最大対称空間中の
余等質1ストリングの可積分性

Hideki Ishihara

Department of Physics,

Osaka City University

Collaboration with S.Hasegawa

On going with T.Koike and Y.Morisawa

Introduction

Physical significance of extended objects

- Topological defects e.g. cosmic strings etc
- D-branes
- Braneworld universe model
- AdS/CFT correspondence

Extended objects (cosmic strings) are described by PDE

- Nambu-Goto equation, etc.

余等質 1 物体
Cohomogeneity-one object

だいたい一様だが，１次元方向だけ非一様な物体

e.g. 一様宇宙モデル

Friedmann universe model,
Bianchi universe model

Einstein方程式 → Friedmann方程式
P.D.E. → O.D.E.

Cohomogeneity-one string を考える

Advantage of C-1 Objects

Tractable and physically interesting

	Homogeneous	Cohomogeneity-1	No symmetry
To solve	Simplest (algebraic)	Simple (ODE)	Difficult (PDE)
Variety	Poor	Rich	Richest
Physics	Trivial	Non-trivial	General

C1 string in a maximally symmetric space

- 対称性の高い時空中のC1 ストリングにはいろんな種類がある.
- 運動方程式は, 連立O.D.E.になるが,

  それらは積分可能か？ ⟺ Chaos?

最大対称時空:

