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Liouville integrability of Hamiltonian systems and spacetime symmetry

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Hamilton formalism

- Many dynamical systems in physics are described in this framework.
- A dynamical system is governed by a function of canonical coordinates q^i and momenta p_i, called Hamiltonian $H(q^i, p_i)$
- Equations of motion

$$\frac{dq^i}{d\tau} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{d\tau} = -\frac{\partial H}{\partial q^i}$$

Hamilton formalism

- Poisson bracket

$$\{A,B\}_P := \frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i}$$

- Conserved quantity (First integral)

$$\frac{dF}{d\tau} = \frac{\partial F}{\partial q^i}\frac{dq^i}{d\tau} + \frac{\partial F}{\partial p_i}\frac{dp_i}{d\tau}$$

$$= \frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i} = \{F,H\}_P$$

F is a conserved quantity $\Leftrightarrow$ $\{F,H\}_P=0$

Hamilton formalism

- Liouville integrability

If there exist D independent Poisson-commuting constants α_i (including Hamiltonian) in a D-dim Hamiltonian system,

$$\{\alpha_i, \alpha_j\}_P = 0, \ (i, j = 1, \ldots, D, \ \alpha_D = H)$$

then the system is said to be completely integrable.

Namely, one can prove that there exists a canonical transf. *(x, p) → (φ, I(α))*, and then easily solve the Hamilton's eq.:

$$\dot{\varphi}^\mu = \frac{\partial H'}{\partial I_\mu}, \qquad \dot{I}_\mu = -\frac{\partial H'}{\partial \varphi^\mu}.$$

Hamiltonian

$$H = \frac{1}{2}\sum_{i,j} g^{ij}(\boldsymbol{q}) p_i p_j + V(\boldsymbol{q})$$

(q^i, p_i) : *canonical coordinates*

- $g_{ij}(\boldsymbol{q})$: *metric*

$$ds^2 = g_{ij} dq^i dq^j$$

- $V(\boldsymbol{q})$: *potential*

$V \neq 0$ Natural Hamiltonian

$V = 0$ Geodesic Hamiltonian

The purpose of this talk

To show a systematic approach for investigating polynomial conserved quantities for any natural Hamiltonian system

Keywords

① Geometrisation

Any natural Hamiltonian system can be translated to the geodesic problem in a corresponding spacetime.

② Prolongation

Equations describing spacetime symmetry can be translated to a first-order linear PDE system.

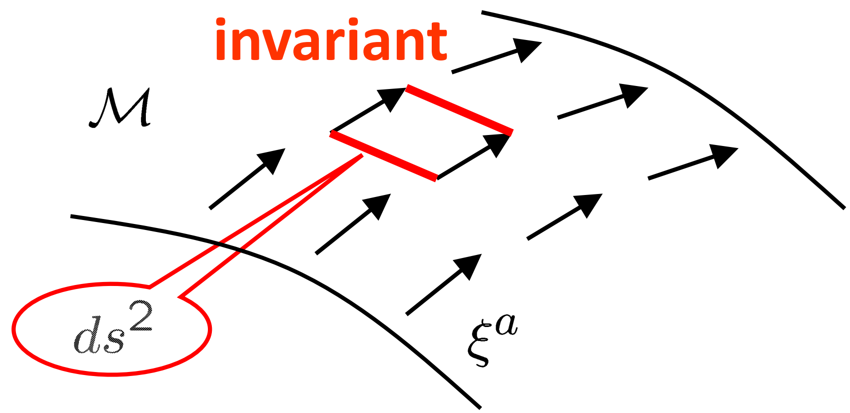
Geodesic problem and spacetime symmetry

Spacetime symmetry

Isometry

Killing equation

$$\nabla_{(a}\xi_{b)} = 0$$

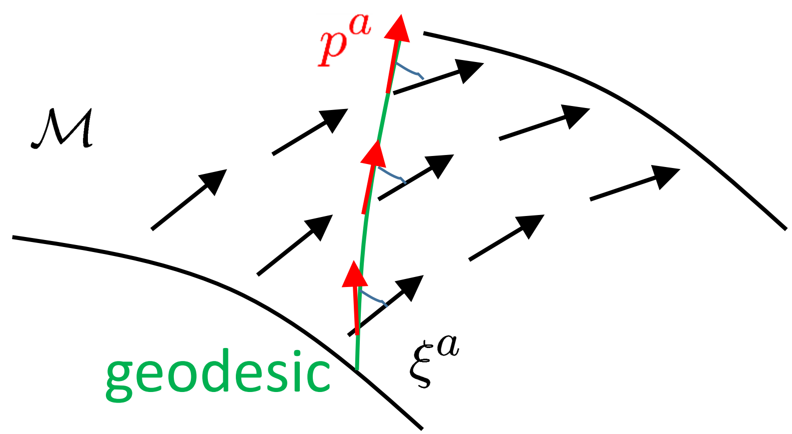


⬍

Constant along geodesics

$$F \equiv \xi_a p^a = g_{ab}\xi^a p^b$$

$$(\because \;\; p^a \nabla_a F = 0)$$



Spacetime symmetry ⬌ Conserved quantities along geodesics

For a geodesic Hamiltonian $H = g^{\mu\nu} p_\mu p_\nu$, when *F* is a *n*-order homogeneous polynomial in p_μ,

$$F = K^{a_1 \cdots a_n}(x)\, p_{a_1} \cdots p_{a_n}$$

then we find that

$$\{F, H\} = 0 \quad \Leftrightarrow \quad \nabla_{(a} K_{b_1 \cdots b_n)} = 0$$

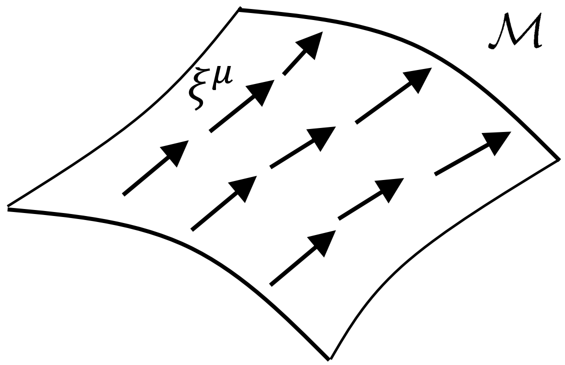
Killing-Stackel Eq.

