \left(-\Box + \frac{R}{d} + m^2 \right) g^{\mu\nu} + \nabla^\mu \nabla^\nu$$

$$\Gamma[\bar{A}_\mu] = W[J] - \int \mathrm{d}^d x \sqrt{g} J_\mu \bar{A}^\mu = -\frac{1}{2} \int \mathrm{d}^d x \sqrt{g} \bar{A}_\mu \Delta_F^{\mu\nu} \bar{A}_\nu + \frac{1}{2} \mathrm{Tr} \ln \Delta_F^{\mu\nu} \,.$$

Massive Vector

$$\int \mathrm{d}^d y\sqrt{g(y)}\Big(\frac{1}{\sqrt{g(x)}}\,\frac{1}{\sqrt{g(y)}}\,\frac{\delta^2\Gamma}{\delta\bar{A}_\mu(x)\delta\bar{A}_\rho(y)}\Big)\Big(\frac{1}{\sqrt{g(y)}}\,\frac{1}{\sqrt{g(x')}}\,\frac{\delta^2 W}{\delta J^\rho(y)\delta J_{\nu'}(x')}\Big) = -\frac{\delta_\mu{}^{\nu'}\,\delta(x-x')}{\sqrt{g(x)}}\,.$$

$$\frac{1}{\sqrt{g}}\,\frac{1}{\sqrt{g'}}\,\frac{\delta^2 W}{\delta J_\mu(x)\delta J_{\nu'}(x')} = \langle A_\mu(x)A_{\nu'}(x')\rangle = \frac{1}{\sqrt{g'}}\,\Big(\Delta_F^{-1}\Big)^{\mu\nu'}\,\delta(x-x') = G^{\mu\nu'}(x,x')\,,$$

$$\frac{1}{\sqrt{g}}\,\frac{1}{\sqrt{g'}}\,\frac{\delta^2\Gamma}{\delta\bar{A}_\mu(x)\delta\bar{A}_{\nu'}(x')} = -\frac{1}{\sqrt{g'}}\Delta_F^{\mu\nu'}\,\delta(x-x')\,.$$

$$\left[\left(-\Box+\frac{R}{d}+m^2\right)\delta_\mu{}^\alpha+\nabla_\mu\nabla^\alpha\right]G_{\alpha\nu'} = \frac{g_{\mu\nu'}\delta(x-x')}{\sqrt{g'}}\,.$$

Massive Vector

$$\left[\left(-\Box + \frac{R}{d} + m^2\right) g^{\mu\nu} + \nabla^\mu \nabla^\nu\right] \langle A_\nu A_{\nu'} \rangle = 0\,.$$

$$\nabla^\mu \langle F_{\mu\nu} A_{\nu'} \rangle = m^2 \langle A_\nu(x) A_{\nu'}(x') \rangle\,.$$

$$G_{\mu\nu'}(x,x') = G^T_{\mu\nu'} + \nabla_\mu \nabla_{\nu'} G\,,$$

$$\nabla^\mu G^T_{\mu\nu'} = \nabla^{\nu'} G^T_{\mu\nu'} = 0\,.$$

$$A_\mu = A^T_\mu + \nabla_\mu a$$

$$G^T_{\mu\nu'} = \langle A^T_\mu A^T_{\nu'} \rangle\,, \qquad G(x,x') = \langle a(x) a(x') \rangle\,.$$

$$\left[-\Box + \frac{R}{d} + m^2\right]\left[G^T_{\mu\nu'} + \nabla_\mu \nabla_{\nu'} G(x,x')\right] = 0\,.$$

Solving For Green's Function: Transverse Part

$$G^T_{\mu\nu'} = \alpha(\sigma) g_{\mu\nu'} + \beta(\sigma)\sigma_\mu \sigma_{\nu'}\,.$$

Then instead of solving for α and β directly, we follow a in-direct path.