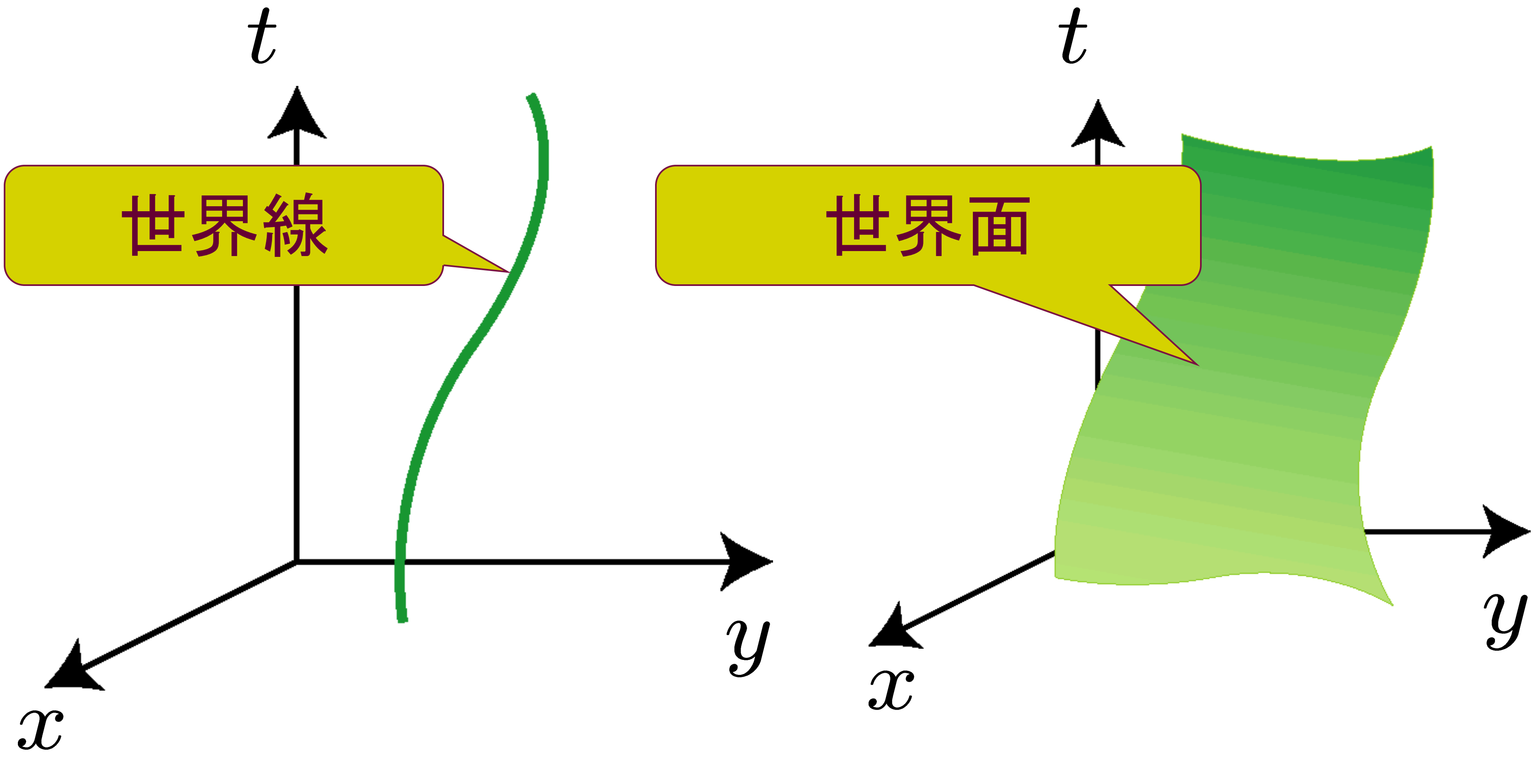
Minkowski, de Sitter, anti-de Sitter

の中のC1 ストリングは積分可能か？

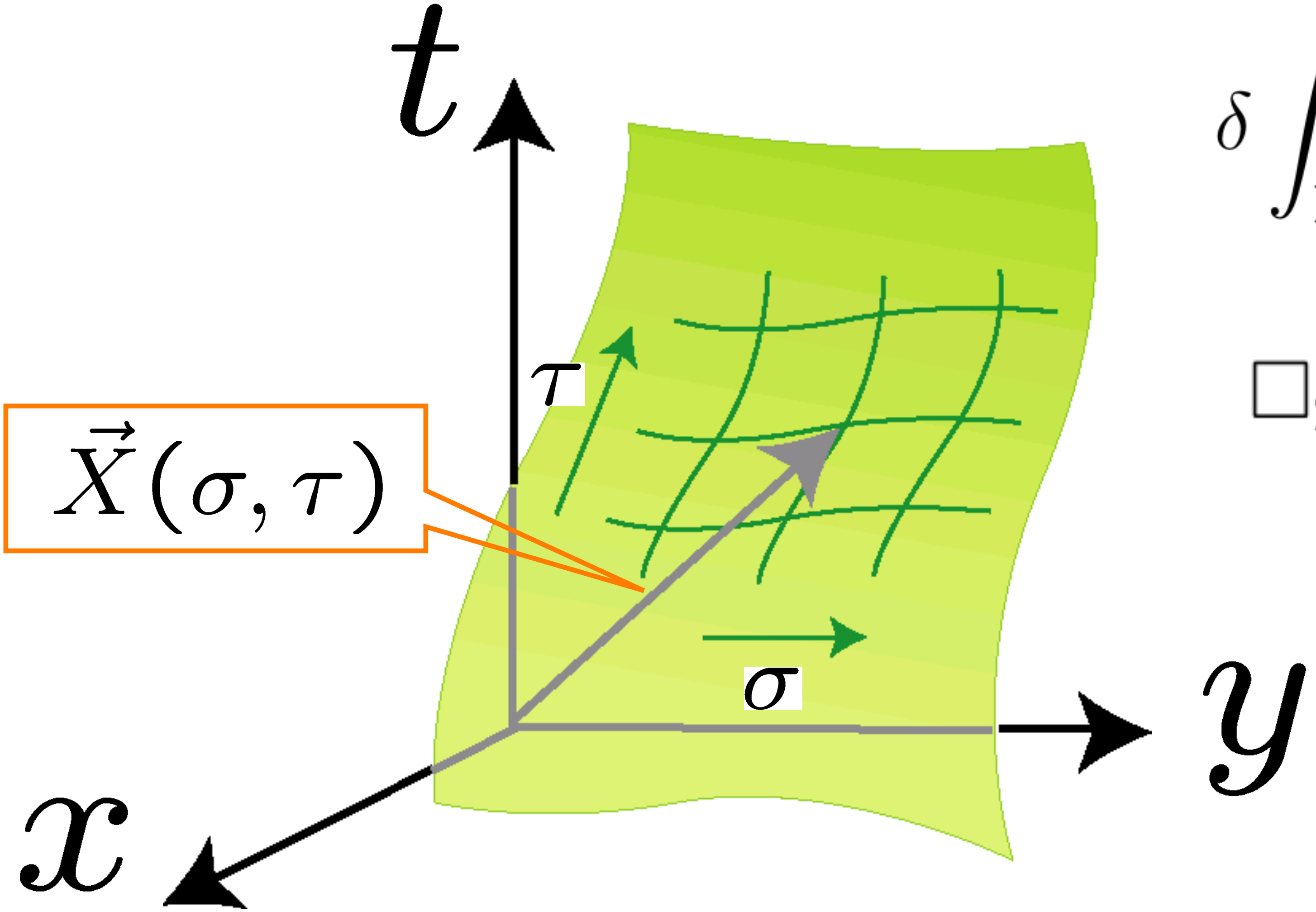
（求積問題に帰着）

Cohomogeneity-one strings

Trajectory of a String



World Sheet



$$\delta \int_{\Sigma} dA = 0$$

$$\Box_2 \vec{X} = 0$$

Equations of Motion

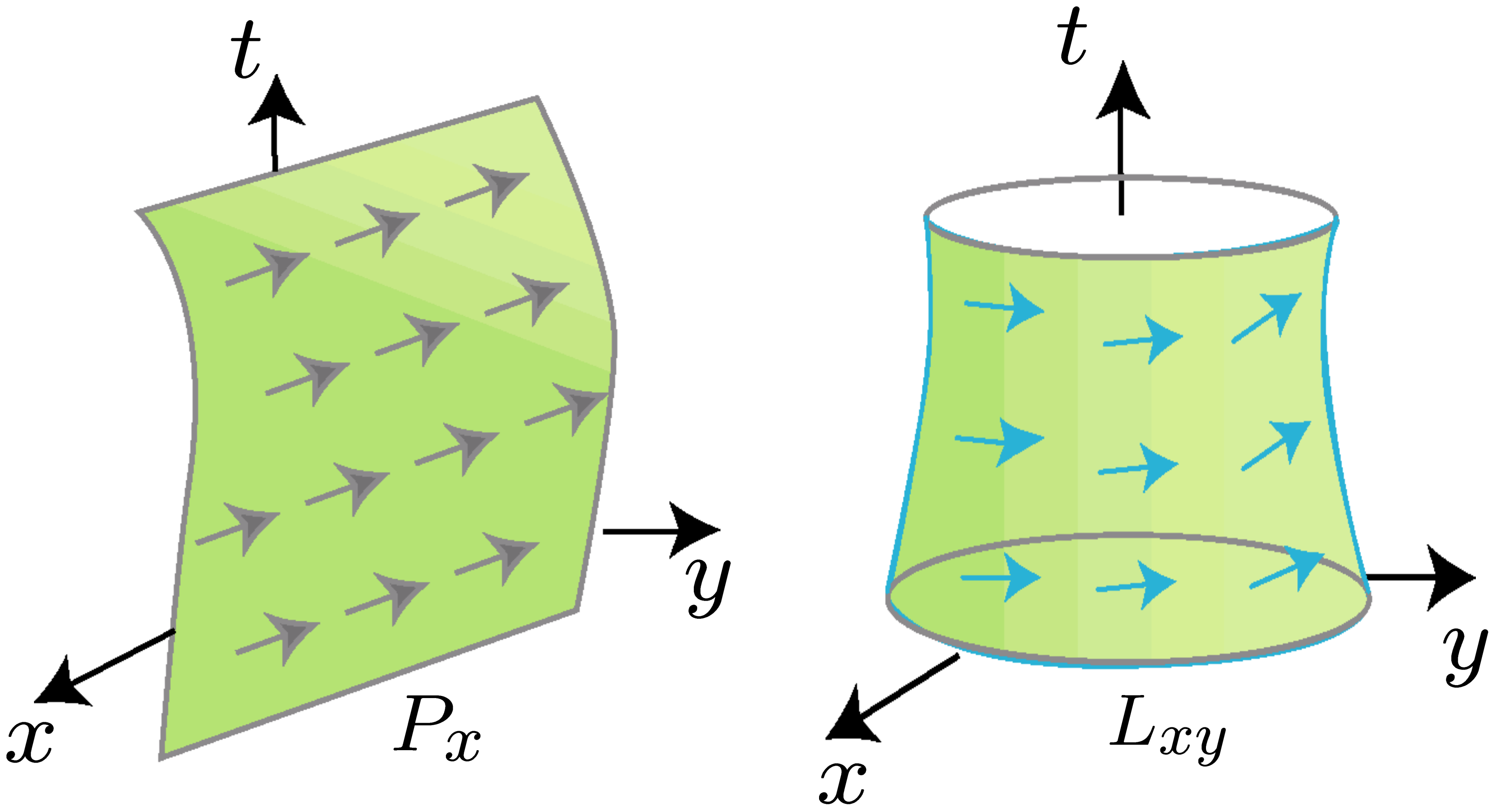
Nambu-Goto string：面積極小　波動方程式

symmetry

$$\partial_i\{\sqrt{-\gamma}\gamma^{ij}\partial_j\vec{X}(\sigma,\tau)\} = 0$$

測地線：長さ極小　常微分方程式

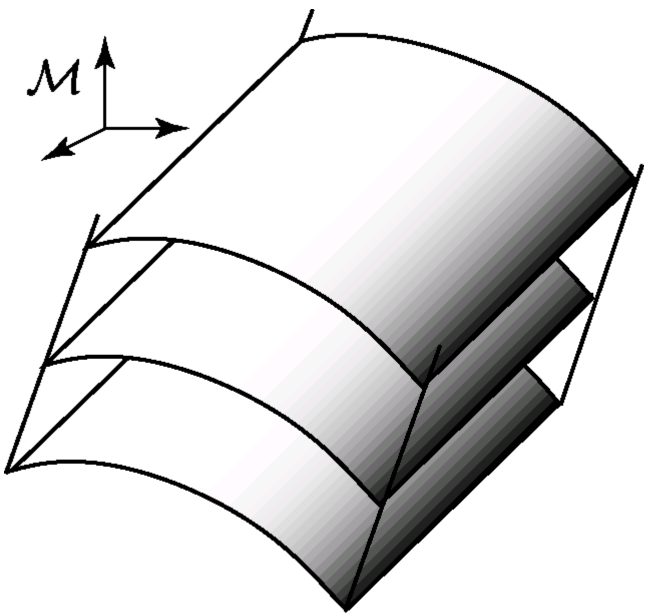
Strings with Spacelike Symmetry



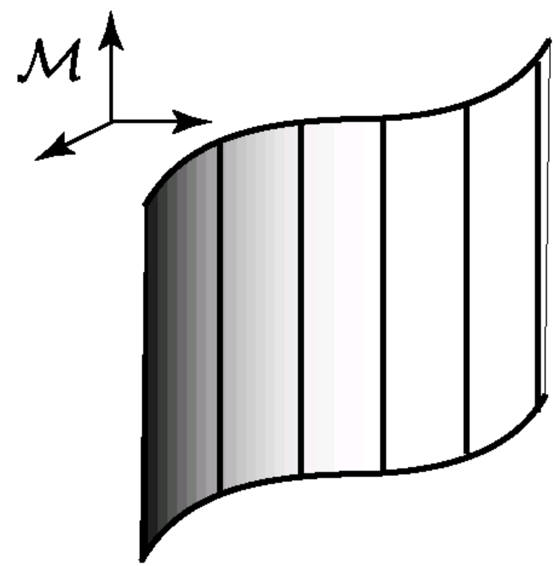
Cohomogeneity-one (C-1) object

(k+1)-dim. C-1 object

= foliation by k-dim. Homogeneous Hypersurfaces



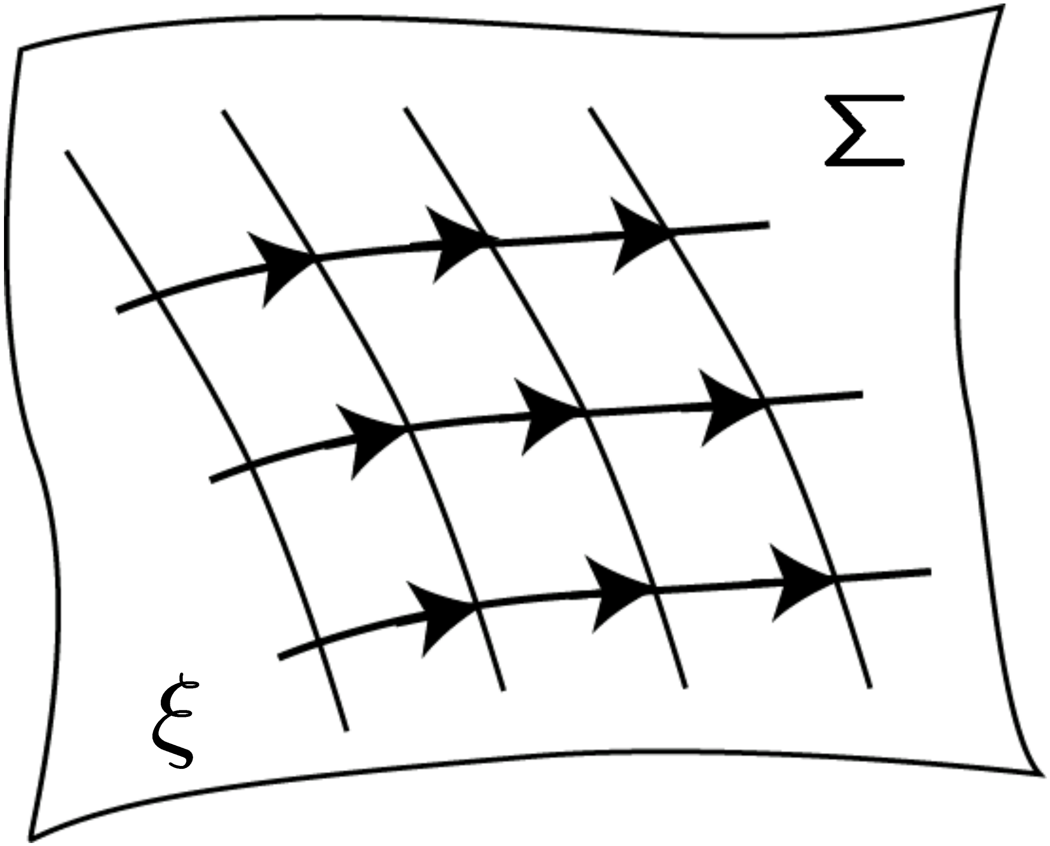
e.g. homogeneous universe



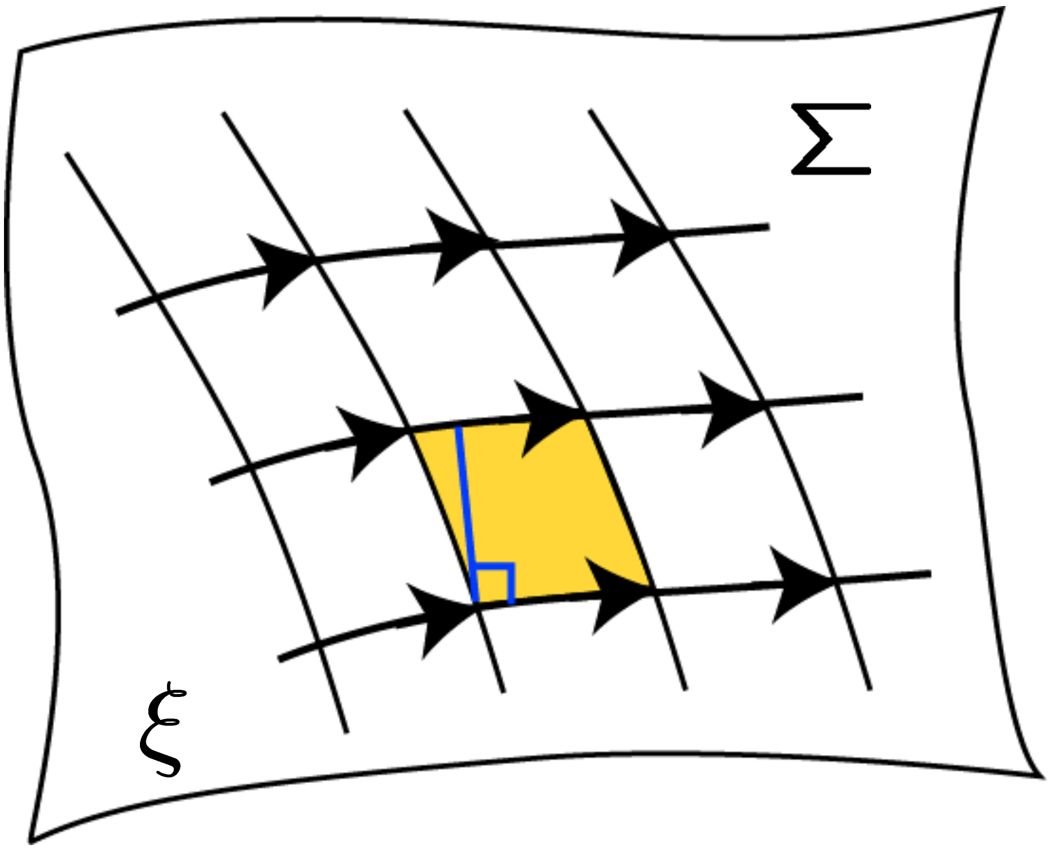
e.g. string with symm.