Spacetime symmetry	↔	Conserved quantities along geodesics
Isometry (Killing vector)	↔	first-order polynomial
Hidden symmetry (Killing tensor)	↔	higher-order polynomial

Spacetime symmetry

- Killing vector fields:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$



- Killing-Stackel tensors [Stackel 1895]

$$\nabla_{(\mu} K_{\nu_1 \nu_2 \dots \nu_n)} = 0 \qquad K_{(\mu_1 \mu_2 \dots \mu_n)} = K_{\mu_1 \mu_2 \dots \mu_n}$$

- Killing-Yano tensors [Yano 1952]

$$\nabla_{(\mu} f_{\nu_1) \nu_2 \dots \nu_n} = 0 \qquad f_{[\mu_1 \mu_2 \dots \mu_n]} = f_{\mu_1 \mu_2 \dots \mu_n}$$

vector fields	Killing	Conformal Killing
symmetric	Killing-Stackel **Stackel 1895**	Conformal Killing-Stackel
anti-symmetric	Killing-Yano **Yano 1952**	Conformal Killing-Yano **Tachibana 1969, Kashiwada 1968**

Why spacetime symmetry?

- Conserved quantities along geodesics
- Integrability of EOMs for matter fields

  Klein-Gordon and Dirac equations

- Classification of spacetimes

  Stationary, axially symmetric, Bianchi type, etc.

- Application to Hamiltonian dynamics

Geometrisation

Basic idea

The dynamical trajectories of a Hamiltonian system of the form

$$H = \frac{1}{2} \sum_{i,k} g^{ik}(q) p_i p_k + U(q) \ ,$$

can be seen as geodesics of a corresponding configuration space, or of enlargement of it, under some constraints.

Examples

① Maupertuis' principle

② Canonical transformations

Ex. ②－1 3D Kepler problem

Ex. ②－2 N=3 open Toda

③ Eisenhart's lifts

Ex. ③－1 Eisenhart lift

Ex. ③－2 Generalised Eisenhart lift

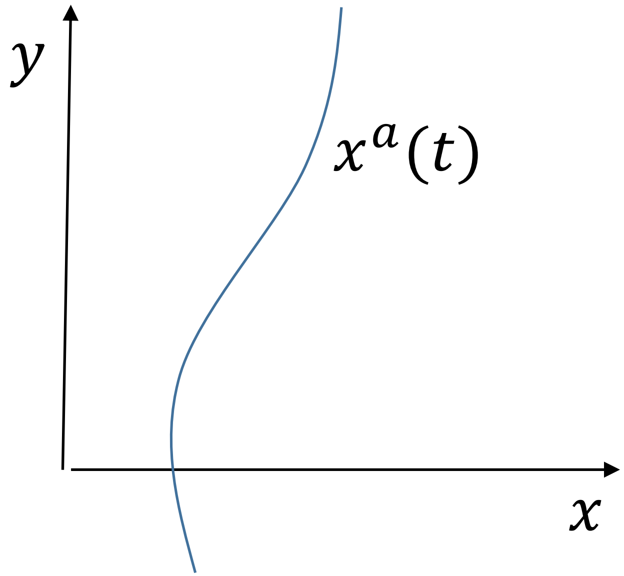
Ex. ③－3 Light-like Eisenhart lift

Maupertuis' principle

Maupertuis 1744, 1746, 1756

One obtains an integral equation that determines the path followed by a physical system without specifying the time parameterization of that path.

Newtonian



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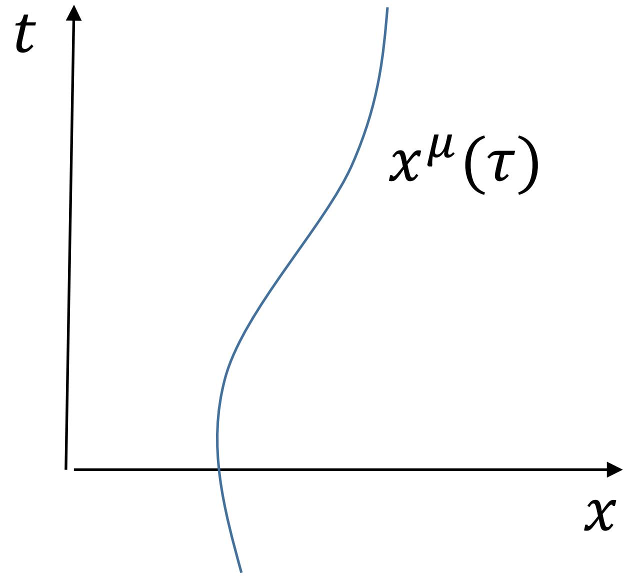


Maupertuis' principle

Maupertuis 1744, 1746, 1756

One obtains an integral equation that determines the path followed by a physical system without specifying the time parameterization of that path.

$$q = q(t), p = p(t)$$

<u>action</u> $$S = \int \left(p_i dq^i - H(q,p)dt\right)$$

↓

<u>abbreviated action</u> $$S_0(E) = \int p_i dq^i$$

Jacobi's formulation

- momentum

$$p_i \equiv \frac{\partial L}{\partial \dot{q}^i} = \sum_k g_{ik}\dot{q}^k$$

- energy

$$E = \frac{1}{2}\sum_{i,j} g_{ij}\dot{q}^i\dot{q}^j + U(q)$$

$$\therefore \ dt = \sqrt{\frac{\sum g_{ij}dq^i dq^j}{2(E-U)}}$$

Lagrangian

$$L = \frac{1}{2}\sum_{i,j} g_{ij}\dot{q}^i\dot{q}^j - U(q)$$

abbreviated action

$$S_0 \equiv \int \sum_i p_i dq^i = \int \sum_{i,k} g_{ik}\frac{dq^k}{dt}dq^i = \int \sqrt{2(E-U)\sum_{i,k} g_{ik}dq^i dq^k}$$