$$\langle F_{\mu\nu} F^{\mu'\nu'}\rangle = 4\nabla_{[\mu}\nabla^{[\mu'}\langle A_{\nu]} A^{\nu']}\rangle = \theta(\sigma) g_{[\mu}{}^{[\mu'} g_{\nu]}{}^{\nu']} + \tau(\sigma)\sigma_{[\mu}\sigma^{[\mu'} g_{\nu]}{}^{\nu']}\,.$$

$$\theta = 4C\left[\alpha' + \frac{A+C}{2\sigma}\alpha - \beta C\right]\,, \qquad \tau = C^{-1}\left[\theta' + \frac{A+C}{\sigma}\theta\right]\,.$$

$$\nabla^\mu\langle F_{\mu\nu} F^{\mu'\nu'}\rangle = 2m^2\nabla^{[\mu'}\langle A^T_\nu A^{T\nu']}\rangle\,.$$

$$2\sigma\theta'' + [(d+1)A + 1]\theta' - \frac{2R}{d}\theta - m^2\theta = 0\,.$$

Transverse Part: Massive Fields

$$z(x,x') = \cos^2\sqrt{\frac{\sigma R}{2d(d-1)}}\,.$$

$$z(1-z)\frac{\mathrm{d}^2\theta}{\mathrm{d}z^2} + \left[\frac{d}{2}+1-(d+2)z\right]\frac{\mathrm{d}\theta}{\mathrm{d}z} - \frac{d(d-1)}{R}\left(m^2+\frac{2R}{d}\right) = 0$$

This is Hyper-Geometric differential equation. It has two independent solutions

$${}_2F_1(a_1,b_1;c_1;z) \text{ and } {}_2F_1(a_1,b_1;c_1;1-z)\,.$$

$$a_1 = \frac{1}{2}\left[d+1+\sqrt{(d-3)^2-\frac{4d(d-1)m^2}{R}}\right],$$

$$b_1 = \frac{1}{2}\left[d+1-\sqrt{(d-3)^2-\frac{4d(d-1)m^2}{R}}\right],$$

$$c_1 = \frac{d}{2}+1\,.$$

Choice of Vacuum

- Generically any solution of the equation will be a linear combination of the two solution. However the choice of vacuum (choice of boundary condition) helps in finding the correct solution.
- The range of z is $0 \leq |z| < 1$.
- There is one parameter family of deSitter invariant fock vacuum state
- one special vacuum called the "Bunch-Davies" vacuum
  - only one singular point at $z = 1$ and is regular at $z = 0$
  - the strength of singularity for $\sigma \to 0$ is the same as in flat case
- Green's function for all other vacuum state can be derived from this one

$$\theta(z) = \frac{2}{(4\pi)^{d/2}} \frac{\Gamma(a_1)\Gamma(b_1)}{\Gamma(d/2+1)} \left(\frac{R}{d(d-1)} \right)^{d/2} \times {}_2F_1(a_1, b_1; c_1; z)$$

Once $\theta(z)$ is know, $\tau(z)$ can be worked out. one can obtain equation for determining $\alpha(z)$ and $\beta(z)$, using the transversality constraint.

Determining Transverse Part of Green's Function

One can obtain equation for determining $\alpha(z)$ and $\beta(z)$, using the transversality constraint.

$$\alpha' - 2\sigma\beta' - 2\beta + \frac{(d-1)(A+C)}{2\sigma}\alpha - (d-1)A\beta = 0\,.$$

$$2\sigma\alpha'' + [(d+1)A+1]\alpha' - \frac{R}{d-1}\alpha = \frac{\sigma\theta'}{2C} + \frac{(d+1)A\theta}{4C}\,.$$

The function $\alpha(z)$ is determined from,

$$z(1-z)\frac{\mathrm{d}^2\alpha}{\mathrm{d}z^2} + \left[\frac{d}{2} + 1 - (d+2)z\right]\frac{\mathrm{d}\alpha}{\mathrm{d}z} - d\alpha = \frac{d(d-1)}{R}\left[\frac{z(1-z)}{2}\frac{\mathrm{d}\theta}{\mathrm{d}z} + \frac{d+1}{4}(1-2z)\theta\right].$$