C-1 String

A Killing vector field is tangent to the worldsheet



Area elements



A Killing vector field is tangent to the worldsheet

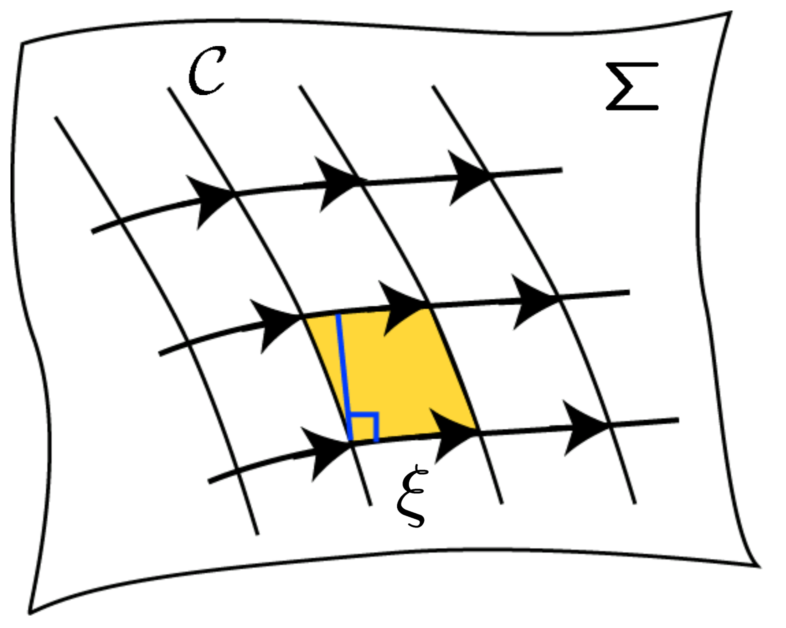
$$dA = |\xi| dl_\perp$$

$$dl_\perp^2 = (g_{\mu\nu} - \xi_\mu \xi_\nu / |\xi|^2) dx^\mu dx^\nu$$

Nambu-Goto action

$$S \propto \int_\Sigma dA = \int_\Sigma |\xi| dl_\perp$$
$$= \int_\mathcal{C} \sqrt{(\xi \cdot \xi) h_{\mu\nu} dx^\mu dx^\nu}$$

$$h_{\mu\nu} = g_{\mu\nu} - \frac{\xi_\mu \xi_\nu}{\xi \cdot \xi}$$



Dynamics of C-1 String

C-1 Nambu-Goto string associated with a Killing vector ξ

$$\pi : (\mathcal{M}, g) \to (\mathcal{O}, h)$$

$\mathcal{M}$: target spacetime

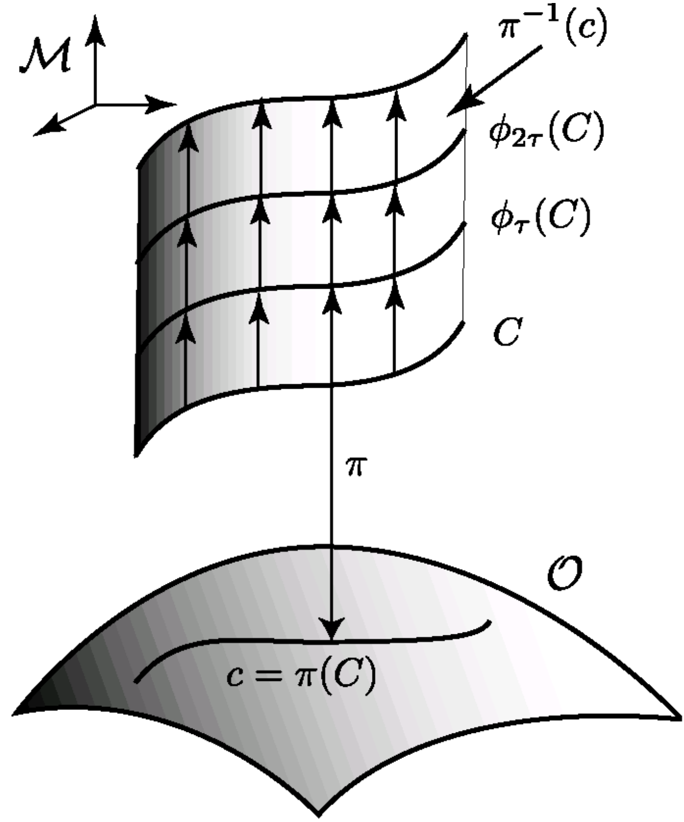
$\mathcal{O} = \mathcal{M}/e^{\xi}$: orbit space

$$h_{ab} = g_{ab} - \xi_a \xi_b / f, \quad f = \xi_a \xi^a$$

N-G action

$$S = \int_c \sqrt{f h_{ab} dx^a dx^b}$$

Geodesic equation (ODE)



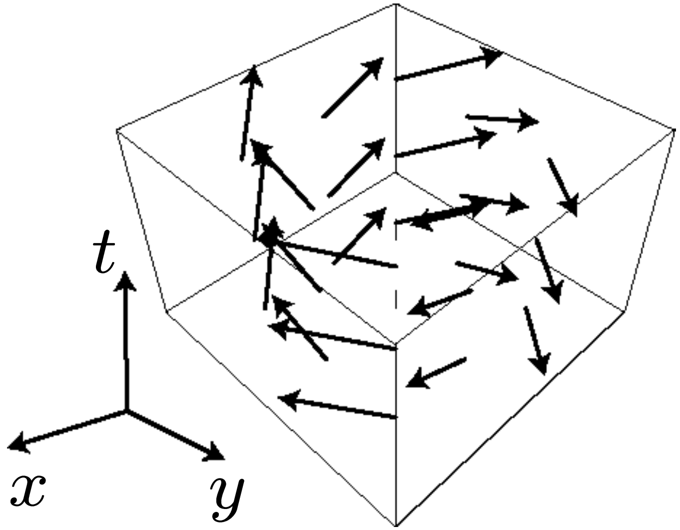
Example: Stationary Rotating Strings in 4D Minkowski

Target space

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$
$$= -dt^2 + d\rho^2 + \rho^2 d\phi^2 + dz^2$$

Consider a Killing vector

$$\xi = \partial_t + \Omega\partial_\phi$$



Ogawa, Ishihara,Kozaki,Nakano,Saitoh, PRD78, 023525(2008)

定常回転ストリング

Target space

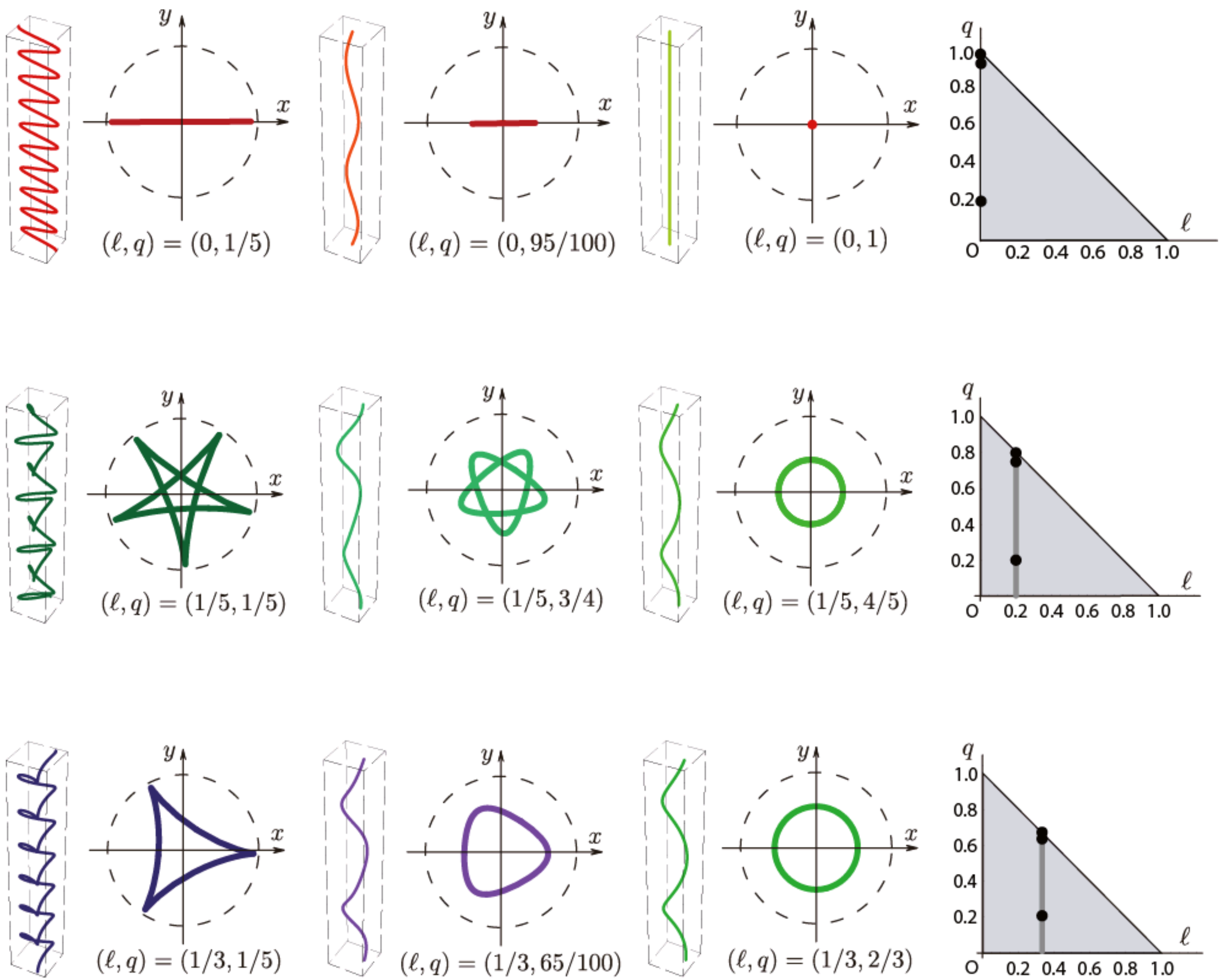
$$
\begin{aligned}
ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\
&= -dt^2 + d\rho^2 + \rho^2 d\phi^2 + dz^2
\end{aligned}
$$