$$\tilde{g}_{ik} = (E-U)g_{ik}$$

$$S_0 = \int \sqrt{2\sum_{i,k} \tilde{g}_{ik}dq^i dq^k}$$

Jacobi's formulation

$$L = \frac{1}{2}\sum_{i,k} g_{ik}\dot{q}^i\dot{q}^k - U(q) \ , \ E = \frac{1}{2}\sum_{i,k} g_{ik}\dot{q}^i\dot{q}^k + U(q) \ ,$$

$$S_0 = \int \sqrt{2(E-U)\sum_{i,k} g_{ik}dq^i dq^k} \ .$$

$$\tilde{L} = \frac{1}{2}\sum_{i,k} \tilde{g}_{ik}\dot{q}^i\dot{q}^k \ , \quad \tilde{E} = \frac{1}{2}\sum_{i,k} \tilde{g}_{ik}\dot{q}^i\dot{q}^k \ ,$$

$$\tilde{S}_0 = \int \sqrt{2\tilde{E}\sum_{i,k} \tilde{g}_{ik}dq^i dq^k} \ .$$

Jacobi's formulation

Theorem *Given a dynamical system on a manifold* (M, g_{ik}) *i.e., a dynamical system whose Lagrangian is*

$$L = \frac{1}{2}\sum_{i,k} g_{ik}\dot{q}^i\dot{q}^k - U(q) \ ,$$

*then it is always possible to find a conformal transformation of the metric (**Jacobi metric**)*

$$\boxed{\tilde{g}_{ik} = (E-U)g_{ik}},$$

such that the geodesics of $(M, \tilde{g}_{ik})$ *with the energy* $\tilde{E} = 1$ *are equivalent to the trajectories of the original dynamical system.*

Comparison of Hamiltonians

Natural Hamiltonian

$$H = \frac{1}{2} g^{ik} p_i p_k + U$$

with $H = E$

$$E = H_2 + U$$

equivalent！

Jacobi's Hamiltonian

$$\widetilde{H} = \frac{1}{2} \tilde{g}^{ik} p_i p_k \quad \text{with} \quad \widetilde{H} = 1$$

$$\tilde{g}_{ik} = (E - U) g_{ik}$$

$$1 = \frac{H_2}{E - U}$$

Comparison of first integrals

- **Natural Hamiltonian**

$$H = H_2 + U$$

$$^{\exists}K \;\; s.t. \;\; \{H, K\} = 0 \; .$$

- **Jacobi's Hamiltonian**

$$\tilde{H} = \frac{H_2}{E - U}$$

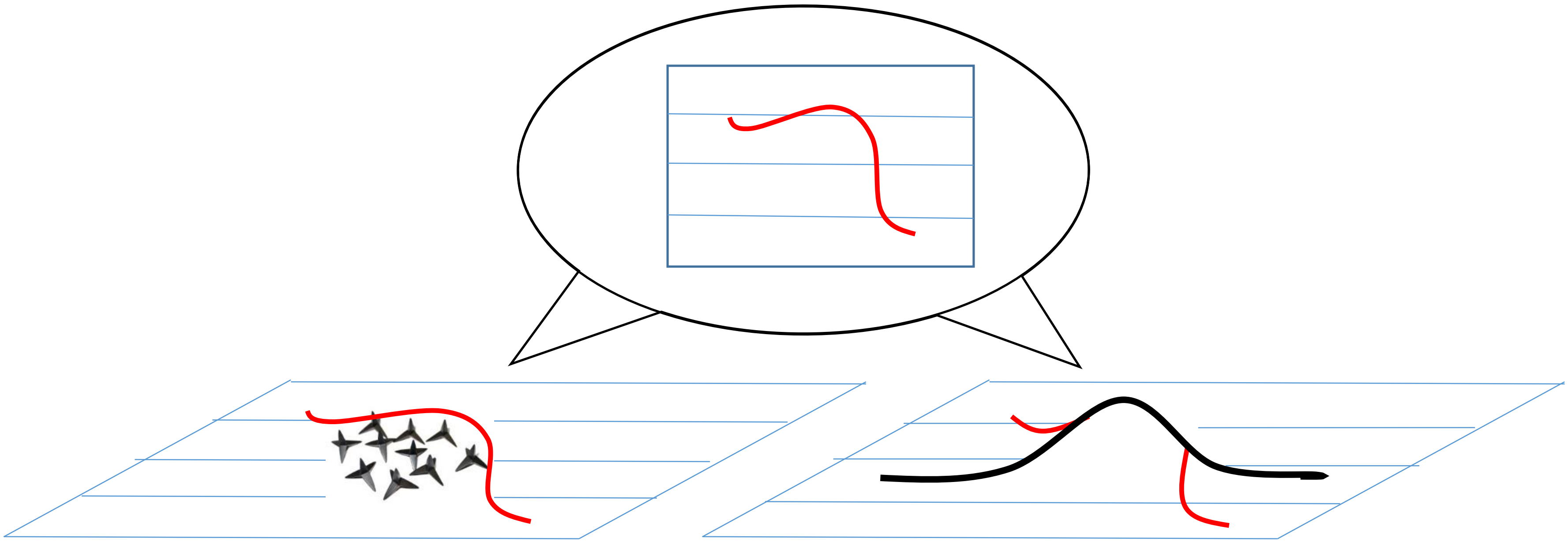
$$^{\exists}\tilde{K} \;\; s.t. \;\; \{\tilde{H}, \tilde{K}\} = 0 \; .$$

For instance,

$$K = K_2 + K_0 \; .$$

$$\tilde{K} = K_2 + K_0 \tilde{H} \; .$$

Maupertuis' principle



Potential system

Geodesic system

Natural Hamiltonian

Qudartic + potential

$$H = \frac{1}{2} g^{ik}(q) p_i p_k + U(q)$$

Geodesic Hamiltonian

Homogeneously quadratic

$$\widetilde{H} = \frac{1}{2} \tilde{g}^{\mu\nu}(\tilde{q}) \tilde{p}_\mu \tilde{p}_\nu$$

Point!