This has both homogenous and particular solution. The particular solution is

$$\alpha_p(z) = \frac{r_d^2}{[z(z-1)]^{d/2}}\int_0^z \mathrm{d}z'\,\left[z'(1-z')\right]^{d/2-1}\int_0^{z'}\mathrm{d}z''\left[z''(1-z'')\frac{\mathrm{d}\theta}{\mathrm{d}z''} + \frac{d+1}{2}(1-2z'')\theta\right].$$

Transverse Part: Massive Fields

The homogenous solution is determined by requiring that the short distance singularity structure of the full solution should match the flat space-time solution.

$$G^{T}_{\mu\nu'}(x,x')\Big|_{z\to 0} \sim \frac{R}{1536\pi^2\gamma}\left[12\pi\gamma(1+6\gamma)\times \sec\left(\frac{\pi}{2}\sqrt{1-48\gamma}\right)-1\right]\left[g_{\mu\nu'}+\frac{R}{6\pi^2}\sigma_\mu\sigma_{\nu'}\right].$$

where $\gamma = m^2/R$. This will give an impression that the massless limit is not well defined ($\gamma \to 0$), but this is not the case. Expanding the above in powers of γ it is seen that this limit is given by,

$$G^{T}_{\mu\nu'}(x,x')\Big|_{z\to 0,\gamma\to 0} \sim -\frac{R}{256\pi^2}\left[g_{\mu\nu'}+\frac{R}{6\pi^2}\sigma_\mu\sigma_{\nu'}\right].$$

- Matches with the $z \to 0$ limit of the massless transverse propagator
- This correlation is negative: signalling that there might be some edge states on the boundary.
- In the flat space-time limit $R \to 0$ this correlation goes to zero.

Longitudinal part of Green's Function

The longitudinal part is determined using,

$$\nabla_\mu \nabla_{\nu'} G = C G' g_{\mu\nu'} + \left(G'' g_{\mu\nu'} + \frac{1+C}{2\sigma} G' \right) \sigma_\mu \sigma_{\nu'} = \alpha_L g_{\mu\nu'} + \beta_L \sigma_\mu \sigma_{\nu'} \, .$$

$$\beta_L = C^{-1} \left(\alpha_L' + \frac{A+C}{2\sigma} \alpha_L \right) \, .$$

$$\nabla^\mu \nabla_\mu \nabla_{\nu'} G(\sigma) = 0 \, .$$

$$\alpha_L' - 2\sigma \beta_L' + \frac{(d-1)(A+C)}{2\sigma} \alpha_L - \left((d-1)A + 2 \right) \beta_L = 0 \, .$$

$$2\sigma \alpha_L'' + [(d+1)A + 1]\alpha' - \frac{R}{d-1} \alpha_L = 0 \, . \qquad (2)$$

This is homogenous equation, solved as before by comparing the solution with the strength of singularity in flat space-time

$$\alpha_L(z) = \frac{R}{4608\pi^2} \frac{1}{\gamma} \frac{2z-3}{(1-z)^2} \, ,$$

$$\beta_L(z) = \frac{R^2}{110592\pi^2} \frac{1}{\gamma} \frac{z-3}{(1-z)^2 (\cos^{-1}\sqrt{z})^2} \, .$$

Massless Limit

$$G_{\mu\nu'}(z)\Big|_{z\to 0} \sim -\frac{R}{256\pi^2}\left[1+\frac{1}{6\gamma}\right]\left[g_{\mu\nu'}+\frac{R}{6\pi^2}\sigma_\mu\sigma_{\nu'}\right].$$

- Term proportional to the $1/\gamma$ coming from the longitudinal part.
- The massless limit and $z \to 0$ limit commute in case of transverse part, while the same is not true for longitudinal part.
- This will also imply that massless limit of the full propagator cannot be taken
- The source of the problem is arising from the longitudinal part of the massive green's function.