$$\xi = \partial_t + \Omega \partial_\phi$$

Metric on the orbit space

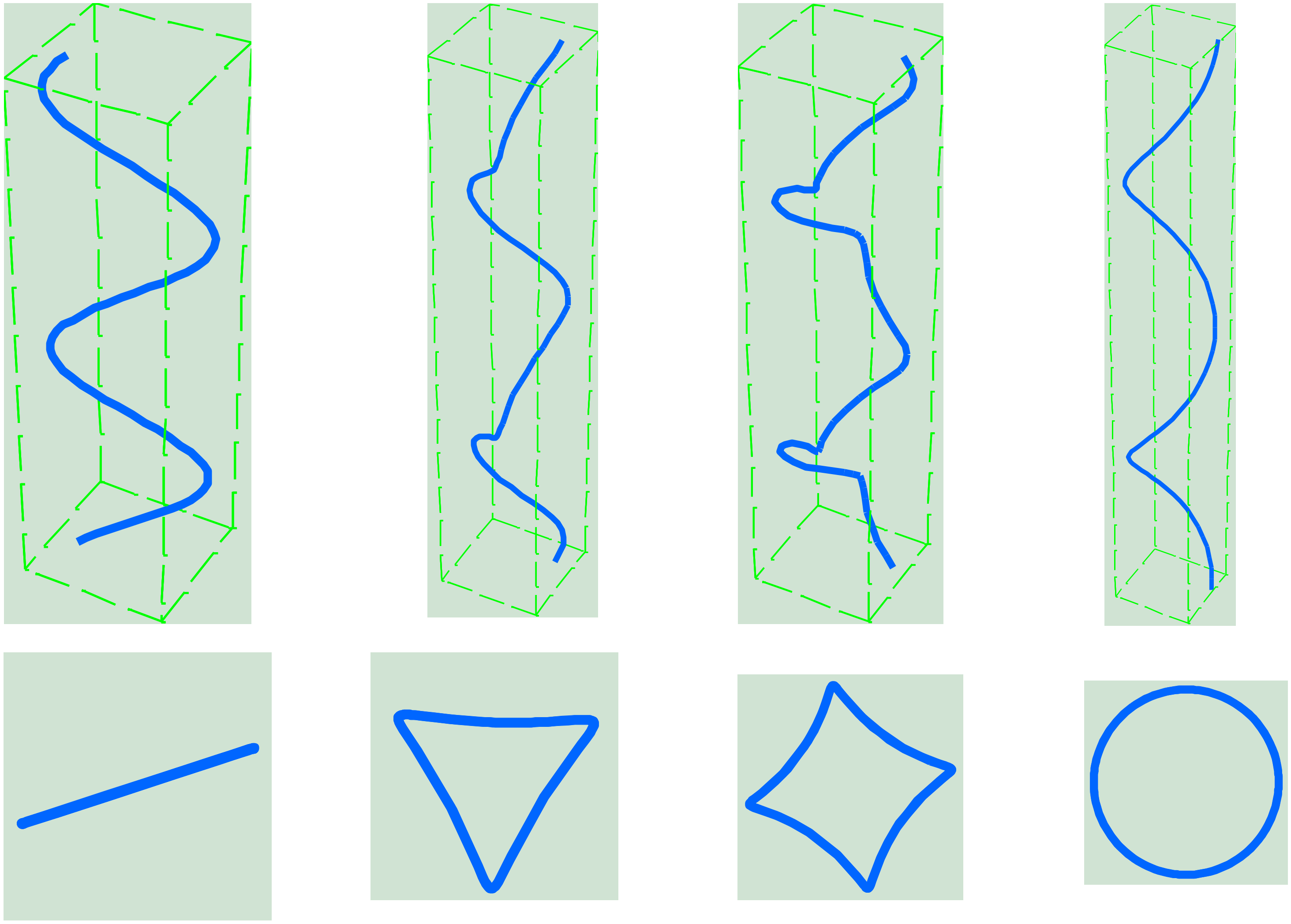
$$ds^2 = (1-\Omega^2\rho^2)(d\rho^2 + dz^2) + \rho^2 d\varphi^2$$

Stationary rotating strings = geodesics on this metric

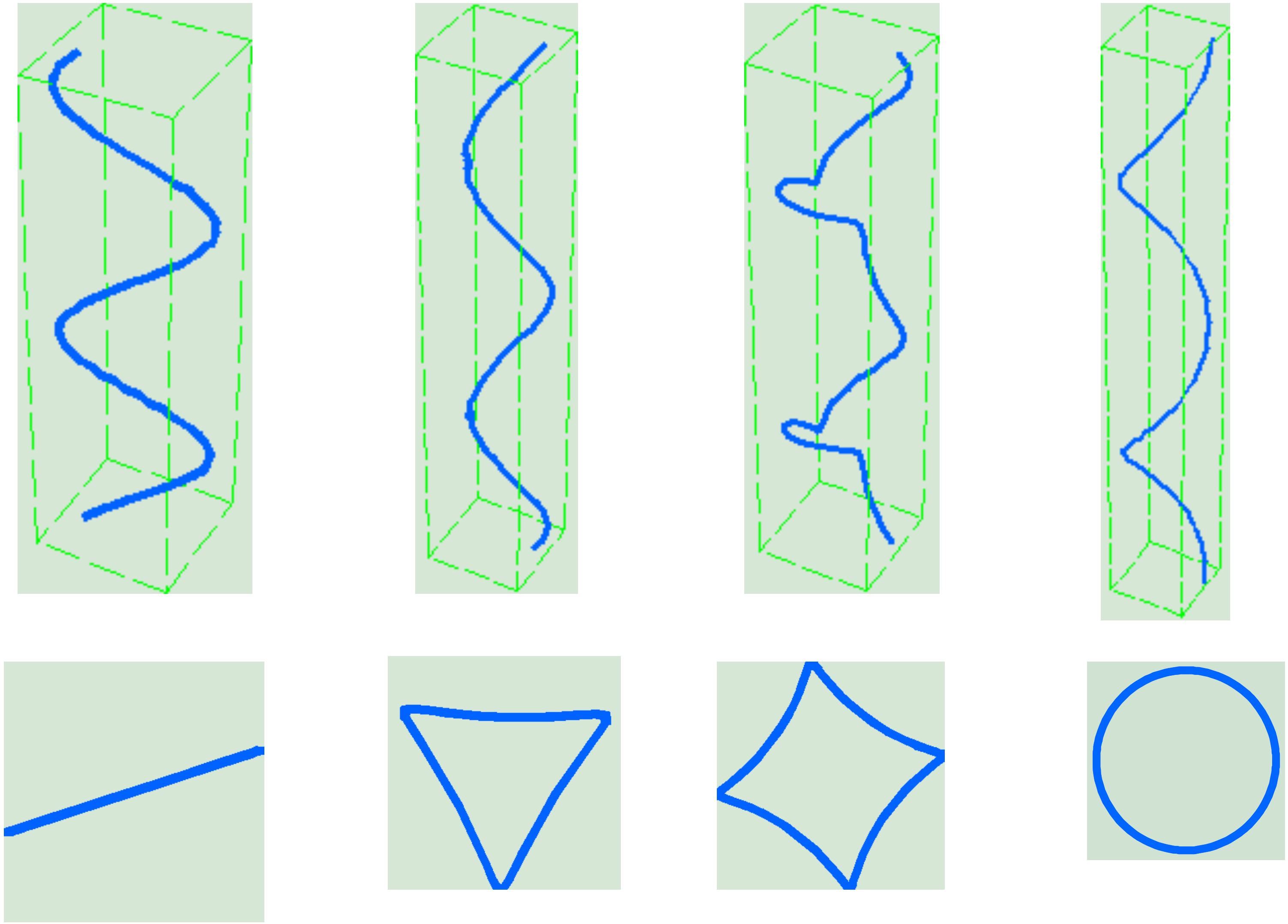


x
y
(ℓ, q) = (0, 1/5)
(ℓ, q) = (0, 95/100)
(ℓ, q) = (0, 1)
(ℓ, q) = (1/5, 1/5)
(ℓ, q) = (1/5, 3/4)
(ℓ, q) = (1/5, 4/5)
(ℓ, q) = (1/3, 1/5)
(ℓ, q) = (1/3, 65/100)
(ℓ, q) = (1/3, 2/3)
q
ℓ
O
0.2 0.4 0.6 0.8 1.0

Solutions



Strings are Rotating



Example: Toroidal Spirals in 5D Minkowski

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -dt^2 + d\rho^2 + \rho^2 d\phi^2 + d\zeta^2 + \zeta^2 d\psi^2. \end{aligned}$$

$\partial_t, \partial_\phi, \partial_\psi$ are commutable Killing vectors

We consider C-1 strings with

$$\xi = \partial_\phi + \alpha \partial_\psi$$

T. Igata, and H. Ishihara (2010)
T. Igata, H. Ishihara and K.Nishiwaki (2012)

Orbit Space for Toroidal Spirals

Killing vector

$$\xi = \partial_\phi + \alpha \partial_\psi$$

Projection tensor

$$h_{\mu\nu} = g_{\mu\nu} - \frac{\xi_\mu \xi_\nu}{\xi \cdot \xi}$$

C-1 string associated with ξ is equivalent to geodesics in the metric of 4-dimensional orbit space

$$\begin{aligned} ds_4^2 &= (\xi \cdot \xi) h_{ij} dx^i dx^j \\ &= (\rho^2 + \alpha^2 \zeta^2)\left(-dt^2 + d\rho^2 + d\zeta^2\right) + \rho^2 \zeta^2 d\bar{\varphi}^2. \end{aligned}$$

$$\bar{\varphi} := \psi - \alpha\phi$$

Killing vector の数が足りない！

Geodesic Particle in Orbit Space

Hamiltonian

$$H = \frac{N}{2}\left((\xi\cdot\xi)^{-1}h^{ij}p_ip_j + 1\right)$$
$$= \frac{N}{2}\left(\frac{1}{\rho^2+\alpha^2\zeta^2}\left(-p_t^2+p_\rho^2+p_\zeta^2\right)+\frac{1}{\rho^2\zeta^2}p_{\bar{\varphi}}^2+1\right)$$