We need to construct a geodesic Hamiltonian, i.e., a homogeneously quadratic Hamiltonian which reduces to the original natural Hamiltonian under some transformation or constraints.

Examples

① Maupertuis' principle

② Canonical transformations

Ex. ②－1 3D Kepler problem

Ex. ②－2 N=3 open Toda

③ Eisenhart's lifts

Ex. ③－1 Eisenhart lift

Ex. ③－2 Generalised Eisenhart lift

Ex. ③－3 Light-like Eisenhart lift

3D Kepler problem

Keane-Barrett-Simmons 2000

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \frac{\alpha}{r} \qquad \text{with} \qquad H = E$$

$$r = \sqrt{(q^1)^2 + (q^2)^2 + (q^3)^2}$$

Transf. $\tilde{q}^i = p_i\,,\ \ \tilde{p}_i = q^i$

$$\widetilde{H} = \left(E - \frac{1}{2}\tilde{r}^2\right)^2 (\tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2) \qquad \text{with} \qquad \widetilde{H} = \alpha^2$$

$$\tilde{r} = \sqrt{(\tilde{q}^1)^2 + (\tilde{q}^2)^2 + (\tilde{q}^3)^2}$$

$$ds^2 = \left(E - \frac{1}{2}\tilde{r}^2\right)^{-2} [(d\tilde{q}^1)^2 + (d\tilde{q}^2)^2 + (d\tilde{q}^3)^2]$$

(constant curvature $-4E$)

Ex. ②-2

N=3 open Toda

Baleanu-Karasu-Makhaldiani 1999

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + a_1^2 + a_2^2$$

$$a_1 = e^{q^1 - q^2}, \quad a_2 = e^{q^2 - q^3}$$

Transf.

$$\tilde{q}^1 = q^1 + \ln p_1, \quad \tilde{q}^2 = q^2, \quad \tilde{q}^3 = q^3 - \ln p_3$$

$$\tilde{p}_1 = p_1, \quad \tilde{p}_2 = p_2, \quad \tilde{p}_3 = p_3$$

$$\widetilde{H} = \frac{1}{2}\{(1 + 2\tilde{a}_1^2)\tilde{p}_1^2 + \tilde{p}_2^2 + (1 + 2\tilde{a}_2^2)\tilde{p}_3^2\}$$

$$\tilde{a}_1 = e^{\tilde{q}^1 - \tilde{q}^2}, \quad \tilde{a}_2 = e^{\tilde{q}^2 - \tilde{q}^3}$$

$$ds^2 = \frac{(d\tilde{q}^1)^2}{1 + 2\tilde{a}_1^2} + (d\tilde{q}^2)^2 + \frac{(d\tilde{q}^3)^2}{1 + 2\tilde{a}_2^2}$$

Examples

① Maupertuis' principle

② Canonical transformations

Ex. ②−1 3D Kepler problem

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Ex. ③−1 Eisenhart lift

Ex. ③−2 Generalised Eisenhart lift

Ex. ③−3 Light-like Eisenhart lift

(Standard) Eisenhart lift

- Natural Hamiltonian on *n*-dim space *(M, g)*

$$H = \frac{1}{2} g^{ik}(q) p_i p_k + U(q)$$

- Geodesic Hamiltonian on *(n+1)*-dim space *(M x R, g_E)*

$$H_E = \frac{1}{2} g^{ik} p_i p_k + U(q) p_s^2$$
$$= \frac{1}{2} g_E^{\mu\nu} p_\mu p_\nu$$

Eisenhart metric

$$ds_E^2 = 2U(q)^{-1} ds^2 + g_{ik} dq^i dq^k$$

Generalised Eisenhart lift

- Natural Hamiltonian on *n*–dim space *(M, g)*

$$H = \frac{1}{2} g^{ik}(q) p_i p_k + U(q)\,, \qquad U(q) = \sum_{\ell=1}^{m} a_\ell U_\ell(q)$$

- Geodesic Hamiltonian on *(n+m)*-dim space *(M x R^m^, g~E~)*

$$H_E = \frac{1}{2} g^{ik} p_i p_k + \sum_{\ell=1}^{m} U_\ell(q)\, p_{s^\ell}^2$$
$$= \frac{1}{2} g_E^{\mu\nu} p_\mu p_\nu$$

Eisenhart metric

$$ds_E^2 = 2U_\ell(q)^{-1}(ds^\ell)^2 + g_{ik} dq^i dq^k$$

Light-like Eisenhart lift

- Natural Hamiltonian on *n*–dim space *(M, g)*

$$H = \frac{1}{2} g^{ik}(q) p_i p_k + U(q)$$

- Geodesic Hamiltonian on *(n+m)*-dim spacetime *(M x R^m, g_E)*

$$H_E = \frac{1}{2} g^{ik} p_i p_k + U(q) p_s^2 + p_s p_t$$

$$= \frac{1}{2} g_E^{\mu\nu} p_\mu p_\nu$$

Eisenhart metric

$$ds_E^2 = -2U(q)dt^2 + 2dt\, ds + g_{ik} dq^i dq^k$$

Comparison

Natural Htn v.s. LL Eisenhart's Htn

- **Natural Hamiltonian**

$$H = H_2 + U$$

$$^{\exists}K \ \ s.t. \ \{H, K\} = 0 \ .$$

For instance,

$$\boxed{K = K_2 + K_0 \ .}$$

- **Eisenhart's Hamiltonian**

$$H_E = H_2 + U p_s^2 + p_s p_t$$

$$^{\exists}K' \ \ s.t. \ \{H', K'\} = 0 \ .$$

$$\boxed{K' = K_2 + K_0 p_s^2 \ .}$$

Prolongation

Review I: Integrability conditions for systems of first order PDEs

A system of first order PDEs

$$\frac{\partial u^\alpha}{\partial x^i} = \psi^\alpha_{i\beta}(x) u^\beta$$

$u = (u^1, u^2, \cdots, u^N)$; unknown functions

$x = (x^1, x^2, \cdots, x^n)$; variables

Questions :

Does the solution exist?

at most N dimensions

If exist, is the solution space finite or infinite? How many dimensions?