Massless Vector Fields

$$S = \int \mathrm{d}^d x \sqrt{g} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\lambda} (\nabla_\mu A^\mu)^2 \right] .$$

$$\begin{aligned} W^{\mu\nu} \langle A_\nu A_{\nu'} \rangle &= \left[\left(-\Box + \frac{R}{d} \right) g^{\mu\nu} + \left(1 - \frac{1}{\lambda} \right) \nabla^\mu \nabla^\nu \right] \langle A_\nu A_{\nu'} \rangle \\ &= \frac{g^\mu{}_{\nu'} \delta(x - x')}{\sqrt{g'}} \end{aligned}$$

$$\nabla^\mu \langle F_{\mu\nu} A_{\nu'} \rangle = -\frac{1}{\lambda} \nabla^\mu \nabla^\nu \langle A_\nu A_{\nu'} \rangle .$$

$$\left(-\Box + \frac{R}{d} \right) \left[\nabla_\mu \nabla_{\nu'} G + \lambda G^T_{\mu\nu'} \right] = 0 .$$

$$\nabla^\mu \langle F_{\mu\nu} F^{\mu'\nu'} \rangle = -\frac{1}{\lambda} \nabla_\nu \Box \langle a F^{\mu'\nu'} \rangle .$$

$$\nabla^\mu \langle F_{\mu\nu} F^{\mu'\nu'} \rangle = 0 .$$

$$z(1-z) \frac{\mathrm{d}^2\theta}{\mathrm{d}z^2} + \left[\frac{d}{2} + 1 - (d+2) z \right] \frac{\mathrm{d}\theta}{\mathrm{d}z} - 2(d-1)\theta = 0 .$$

This is a Hyper-Geometric differential equation. It has two solutions. Only one of them has short distance singularity at $z = 1$.

Transverse and Longitudinal Part

We then solve for the transverse part of the propagator. This is achieved by

$$z(1-z)\frac{\mathrm{d}^2\alpha}{\mathrm{d}z^2}+\left[\frac{d}{2}+1-(d+2)z\right]\frac{\mathrm{d}\alpha}{\mathrm{d}z}-d\alpha=\frac{d(d-1)}{R}\left[\frac{z(1-z)}{2}\frac{\mathrm{d}\theta}{\mathrm{d}z}+\frac{d+1}{4}(1-2z)\theta\right].$$

Its non-homogenous. Its has both particular solution and homogenous solution. The particular solution is determined as before, and the full solution is obtained by comparison with flat space-time limit. It should be noticed that equation has same form as for massive case except this time $\theta(z)$ is for massless fields.

$$\begin{aligned}\alpha(z) &= \frac{R}{384\pi^2}\left[\frac{1}{z(1-z)}+\frac{(2z+1)\ln(1-z)}{z^2}\right],\\ \beta(z) &= -\frac{R^2}{9216\pi^2(\cos^{-1}\sqrt{z})^2}\left[\frac{1}{z(1-z)}+\frac{(1-z)\ln(1-z)}{z^2}\right].\end{aligned}$$

The longitudinal part can be worked out as before and in four dimensions is given by,

$$\begin{aligned}\alpha_L(z) &= \frac{R\lambda}{1152\pi^2}\left[\frac{4z-1}{z(1-z)}-\frac{(2z+1)\ln(1-z)}{z^2}\right]\\ \beta_L(z) &= \frac{R^2\lambda}{27648\pi^2(\cos^{-1}\sqrt{z})^2}\left[\frac{2z^2-4z-1}{z(z-1)}+\frac{(1-z)\ln(1-z)}{z^2}\right]\end{aligned}$$

Summary and Conclusions

- DeSitter metric is an important space-time manifold having great relevance in early and late time universe
- Important to formulate methods of QFT on this background
- Known perturbative methods of flat space-time QFT needs modification
- The first important step in this direction is the construction of the propagator over this background

Summary and Conclusions

- Scalar propagator have been studied: IR problems for the massless propagator. In order to have smooth massless limit of the massive propagator, zero mode contribution should be taken into account.
- Off-shell Vector propagator have been studied using rigorous path-integrals methods
- Two cases have explicitly investigated: massive and massless vectors
- The transverse part of the massive propagator matches with the transverse propagator of the massless propagator in the massless limit
- The longitudinal part doesn't have this well defined limit
- The antipodal point separation limit is well defined, regular but negative. Correlation goes to zero when $R \to 0$.