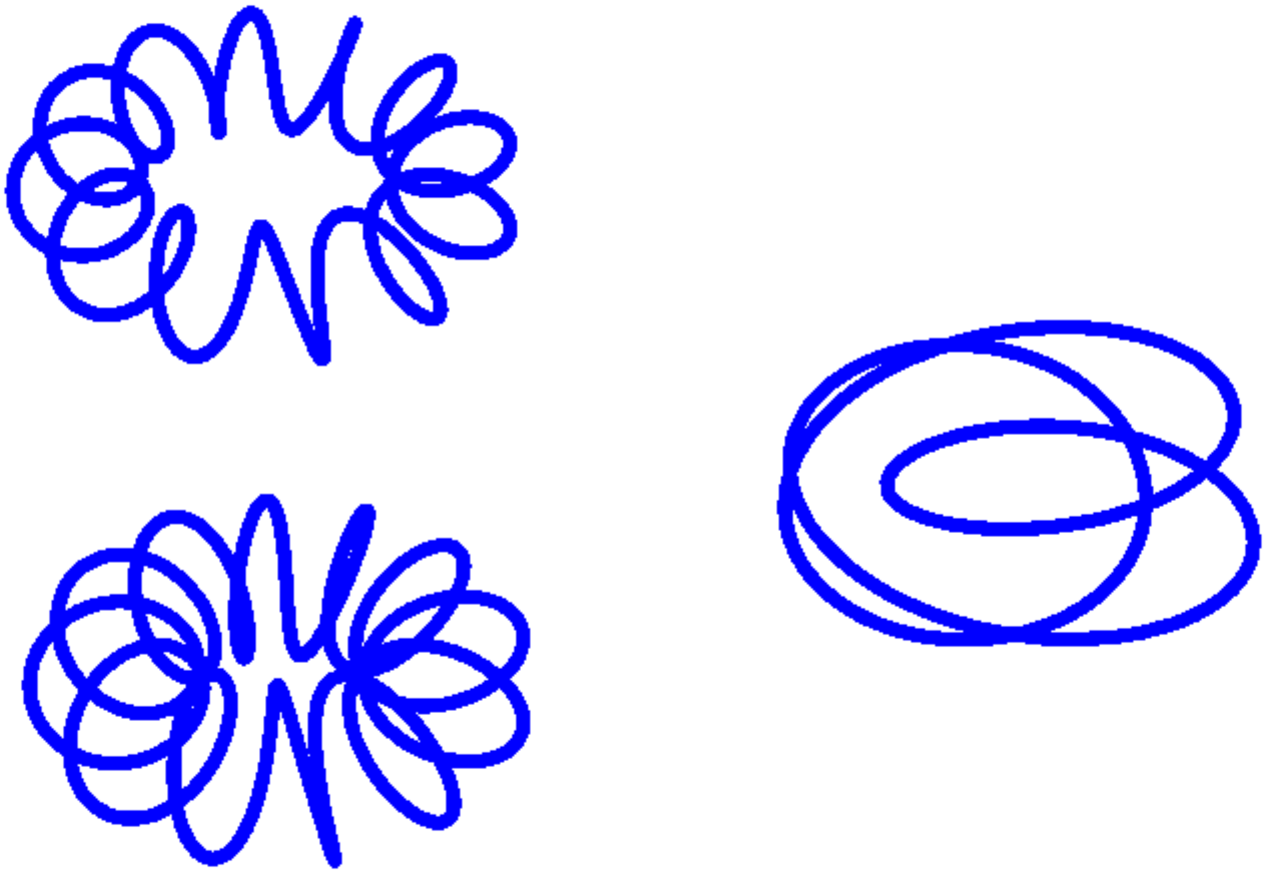
Constants of motion $\quad E = p_t,\, L = p_{\bar{\varphi}}$

Effective Hamiltonian

$$H = \frac{N}{2}\left(\frac{1}{\rho^2+\alpha^2\zeta^2}\left(p_\rho^2+p_\zeta^2\right)-\frac{E^2}{\rho^2+\alpha^2\zeta^2}+\frac{L^2}{\rho^2\zeta^2}+1\right)$$

Killing vector の数が足りないが，解ける！

Solutions



これまでの例は,

Minkowski時空の中の
いくつかのC-1ストリングは積分できる.

de Sitter, anti-de Sitter 時空の中の
すべてのC-1ストリングは積分できるか？

In general

- **Let $(\mathcal{M}, g)$ admits isometry group ϕ generated by a Killing vector ξ**
- **Consider the orbit space $\mathcal{M}/\phi$**
- **We introduce the metric**

$$h_{ab} = |\xi \cdot \xi| \left(g_{ab} - \frac{\xi_a \xi_b}{\xi \cdot \xi} \right)$$

on the orbit space $\mathcal{M}/\phi$.

H.Ishihara and H.Kozaki, Phys.Rev. D72 (2005) 061701.

T. Koike, H. Kozaki, and H. Ishihara, Phys.Rev. D77 (2008) 125003

H. Kozaki, T. Koike, and H. Ishihara, Class.Quant.Grav. 27 (2010) 10500

Results

- Consider the n-dimensional sphere as $\mathcal{M}$, for example,
- We show all possible orbit spaces

  with the metric

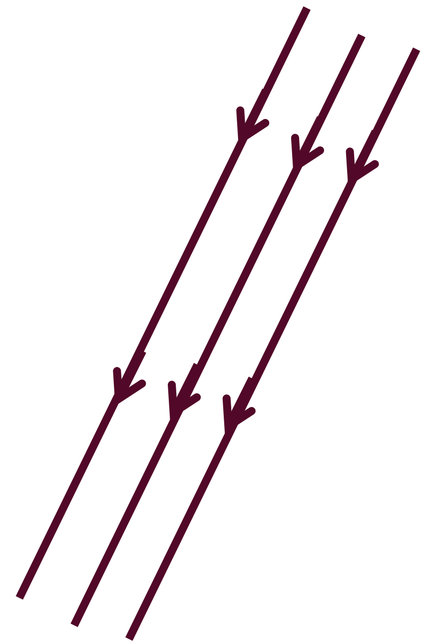
$$h_{ab} = |\xi \cdot \xi| \left(g_{ab} - \frac{\xi_a \xi_b}{\xi \cdot \xi} \right)$$

are geodesically integrable, i.e.,

Hamiltonian system describing the geodesics is integrable in the Liouville sence.

Orbit space

“金太郎あめ” traditional candy

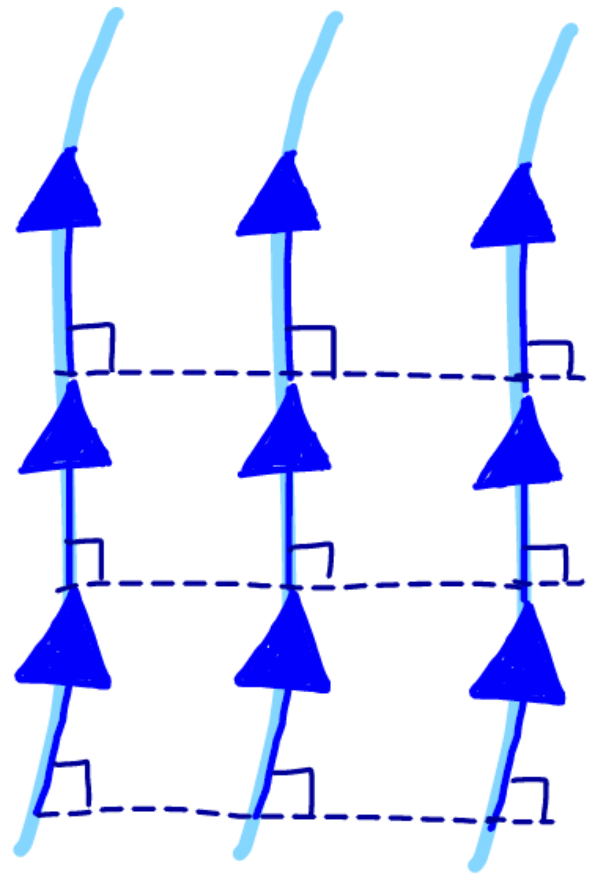


Isometry group acts

“断面”を調べる：orbit space！

Metric on an orbit space

$$h_{ab} = |\xi \cdot \xi| \left(g_{ab} - \frac{\xi_a \xi_b}{\xi \cdot \xi} \right)$$



ξ を分類する

Dynamics of Cohomogeniety-1 String

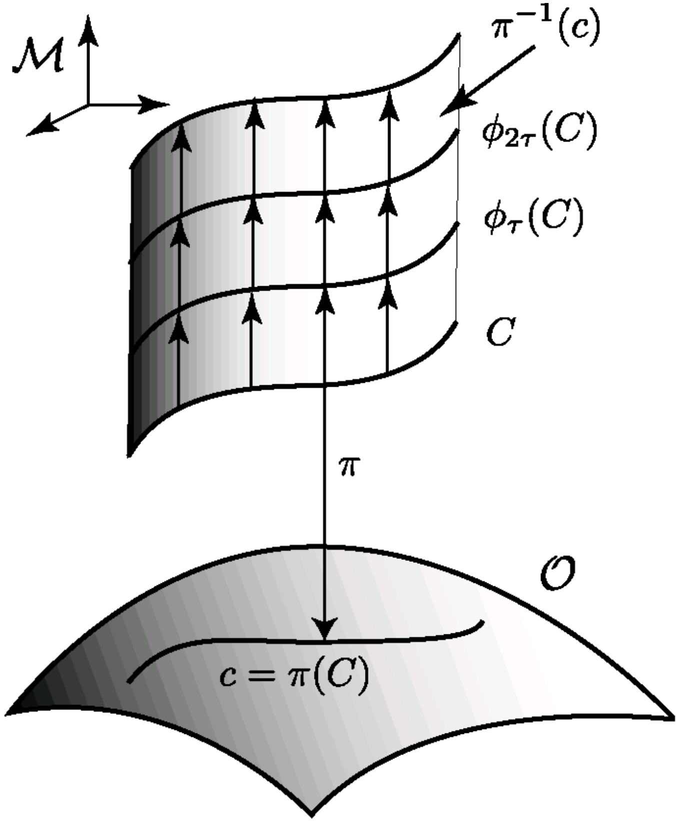
Cohomogeniety-1 Nambu-Goto string
associated with a Killing vector ξ

