# Recurrence Relations for Finite-Temperature Correlators via $AdS_2/CFT_1$

Satoshi Ohya
*Institute of Quantum Science, Nihon University*

Based on:
SO, `arXiv:1309.2939[hep-th]`

# Plan of the talk

# Introduction and Overview

- In $d$-dimensional Minkowski spacetime, $SO(2,d)$ global conformal transformations consist of the followings:

$$\begin{aligned} &\text{(translation)} && x^\mu \longmapsto x'^\mu = x^\mu + a^\mu \\ &\text{(Lorentz transformation)} && x^\mu \longmapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu \\ &\text{(dilatation)} && x^\mu \longmapsto x'^\mu = \lambda x^\mu \\ &\text{(conformal boost)} && x^\mu \longmapsto x'^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2bx + b^2 x^2} \end{aligned}$$

- Basic elements of conformal field theory (CFT) are (quasi-)primary operators $\mathcal{O}_\Delta(x)$, which transform under the $SO(2,d)$ global conformal transformations $x^\mu \longmapsto x'^\mu$ as follows:

$$\mathcal{O}_\Delta(x) \longmapsto \mathcal{O}'_\Delta(x) = \left|\frac{\partial x'}{\partial x}\right|^{\Delta/d} \mathcal{O}_\Delta(x')$$

where $|\partial x'/\partial x|$ stands for the Jacobian of the global conformal transformation. $\Delta$ is called the conformal weight of $\mathcal{O}_\Delta(x)$.

- Let $|0\rangle$ be the conformally invariant vacuum. Then the $n$-point function for the (quasi-)primary operators satisfies the identity

$$\langle 0|\mathcal{O}_{\Delta_1}(x_1)\cdots\mathcal{O}_{\Delta_n}(x_n)|0\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d}\cdots\left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{\Delta_n/d}\langle 0|\mathcal{O}_{\Delta_1}(x'_1)\cdots\mathcal{O}_{\Delta_n}(x'_n)|0\rangle$$

- It has been long known that the $SO(2,d)$ global conformal symmetry completely fixes the possible forms of two- and three-point functions in any spacetime dimension $d$ [Polyakov '70]. Indeed, up to overall normalization factors, they can be determined as follows:

$$\langle 0|\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)|0\rangle = \delta_{\Delta_1\Delta_2}\frac{C_{\Delta_1\Delta_2}}{|x_1-x_2|^{\Delta_1+\Delta_2}}$$

$$\langle 0|\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)|0\rangle = \frac{C_{\Delta_1\Delta_2\Delta_3}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_3-x_1|^{\Delta_3+\Delta_1-\Delta_2}}$$

- Conformal constraints work well in coordinate space; however, they tell us little about momentum-space correlators before performing Fourier transform.

- Indeed, in spite of its simplicity in coordinate space, three-point functions in momentum space are known to be very complicated.

  zero-temperature $CFT_d$: [Corianò-Delle Rose-Mottola-Serino '13] [Bzowski-McFadden-Skenderis '13]

  finite-temperature $CFT_2$: [Becker-Cabrera-Su '14]

- It is desirable to understand how conformal symmetry restricts the possible forms of momentum-space correlators. Here are the reasons:
  - Momentum-space correlators are directly related to physical observables (such as spectral density);
  - Fourier transform of position-space correlators is very hard.

- Today I am going to present a small example that, by using the AdS/CFT correspondence, finite-temperature $CFT_1$ two-point functions in frequency space can also be determined by conformal symmetry.

- **Scope of the talk**: Klein-Gordon equation on charged $AdS_2$ black hole

$$(\Box_{AdS_2} - m^2)\Phi(T,X) = 0$$

and the asymptotic near-boundary behavior of general solution

$$\Phi(T,X) \sim A_\Delta(\omega) X^\Delta e^{-i\omega T} + B_\Delta(\omega) X^{1-\Delta} e^{-i\omega T} \quad \text{as} \quad X \to 0$$

- According to the real-time prescription of AdS/CFT correspondence, the ratio

$$G_\Delta(\omega) = (2\Delta - 1)\frac{A_\Delta(\omega)}{B_\Delta(\omega)}$$

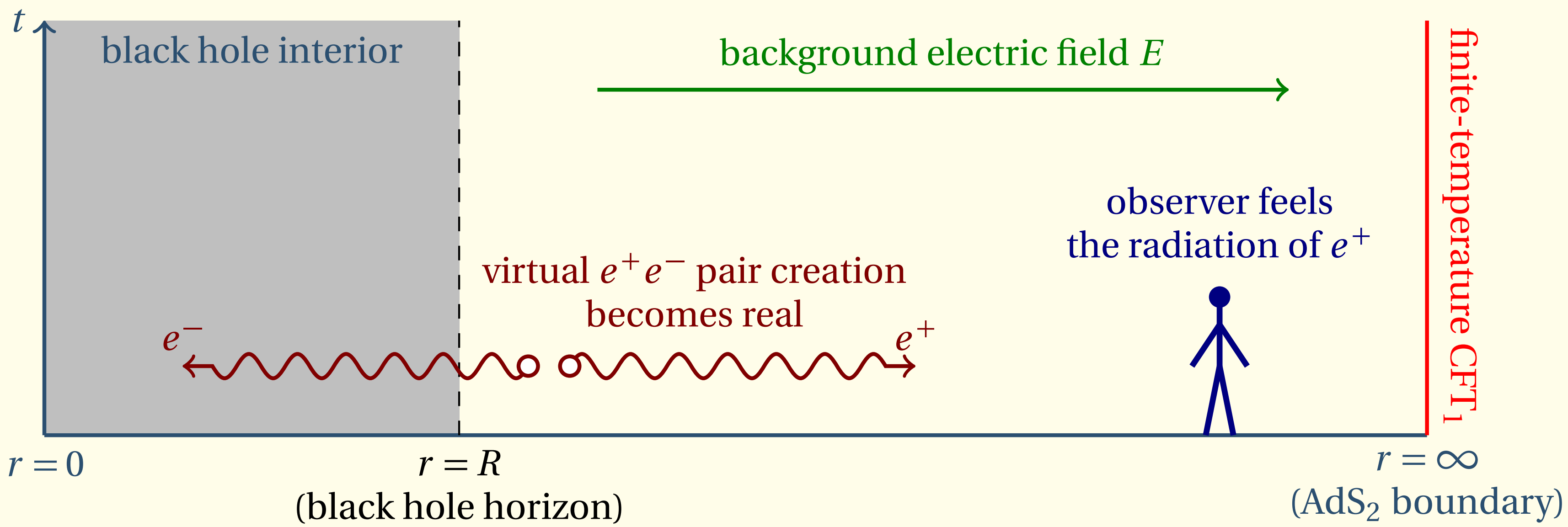
gives the frequency-space two-point function for a charged scalar operator of conformal weight $\Delta$ in dual finite-temperature $CFT_1$ **[Iqbal-Liu '09]**.