Explicit expressions?

Consistency conditions

$$\frac{\partial}{\partial x^j}\frac{\partial u^\alpha}{\partial x^i} - \frac{\partial}{\partial x^i}\frac{\partial u^\alpha}{\partial x^j} = 0$$

$$\frac{\partial}{\partial x^j}\frac{\partial u^\alpha}{\partial x^i} = \frac{\partial \psi^\alpha_{i\beta}}{\partial x^j}u^\beta + \psi^\alpha_{i\beta}\frac{\partial u^\beta}{\partial x^j} = \frac{\partial \psi^\alpha_{i\beta}}{\partial x^j}u^\beta + \psi^\alpha_{i\beta}\psi^\beta_{j\gamma}u^\gamma$$

$$\frac{\partial}{\partial x^i}\frac{\partial u^\alpha}{\partial x^j} = \frac{\partial \psi^\alpha_{j\beta}}{\partial x^i}u^\beta + \psi^\alpha_{j\beta}\frac{\partial u^\beta}{\partial x^i} = \frac{\partial \psi^\alpha_{j\beta}}{\partial x^i}u^\beta + \psi^\alpha_{j\beta}\psi^\beta_{i\gamma}u^\gamma$$

$$\left(\frac{\partial \psi^\alpha_{i\gamma}}{\partial x^j} - \frac{\partial \psi^\alpha_{j\gamma}}{\partial x^i} + \psi^\alpha_{i\beta}\psi^\beta_{j\gamma} - \psi^\alpha_{j\beta}\psi^\beta_{i\gamma}\right)u^\gamma = 0$$

Frobenius' theorem

The necessary and sufficient conditions for the unique solution $u^{\alpha} = u^{\alpha}(x)$ to the system

$$\frac{\partial u^{\alpha}}{\partial x^{i}} = \psi^{\alpha}_{i\beta} u^{\beta}$$

such that $u(x_0) = u_0$ to exist for any initial data (x_0, u_0) is that the relation

$$\frac{\partial \psi^{\alpha}_{i\gamma}}{\partial x^{j}} - \frac{\partial \psi^{\alpha}_{j\gamma}}{\partial x^{i}} + \psi^{\alpha}_{i\beta}\psi^{\beta}_{j\gamma} - \psi^{\alpha}_{j\beta}\psi^{\beta}_{i\gamma} = 0$$

hold.

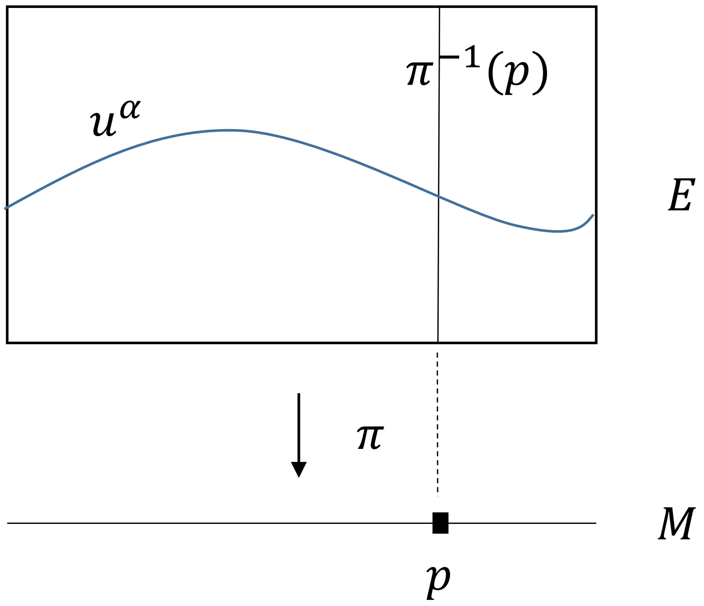
Parallel equation

$$\frac{\partial u^\alpha}{\partial x^i} = \psi^\alpha_{i\beta}(x) u^\beta$$

$$\Leftrightarrow \quad \frac{\partial u^\alpha}{\partial x^i} - \psi^\alpha_{i\beta}(x) u^\beta = 0$$

$$\Leftrightarrow \quad D_i u^\alpha = 0$$

where $D_i u^\alpha := \dfrac{\partial u^\alpha}{\partial x^i} - \psi^\alpha_{i\beta}(x) u^\beta$



The system can be viewed as a parallel equation for sections u^α of a vector bundle $\pi: E \to M$ of rank N.

Curvature conditions

For a connection D_i

$$D_i u^\alpha := \frac{\partial u^\alpha}{\partial x^i} - \psi^\alpha_{i\beta}(x) u^\beta$$

the curvature of D_i is defined by $(D_i D_j - D_j D_i) u^\alpha = -R_{ij\beta}{}^\alpha u^\beta$.

$$D_i u^\alpha = 0 \Rightarrow \boxed{R_{ij\beta}{}^\alpha u^\beta = 0}$$

This is equivalent to the Frobenius integrability condition

Frobenius' theorem (II)

The necessary and sufficient conditions for the unique solution $u^\alpha = u^\alpha(x)$ to the system

$$D_i u^\alpha = 0 \qquad i = 1, \cdots, n \quad \alpha = 1, \cdots, N$$

where

$$D_i u^\alpha := \frac{\partial u^\alpha}{\partial x^i} - \psi^\alpha_{i\beta}(x) u^\beta$$

such that $u(x_0) = u_0$ to exist for any initial data (x_0, u_0) is that the relation

$$R_{ij\beta}{}^\alpha u^\beta = 0$$

hold.