$$\pi : (\mathcal{M}, g) \to (\mathcal{O}, h)$$

$\mathcal{M}$: target spacetime

$\mathcal{O} = \mathcal{M}/e^{\xi}$: orbit space

$$h_{ab} = g_{ab} - \xi_a \xi_b / f, \quad f = \xi_a \xi^a$$



N-G action

$$S = \int_c \sqrt{-f h_{ab} dx^a dx^b}$$

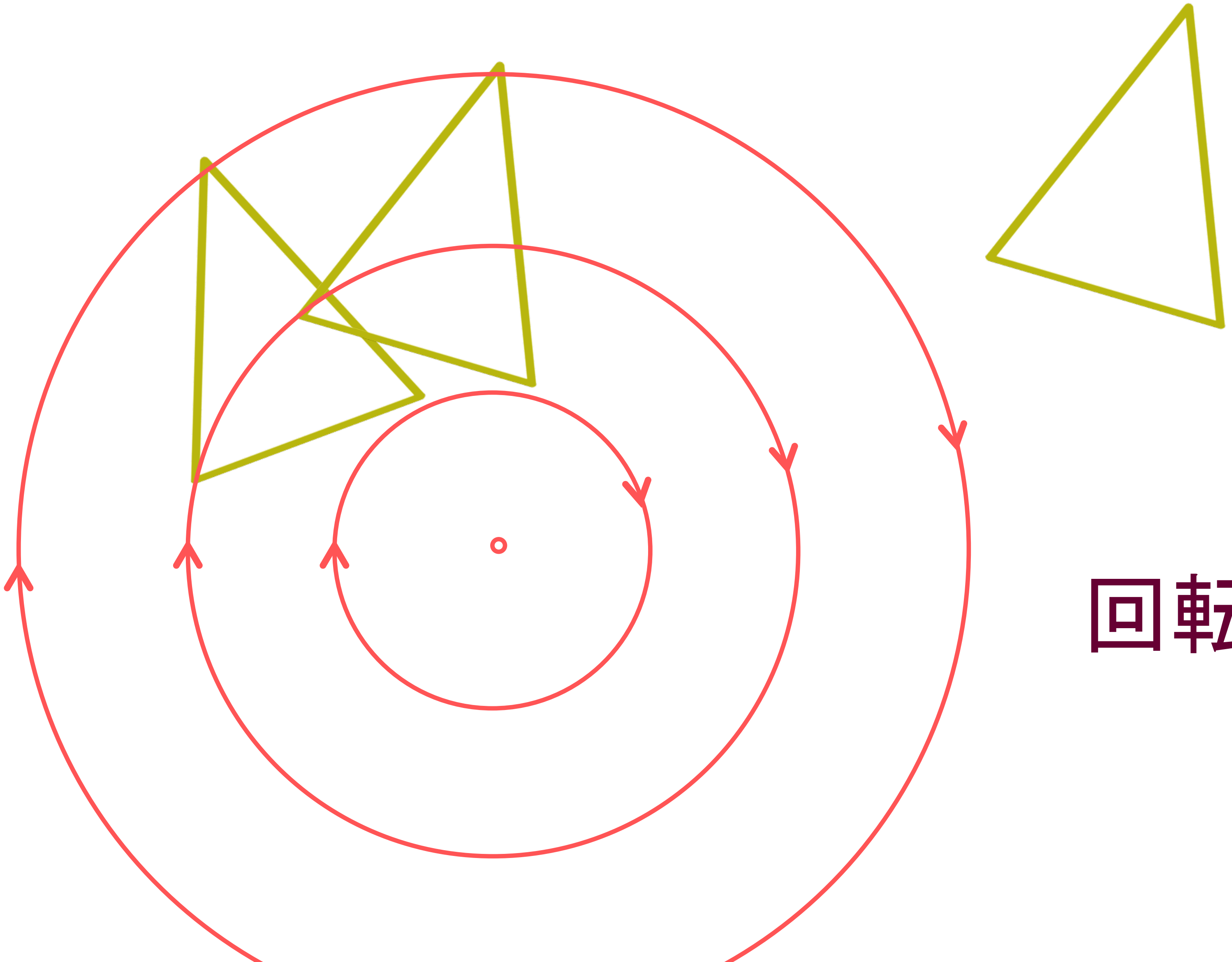
Geodesic equation (ODE)

Classification of Killing vectors

等長変換

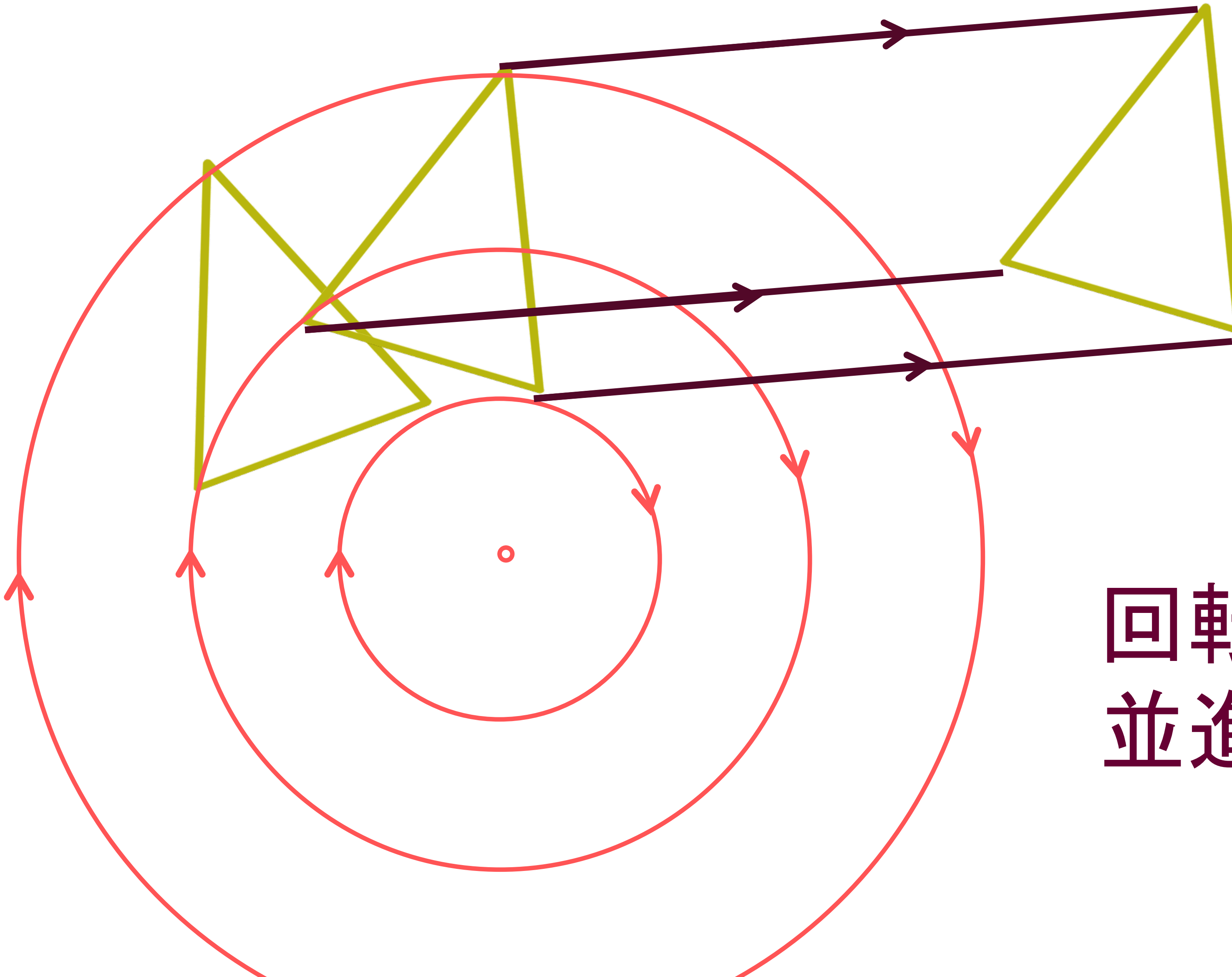
合同

等長変換



回転

等長変換



回転
並進

Equivalence of Killing vectors

Equivalence class of isometry

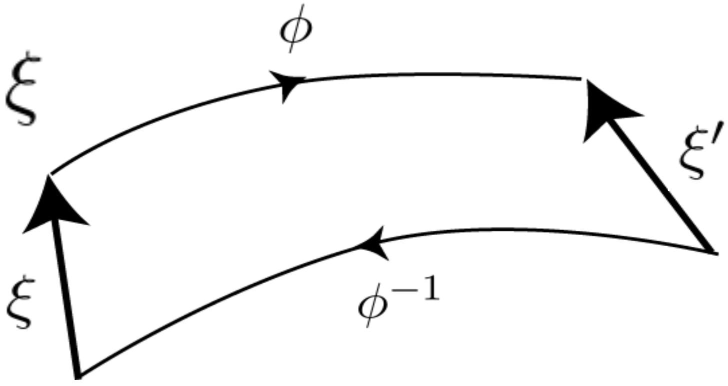
$$g,\ \ g' \in \mathrm{Isom}\mathcal{M} \qquad g \sim g'$$

$$\Leftrightarrow \ \exists \phi \in \mathrm{Isom}\mathcal{M} \ \text{ s.t. } \ g' = \phi g \phi^{-1}$$

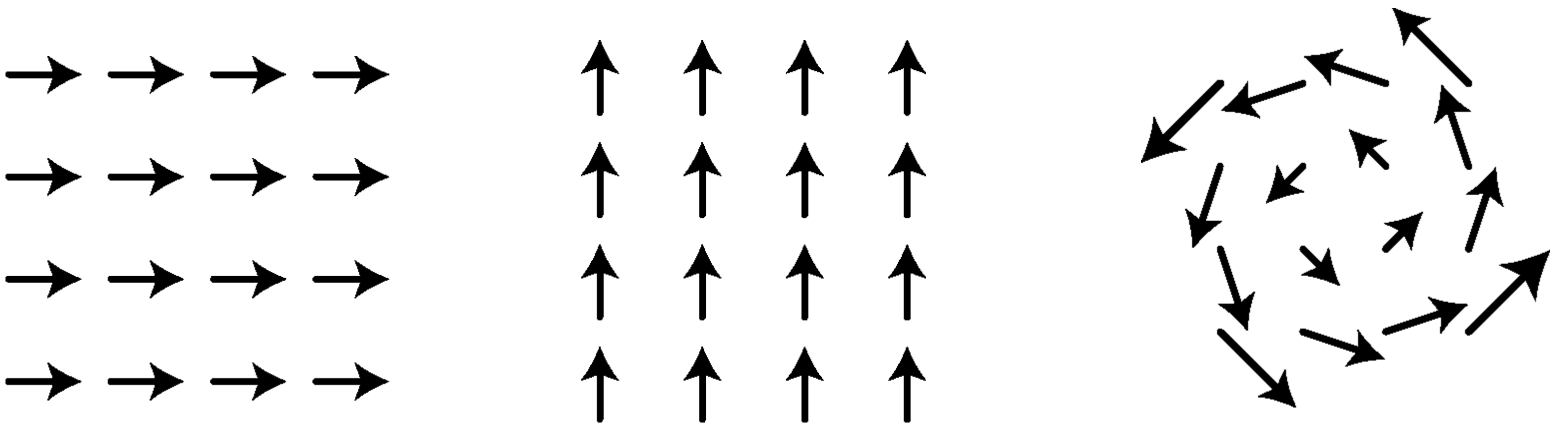
Conjugacy class

Equivalence of Killing vector

$$\xi \sim \xi' \quad \Leftrightarrow \quad \xi' = \phi_* \xi$$



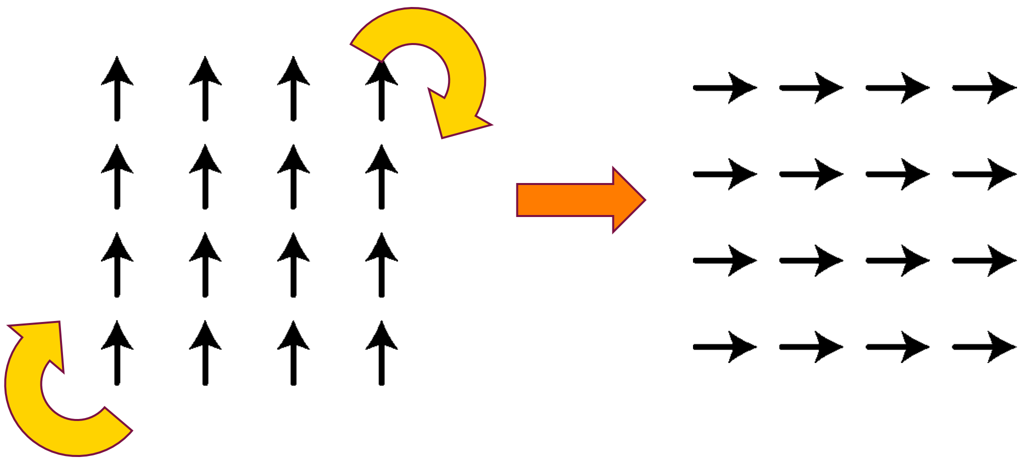
Isometries in x-y Plane



$$P_x = \partial_x \qquad P_y = \partial_y \qquad L_{xy} = x\partial_y - y\partial_x$$

3 linearly independent Klling vector fields

Equivalence Class



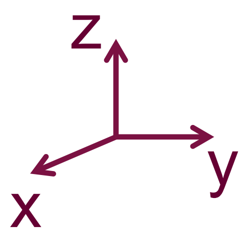
$$P_y \sim P_x$$

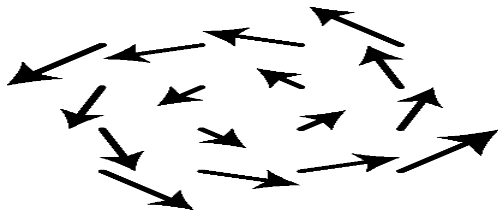
$$\alpha P_x + \beta P_y \sim P_x$$

$$\alpha P_x + \beta P_y + L_{xy} \sim L_{xy}$$

Equivalence classes $\{P_x, L_{xy}\}$

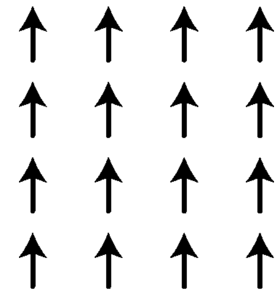
Isometry in R^3



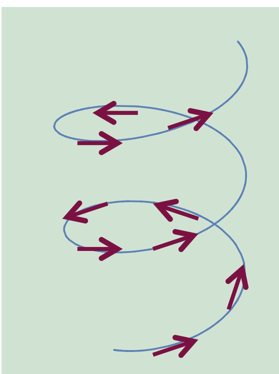


L_{xy}

$+$

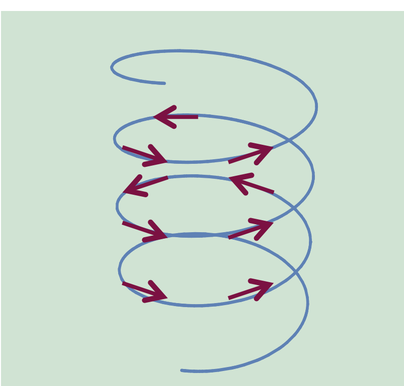


P_z



$$\xi = L_{xy} + aP_z$$

$$a \neq a' \Rightarrow \xi \neq \xi'$$



$$\xi' = L_{xy} + a'P_z$$

Classification of Killing vectors

4-dim. Minkowski

Type	Canonical form
I	$aP_t + bL_{xy}$
II	$a(P_t + P_z) + bL_{xy}$
III	$aP_z + bL_{xy}$
IV	$aP_z + b(K_{ty} + L_{xy})$
V	$aP_z + bK_{ty}$
VI	$aP_x + b(K_{ty} + L_{xy})$
VII	$aK_{tz} + bL_{xy}$

4-dim. Euclid

Type	Canonical form
I	$aP_z + bL_{xy}$
II	$aL_{zw} + bL_{xy}$

P : translation

L : rotation

K : Lorentz boost

Ishihara and Kozaki PRD(2005)

Trivial example :4-dim. Euclid space

R^4 $$ds^2 = dx^2 + dy^2 + dz^2 + dw^2$$

Killing vector for reduction

$$\xi = \partial_w \qquad \xi \cdot \xi = 1$$

Reduced metric

$$d\tilde{s}^2 = (\xi \cdot \xi)\left(g_{\mu\nu} - \frac{\xi_\mu \xi_\nu}{\xi \cdot \xi}\right) dx^\mu dx^\nu$$

$$= dx^2 + dy^2 + dz^2$$

Killing vectors $\partial_x, \quad \partial_y, \quad \partial_z$

Non trivial example :4-dim. Euclid space

R^4 $$ds^2 = dr_1^2 + r_1^2 d\phi_1^2 + dr_2^2 + r_2^2 d\phi_2^2$$

Killing vector for reduction

$$\xi = a_1 \partial_{\phi_1} + a_2 \partial_{\phi_2} \qquad \xi \cdot \xi = a_1^2 r_1^2 + a_2^2 r_2^2$$

Reduced metric

$$d\tilde{s}^2 = (\xi \cdot \xi)\left(g_{\mu\nu} - \frac{\xi_\mu \xi_\nu}{\xi \cdot \xi}\right) dx^\mu dx^\nu$$

$$= (a_1^2 r_1^2 + a_2^2 r_2^2)(dr_1^2 + dr_2^2) - r_1^2 r_2^2 \, d\psi^2$$

Only one Killing vector ∂_ψ

Killing vector と残存対称性

Type	tangential Killing vector ξ	basis of $\mathcal{C}(\xi)$	n
I	$P_t + aL_z\ (a \neq 0)$	P_t, P_z, L_z	2
I	P_t	$P_t, P_x, P_y, P_z, L_x, L_y, L_z$	3
I	L_z	P_t, P_z, L_z, K_z	3
II	$(P_t + P_z) + aL_z\ (a \neq 0)$	P_t, P_z, L_z	2
II	$P_t + P_z$	$P_t, P_x, P_y, P_z, K_y + L_x, K_x - L_y, L_z$	3
III	$P_z + aL_z\ (a \neq 0)$	P_t, P_z, L_z	2
III	P_z	$P_t, P_x, P_y, P_z, L_z, K_x, K_y$	3
IV	$P_z + a(K_y + L_z)$	$P_t - P_x, P_z, P_y + a(K_z - L_y), K_y + L_z$	2
V	$P_z + aK_y\ (a \neq 0)$	P_x, P_z, K_y	2
V	K_y	P_x, P_z, L_y, K_y	2
VI	$K_y + L_z + aP_x\ (a \neq 0)$	$K_y + L_z + aP_x, P_t - P_x, P_z$	2
VII	$K_z + aL_z\ (a \neq 0)$	L_z, K_z	1

n=3: 一様
n=2: あと1歩で一様
n=1: 一様でないが･･･

Example :4-dim. Minkowski時空

$$ds^2 = -dt^2 + t^2 d\psi^2 + dr^2 + r^2 d\phi^2$$

Killing vector for reduction

$$\xi^a = K_z + aL_{xy} = \partial_\psi + a\partial_\phi$$

metric

$$ds_h^2 = (t^2 + a^2 r^2)^\alpha \left(-dt^2 + dr^2 + \frac{t^2 r^2}{t^2 + a^2 r^2} d\sigma^2 \right)$$

$\alpha = 1$ のとき Killing tensor

$$K = a^2 r^2 (t^2 + a^2 r^2) dt^2 + t^2 (t^2 + a^2 r^2) dr^2 + t^2 r^2 (t^2 - a^2 r^2) d\sigma^2$$

が存在する

Example :4-dim. Minkowski時空

$$ds^2 = -dt^2 + t^2 d\psi^2 + dr^2 + r^2 d\phi^2$$

Killing vector for reduction

$$\xi^a = K_z + aL_{xy} = \partial_\psi + a\partial_\phi$$

metric

$$ds_h^2 = (t^2 + a^2 r^2)^\alpha \left(-dt^2 + dr^2 + \frac{t^2 r^2}{t^2 + a^2 r^2} d\sigma^2 \right)$$