Discussion

- If the curvature conditions hold, the general solution depends on N arbitrary constants.

$$u^\alpha(x; a_i) = a_1 u_1^\alpha(x) + a_2 u_2^\alpha(x) + \cdots + a_N u_N^\alpha(x)$$

- If not, they give a set of algebraic equations

$$R_{ij\beta}{}^\alpha u^\beta = 0$$

- Differentiating these equations and eliminating the derivatives of u^α leads to a new set of equations

$$(D_k R_{ij\beta}{}^\alpha) u^\beta = 0$$

$$D_k F_{ij\beta}{}^\alpha := \partial_k F_{ij\beta}{}^\alpha - \psi_{k\gamma}^\alpha F_{ij\beta}{}^\gamma + F_{ij\gamma}{}^\alpha \psi_{k\beta}^\gamma$$

Discussion

- Proceeding in this way we get a sequence of sets of equations

$$R_{ij\beta}{}^{\alpha}u^{\beta} = 0, \quad (D_k R_{ij\beta}{}^{\alpha})u^{\beta} = 0 \ , \quad (D_\ell D_k R_{ij\beta}{}^{\alpha})u^{\beta} = 0, \quad \cdots$$

- If p is the number of independent equations in the first K sets, then the general solution depends on $N - p$ arbitrary constants.

Review II:
Prolongation of PDEs

Prolongation

$$F(x, f, \partial f, \partial\partial f, \cdots) = 0$$

↓

$$\frac{\partial u^\alpha}{\partial x^i} = \psi^\alpha_{i\beta} u^\beta$$

$$i = 1, \cdots, n \quad \alpha = 1, \cdots, N$$

Example 1

Introduce $w = u_y - v_x$

$$u_x = au + bv$$
$$u_y + v_x = cu + dv$$
$$v_y = eu + fv$$

→

$$u_x = au + bv$$
$$u_y = \frac{1}{2}(cu + dv + w)$$
$$v_x = \frac{1}{2}(cu + dv - w)$$
$$v_y = eu + fv$$
$$w_x = w_x(u, v, w)$$
$$w_y = w_y(u, v, w)$$

Example 2: Cauchy-Riemann equation

$$u_x = v_y$$

$$u_y = -v_x$$

Impossible to make a prolongation!

In fact, solution of this system depends on one holomorphic function.

Prolongation

$$F(x, f, \partial f, \partial\partial f, \cdots) = 0$$

↓

Not always possible

When can we make a prolongation successfully?

$$\frac{\partial u^\alpha}{\partial x^i} = \psi_i^\alpha(x, u)$$

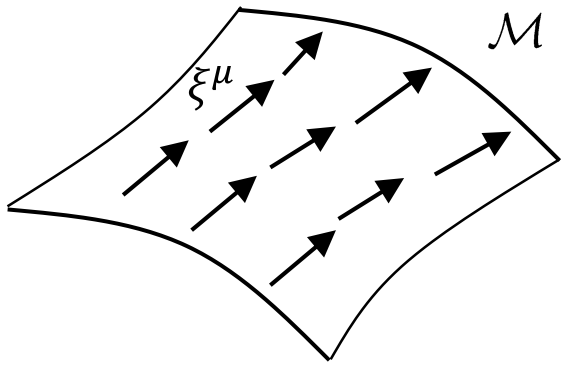
$$i = 1, \cdots, n \quad \alpha = 1, \cdots, N$$

Prolongation of Killing equations

Spacetime symmetry

- Killing vector fields:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$



- Killing-Stackel tensors [Stackel 1895]

$$\nabla_{(\mu} K_{\nu_1 \nu_2 \dots \nu_n)} = 0 \qquad K_{(\mu_1 \mu_2 \dots \mu_n)} = K_{\mu_1 \mu_2 \dots \mu_n}$$

- Killing-Yano tensors [Yano 1952]

$$\nabla_{(\mu} f_{\nu_1) \nu_2 \dots \nu_n} = 0 \qquad f_{[\mu_1 \mu_2 \dots \mu_n]} = f_{\mu_1 \mu_2 \dots \mu_n}$$

Killing vectors

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$

Killing vector equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$

$$\nabla_\mu \xi_\nu = L_{\mu\nu}\,, \quad L_{\mu\nu} = \nabla_{[\mu}\xi_{\nu]}$$

$$\nabla_\mu L_{\nu\rho} = -R_{\nu\rho\mu}{}^{\sigma}\xi_\sigma$$

$$\nabla_\mu \xi_\nu = L_{\mu\nu}\,, \qquad L_{\mu\nu} = L_{[\mu\nu]}$$
$$\nabla_\mu L_{\nu\rho} = -R_{\nu\rho\mu}{}^{\sigma}\xi_\sigma$$

$$\nabla_\mu \begin{pmatrix} \xi_\nu \\ L_{\nu\rho} \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -R_{\nu\rho\mu}{}^{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \xi_\sigma \\ L_{\mu\nu} \end{pmatrix} = 0$$

$$D_\mu \hat{\xi}_A = 0$$

- $\hat{\xi}_A = (\xi_\mu, L_{\mu\nu})$: a section of $\Lambda^1(M) \oplus \Lambda^2(M)$
- D_μ : connection on $\Lambda^1(M) \oplus \Lambda^2(M)$

$$D_\mu \hat{\xi}_A \equiv \nabla_\mu \begin{pmatrix} \xi_\nu \\ L_{\nu\rho} \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -R_{\nu\rho\mu}{}^{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \xi_\sigma \\ L_{\mu\nu} \end{pmatrix}$$

Point ① Prolongation

Killing vectors ⟺ parallel sections of $\Lambda^1(M) \oplus \Lambda^2(M)$

$$\xi^\mu \Rightarrow \hat{\xi}_A = \begin{pmatrix} \xi_\mu \\ \nabla_{[\mu}\xi_{\nu]} \end{pmatrix} \quad s.t. \quad D_\mu \hat{\xi}_A = 0$$

Parallel equation

The number of linearly independent sections of $\Lambda^1(M) \oplus \Lambda^2(M)$ is bound by the rank of the vector bundle.