隠れた対称性
絶妙な美しさ？

$\alpha = 1$ のとき Killing tensor

$$K = a^2 r^2 (t^2 + a^2 r^2) dt^2 + t^2 (t^2 + a^2 r^2) dr^2 + t^2 r^2 (t^2 - a^2 r^2) d\sigma^2$$

が存在する

Killing Vector Fields in AdS^5

Type	Killing vector field ξ
$(4\|0)$	$K_x + \widetilde{K}_y + J_{xy} + L + 2(J_{yz} + K_z)$
$\pm(3,1\|0)$	$K_x + \widetilde{K}_y + J_{yz} \pm J_{xw} + a(J_{xy} - L \mp J_{zw})$
$(2,2\|0)$	$K_x + L + aJ_{yz}$
$(2,-2\|0)$	$K_x + J_{xy} + aJ_{zw}$
$(2,1,1\|0)$	$K_x + \widetilde{K}_y + J_{xy} + L + a\,J_{zw} + b\,(J_{xy} - L)$
$(1,1,1,1\|0)$	$a\,L + b\,J_{xy} + c\,J_{zw}\ \ (a^2 + b^2 + c^2 = 1)$
$(2\|1)$	$K_x + \widetilde{K}_y + L + J_{xy} + aJ_{zw} + b(K_y + \widetilde{K}_x)$
$(1,1\|1)$	$K_x + \widetilde{K}_y + b\,(L - J_{xy}) + c\,J_{zw}$
$(0\|2)$	$K_x + J_{xy} + a\,\widetilde{K}_z\ \ (a \neq 0)$
$(0\|1,1)$	$aK_x + b\,\widetilde{K}_y + c\,J_{zw}\quad (b \neq \pm a,\ \ a^2 + b^2 + c^2 = 1)$

C-1 strings in AdS^5 are classified in 10 families.

Integrability of geodesics on an orbit space

Geodesic Hamiltonian

metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

Hamiltonian

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$$

Canonical equations

$$\dot{x}^\rho = \{x^\rho, H\}$$

$$\dot{p}^\sigma = \{p^\sigma, H\}$$

⟹ geodesic equation

Poisson bracket

$$\{F(p,x), G(p,x)\} := \frac{\partial F}{\partial x^\mu}\frac{\partial G}{\partial p_\mu} - \frac{\partial G}{\partial x^\mu}\frac{\partial F}{\partial p_\mu}$$

Killing field and constant of motion

If the metric admits a Killing vector

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)} = 0$$

The quantity $Q = \xi^\mu p_\mu$ is conserved.

$$\dot{Q} = \{Q, H\}$$
$$= \{\xi^\mu p_\mu, H\} = (\nabla^\mu \xi^\nu) p_\mu p_\nu = 0$$

If the metric admits a Killing tensor

$$\nabla_{(\lambda} K_{\mu\nu)} = 0$$

The quantity $Q_{(2)} = K^{\mu\nu} p_\mu p_\nu$ is conserved.

$$\dot{Q}_{(2)} = \{Q_{(2)}, H\}$$
$$= \{K^{\mu\nu} p_\mu p_\nu, H\} = (\nabla^\lambda K^{\mu\nu}) p_\lambda p_\mu p_\nu = 0$$

Integrability of Hamiltonian system

The Hamiltonian system with the degree of freedom N
is integrable in Liouville's sence
if the number of independent Poisson commuting invariants
(including the Hamiltonian itself) is N.

$$\{H, Q_i\} = \{Q_i, Q_j\} = 0, (i = 1, 2, \cdots N - 1)$$

Restriction of Hamiltonian

$$g^{\mu\nu}p_\mu p_\nu = \left(g^{\mu\nu} - \frac{\xi^\mu \xi^\nu}{\xi \cdot \xi}\right) p_\mu p_\nu + \frac{\xi^\mu \xi^\nu}{\xi \cdot \xi} p_\mu p_\nu$$

$$= h^{\mu\nu}p_\mu p_\nu + \frac{(\xi^\mu p_\mu)(\xi^\nu p_\nu)}{\xi \cdot \xi}$$

$$H_h = \frac{1}{2} h^{\mu\nu} p_\mu p_\nu$$

$$= \frac{1}{2} g^{\mu\nu} p_\mu p_\nu |_{\xi p = 0} = H_g |_{\xi p = 0}$$

If

$$H_g = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$$

is integrable

$$\xi^\mu p_\mu = 0$$

then

$$H_h = \frac{1}{2} h^{\mu\nu} p_\mu p_\nu$$

is integrable

$$H_g = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \longrightarrow \tilde{H}_g = \frac{1}{2} (\xi \cdot \xi)^{-1} g^{\mu\nu} p_\mu p_\nu$$

$$\xi^\mu p_\mu = 0 \Big\downarrow \qquad\qquad \xi^\mu p_\mu = 0 \Big\downarrow$$

$$H_h = \frac{1}{2} h^{\mu\nu} p_\mu p_\nu \longrightarrow \tilde{H}_h = \frac{1}{2} (\xi \cdot \xi)^{-1} h^{\mu\nu} p_\mu p_\nu$$

$$H_g = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$$

If

$$\tilde{H}_g = \frac{1}{2} (\xi \cdot \xi)^{-1} g^{\mu\nu} p_\mu p_\nu$$

is integrable

$\xi^\mu p_\mu = 0$

$\xi^\mu p_\mu = 0$

then

$$H_h = \frac{1}{2} h^{\mu\nu} p_\mu p_\nu$$

$$\tilde{H}_h = \frac{1}{2} (\xi \cdot \xi)^{-1} h^{\mu\nu} p_\mu p_\nu$$

is integrable

Our aim

We show the system with the Hailtonian

$$H_g = \frac{1}{2}(\xi \cdot \xi)^{-1} g^{\mu\nu} p_\mu p_\nu$$

is integrable, in the case of the metric $g_{\mu\nu}$ is S^{2n-1} and ξ is any Killing vector on S^{2n-1}

If it is true, the system with the Hailtonian

$$H_h = \frac{1}{2}(\xi \cdot \xi)^{-1} h^{\mu\nu} p_\mu p_\nu$$

is integrable.

(2n-1)-dimensional sphere

S^{2n-1} is defined by

$$ds^2 = dx_1^2 + dy_1^2 + dx_2^2 + dy_2^2 + \cdots + dx_n^2 + dy_n^2$$

$$x_1^2 + y_1^2 + x_2^2 + y_2^2 + \cdots + x_n^2 + y_n^2 = 1$$

Killing vectors

$$L_{x_i x_j} \ (i \neq j), L_{y_i y_j} \ (i \neq j), L_{x_i y_j}, \quad (i, j = 1, 2, \cdots, n)$$

The most less symmetric $(\xi \cdot \xi)^{-1} g^{\mu\nu}$ case is

$$\xi = \sum_{i=1,\cdots,n} a_i L_{x_i y_i}$$

Dangerous case

In the case of S^{2n-1}

$(\xi \cdot \xi)^{-1} g^{\mu\nu}$ with $\xi = \sum_{i=1,\cdots,n} a_i L_{x_i y_i}$

admits *n* commutable Killing vectors, η ,
and no more.

If we find *(n-1)* commutable Killing tensors,

the metric $(\xi \cdot \xi)^{-1} g^{\mu\nu}$

is geodesically integrable.

The Hamiltonian system

$$\tilde{H}_g = \frac{1}{2}(\xi \cdot \xi)^{-1} g^{\mu\nu} p_\mu p_\nu = E$$

is equivalent to the system

$$H'_g = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu - E(\xi \cdot \xi) \approx 0$$

We find constants of motion of the system H'_g .

"Jaccobi's Hamiltonian"

Killing hierarchy

We assume the constant of motion in the form

$$Q = \frac{1}{2} K^{\mu\nu}_{(2)} p_\mu p_\nu + E K_{(0)}$$

$$\{Q, H'_g\} = \{\frac{1}{2} K^{\mu\nu}_{(2)} p_\mu p_\nu + E K_{(0)}, \frac{1}{2} g^{\mu\nu} p_\mu p_\nu - E(\xi \cdot \xi)\}$$

$$\cdots$$

$$= \nabla^\lambda K^{\mu\nu}_{(2)}\, p_\lambda p_\mu p_\nu + E \left(K^{\mu\nu}_{(2)} \partial_\mu (\xi \cdot \xi) - \partial_\mu K_{(0)} g^{\mu\nu} \right) p_\nu$$

$$= 0$$

We have

$$\nabla^{(\lambda} K^{\mu\nu)}_{(2)} = 0$$

$$\partial^\mu K_{(0)} - K^{\mu\nu}_{(2)} \partial_\nu (\xi \cdot \xi) = 0$$

$$\nabla^{(\lambda} K_{(2)}^{\mu\nu)} = 0$$

Killing tensor eqs. for S^{2n-1}

$$\partial^{\mu} K_{(0)} - K_{(2)}^{\mu\nu} \partial_{\nu} (\xi \cdot \xi) = 0$$

Reducible Killing tensor

$$\partial^{[\lambda} \partial^{\mu]} K_{(0)} = \partial^{[\lambda} K_{(2)}^{\mu]\nu} \partial_{\nu} (\xi \cdot \xi) = 0$$

Integrability condition for $K_{(0)}$

$$K_{(2)} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij}(L^2_{x_i x_j} + L^2_{x_i y_j} + L^2_{y_i x_j} + L^2_{y_i y_j})$$

$$(a_i^2 - a_j^2)c_{ij} + (a_j^2 - a_k^2)c_{jk} + (a_k^2 - a_i^2)c_{ik} = 0, \quad \text{(no sum)}$$

(n-1) independent c_{ij}

$$\partial^\mu K_{(0)} - K^{\mu\nu}_{(2)} \partial_\nu (\xi \cdot \xi) = 0$$

$$Q = \frac{1}{2} \left(K^{\mu\nu}_{(2)} + K_{(0)} g^{\mu\nu} \right) p_\mu p_\nu$$

Q *:(n-1)* , η *: (n)*, and $\tilde{H}_g$ are commutable.

The metric $(\xi \cdot \xi)^{-1} g^{\mu\nu}$

is geodesically integrable.

Results

- We consider the n-dimensional sphere, for example.
- We show all possible orbit spaces with the metric

$$\tilde{h}_{\mu\nu} = (\xi \cdot \xi)\left(g_{\mu\nu} - \frac{\xi_\mu \xi_\nu}{\xi \cdot \xi}\right)$$

are geodesically integrable.

It suggests that cohomogeneity-one strings in (A-) dS space are integrable.