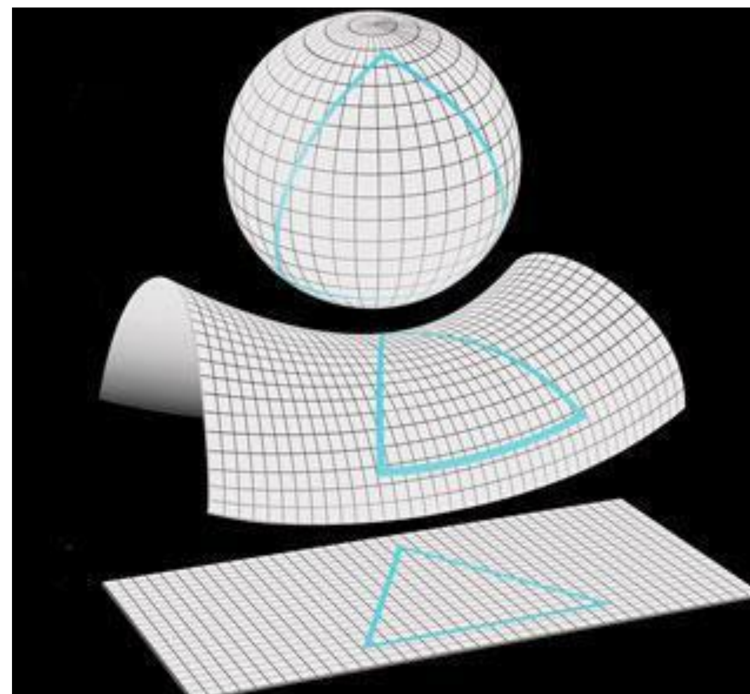
$$N = \binom{n}{1} + \binom{n}{2} = \frac{n(n+1)}{2}$$

Maximally symmetric spaces

Spaces that have the maximum number of KVs

⇔ constant curvature spaces

n	$N = \frac{n(n+1)}{2}$
2	3
3	6
4	10
5	15
…	…



Point ② Curvature conditions

# of parallel sections ＝ rank of Ep － # of curv. cond.

$$D_\mu \hat{\xi}_A = 0 \quad \Leftrightarrow \quad \begin{aligned} &[D_\mu, D_\nu]\hat{\xi}_A = 0 \\ &\left[D_\mu, [D_\nu, D_\rho]\right]\hat{\xi}_A = 0 \\ &\left[D_\mu, \left[D_\nu, [D_\rho, D_\sigma]\right]\right]\hat{\xi}_A = 0 \\ &\qquad\cdots \end{aligned}$$

Killing-Yano tensors

$$\nabla_{(\mu} f_{\nu_1)\nu_2 \ldots \nu_n} = 0$$

$$f_{[\mu_1 \mu_2 \ldots \mu_n]} = f_{\mu_1 \mu_2 \ldots \mu_n}$$

KY tensor equation

$$\nabla_{(\mu} f_{\nu)\rho} = 0 \qquad f_{\mu\nu} = -f_{\nu\mu}$$

$$\nabla_\mu f_{\nu\rho} = \nabla_{[\mu} f_{\nu\rho]}$$

$$\nabla_\mu \left(\nabla_{[\nu} f_{\rho\sigma]} \right) = -R_{[\nu\rho|\mu}{}^{\alpha} f_{\alpha|\sigma]}$$

Rank-2

$$\nabla_\mu f_{\nu\rho} = \nabla_{[\mu} f_{\nu\rho]}$$

$$\nabla_\mu(\nabla_{[\nu} f_{\rho\sigma]}) = -R_{[\nu\rho|\mu}{}^{\alpha} f_{\alpha|\sigma]}$$

Rank-p

$$\nabla_\mu f_{\nu_1 \cdots \nu_p} = \nabla_{[\mu} f_{\nu_1 \cdots \nu_p]}$$

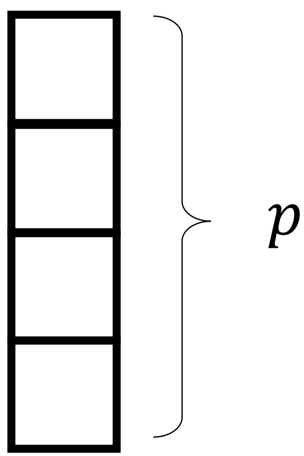
$$\nabla_\mu(\nabla_{[\nu} f_{\rho_1 \cdots \rho_p]}) = -R_{[\nu\rho_1|\mu}{}^{\alpha} f_{\alpha|\rho_2 \cdots \rho_p]}$$

Prolongation of KY tensors

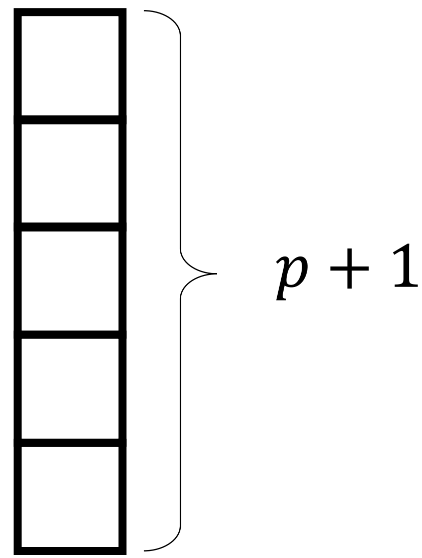
rank-p KY tensors $\Longleftrightarrow$ parallel sections of E^p

$$E^p = \Lambda^p(M) \oplus \Lambda^{p+1}(M)$$

$=$



$\oplus$



$$rank(E^p) = \binom{n+1}{p+1}$$

The number of KY tensors in maximally symmetric spaces

$$N = \binom{n+1}{p+1}$$

Semmelmann 2002

	p=1	p=2	p=3	p=4
2D	3			
3D	6	4		
4D	10	10	5	
5D	15	20	15	6

Examples in four dimensions

TH-Yasui 2014

4D metrics	$p = 1$	$p = 2$	$p = 3$
Maximally symmetric	10	10	5
Plebanski-Demianski	2	0	0
Kerr	2	1	0
Schwazschild	4	1	0
FLRW	6	4	1
Self-dual Taub-NUT	4	4	0
Eguchi-Hanson	4	3	0

Examples in five dimensions

TH-Yasui 2014

5D metrics	$p = 1$	$p = 2$	$p = 3$	$p = 4$
Maximally symmetric	15	20	15	6
Myers-Perry	3	0	1	0
Emparan-Reall	3	0	0	0
Kerr string	3	1	0	1

Killing-Stackel tensors

$$\nabla_{(\mu} K_{\nu_1 \nu_2 \dots \nu_n)} = 0$$

$$K_{(\mu_1 \mu_2 \dots \mu_n)} = K_{\mu_1 \mu_2 \dots \mu_n}$$

$$\nabla_{(\mu} K_{\nu\rho)} = 0 \qquad K_{\mu\nu} = K_{\nu\mu}$$

$$\nabla_\mu K_{\nu\rho} = \frac{2}{3}\left(\nabla_{[\mu} K_{\nu]\rho} + \nabla_{[\mu} K_{\rho]\nu}\right)$$

$$\nabla_\mu\left(\nabla_{[\nu} K_{\rho]\sigma}\right) = -R_{\nu\rho(\mu}{}^{\alpha} K_{\alpha|\sigma)} - R_{(\mu|[\nu\rho]}{}^{\alpha} K_{\alpha|\sigma)}$$
$$- \frac{1}{4} R_{\nu\rho[\mu}{}^{\alpha} K_{\alpha|\sigma]} - \frac{1}{2} R_{(\mu|[\nu\rho]}{}^{\alpha} K_{\alpha|\sigma)} + \phi_{[\mu|[\nu\rho]|\sigma]}$$

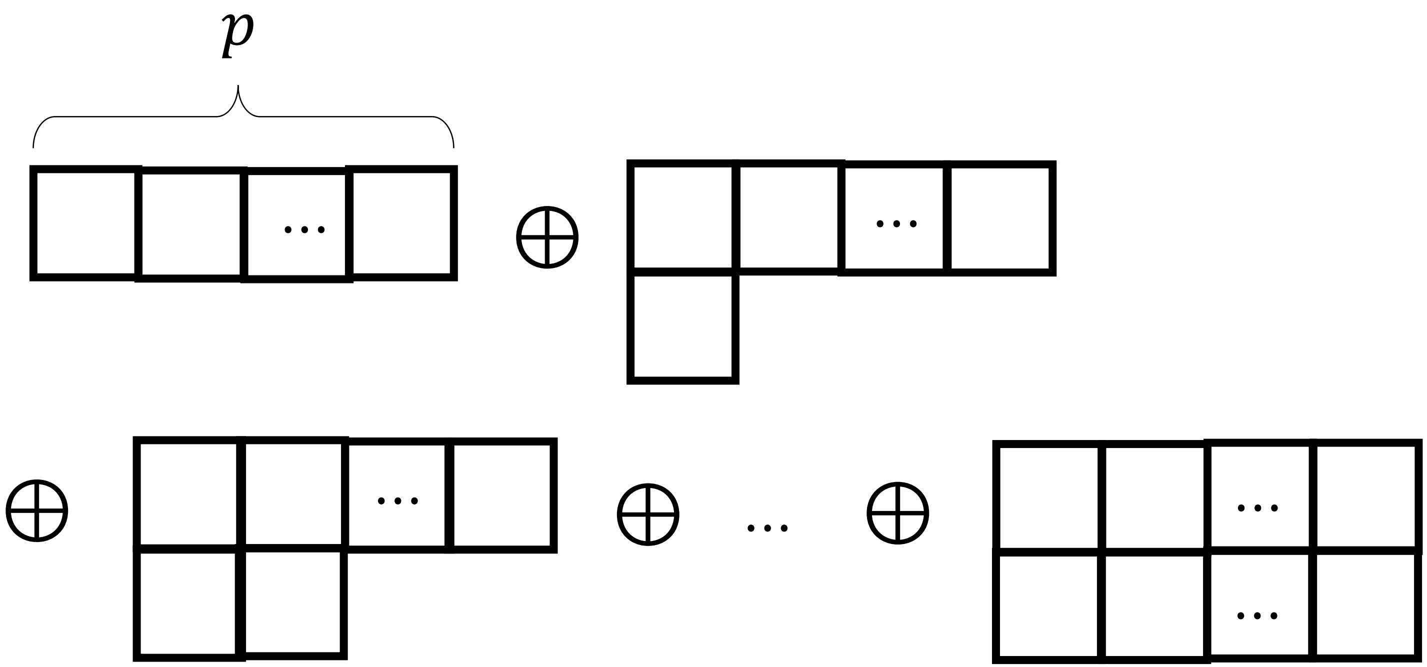
where $\phi_{\mu\nu\rho\sigma} \equiv \nabla_{(\mu}\nabla_{\nu)} K_{\rho\sigma}$

$$\nabla_\mu\left(\phi_{[\nu|[\rho\sigma]|\kappa]}\right) = (R_1 \cdot K_{**})_{\mu\nu\rho\sigma\kappa} + \left(R_2 \cdot \nabla_{[*} K_{*]*}\right)_{\mu\nu\rho\sigma\kappa}$$

Prolongation of KS tensors

rank-p KS tensors $\Longleftrightarrow$ parallel sections of E^p

$E^p =$



$$rank(E^p) = \frac{1}{n}\binom{n+p}{p+1}\binom{n+p-1}{p}$$

The number of KS tensors in maximally symmetric spaces

$$N = \frac{1}{n}\binom{n+p}{p+1}\binom{n+p-1}{p}$$

Barbance 1973, Michel et al 2012

	p=1	p=2	p=3	p=4	
2D	3	6	10	15	⋯
3D	6	20	50	105	⋯
4D	10	50	175	490	⋯
5D	15	105	490	1764	⋯

On-going tasks

- Analysis of curvature conditions

  - Compute the curvature conditions

  - Construct the package of Mathematica which compute and solve the curvature conditions

  - Investigate the curvature conditions for various metrics

Conjecture **No non-trivial quadratic constant for geodesic motion in the Kerr spacetime exists, with the exception of Carter constant.**

Foresight into the future

- **CKY and CKS**

  Cotton tensor, Bach tensor, Q-curvature, conformal geometry

- **PDE theory**

  Prolongation

- **Differential geometry**

  Generalised gradients, Weitzenbock formula, twisted Dirac

- **Hamiltonian dynamics**

  Integrable systems, Chaos, Lax pairs, Painleve systems

- **GR, SUGRA, ...**

  Exact solutions, strings, branes