- **My small findings:** The ratio $A_\Delta(\omega)/B_\Delta(\omega)$ can be computed without solving the Klein-Gordon equation: it is determined only through symmetry of the $AdS_2$ black hole.

## $AdS_2$ in a Nutshell

- Charged $AdS_2$ black hole is a *locally* $AdS_2$ spacetime and described by the following metric and gauge field:

$$ds^2_{AdS_2} = -\left(\left(\frac{r}{R}\right)^2 - 1\right) dt^2 + \frac{dr^2}{(r/R)^2 - 1} \quad \text{and} \quad A = -E(r-R)dt$$



- Charged $AdS_2$ black hole is a *locally* $AdS_2$ spacetime and described by the following metric and gauge field:

$$ds^2_{AdS_2} = -\left(\left(\frac{r}{R}\right)^2 - 1\right)dt^2 + \frac{dr^2}{(r/R)^2 - 1} \quad \text{and} \quad A = -E(r-R)dt$$

- It is known that any black holes have an $AdS_2$ factor in the near-horizon limit [**$AdS_2$ structure theorem** [Kunduri-Lucietti-Reall '07]]. In this sense the $AdS_2$ black hole plays a special role in black hole physics and holography.

- For the following discussions it is convenient to introduce a new coordinate system $(t, x)$ via

$$r = R\coth(x/R), \quad x \in (0, \infty)$$

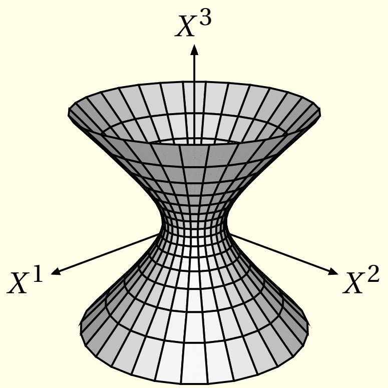
in which the metric and gauge field become

$$ds^2_{AdS_2} = \frac{-dt^2 + dx^2}{\sinh^2(x/R)} \quad \text{and} \quad A = -ER(\coth(x/R) - 1)dt$$

- **Below I will work in the units $R = 1$.**

- $AdS_2$ spacetime is a 2d spacetime with negative constant curvature $-1$. It is embedded into the ambient space $\mathbb{R}^{2,1}$ and defined as the 2d hyperboloid

$$\mathrm{AdS}_2 = \left\{ (X^1, X^2, X^3) \in \mathbb{R}^{2,1} : -(X^1)^2 - (X^2)^2 + (X^3)^2 = -1 \right\}$$




- $AdS_2$ metric is then given by the following induced metric

$$ds^2_{\mathrm{AdS}_2} = -(dX^1)^2 - (dX^2)^2 + (dX^3)^2 \Big|_{-(X^1)^2-(X^2)^2+(X^3)^2=-1}$$

- The hyperboloid is obviously invariant under the $SO(2,1)$ transformations

$$X^\mu \mapsto X'^\mu = \Lambda^\mu{}_\nu X^\nu, \quad \Lambda \in SO(2,1) \cong SL(2,\mathbb{R})/\mathbb{Z}_2$$

**isometry group of $AdS_2$ = $SO(2,1)(\cong SL(2,\mathbb{R})/\mathbb{Z}_2)$**

- The Lie algebra $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$ of the Lie group $SO(2,1) \cong SL(2,\mathbb{R})/\mathbb{Z}_2$ is spanned by the following traceless matrices:

$$iJ_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad iJ_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad iJ_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which satisfy the commutation relations

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = -iJ_1, \quad [J_3, J_1] = -iJ_2$$

- These generators generate the following one-parameter subgroups:

$$\exp(i\epsilon J_1) = \begin{pmatrix} \cosh\frac{\epsilon}{2} & \sinh\frac{\epsilon}{2} \\ \sinh\frac{\epsilon}{2} & \cosh\frac{\epsilon}{2} \end{pmatrix} \in SO(1,1)$$

$$\exp(i\epsilon J_2) = \begin{pmatrix} \mathrm{e}^{\epsilon/2} & 0 \\ 0 & \mathrm{e}^{-\epsilon/2} \end{pmatrix} \in SO(1,1)$$

$$\exp(i\epsilon J_3) = \begin{pmatrix} \cos\frac{\epsilon}{2} & \sin\frac{\epsilon}{2} \\ -\sin\frac{\epsilon}{2} & \cos\frac{\epsilon}{2} \end{pmatrix} \in SO(2)$$

- The linear combinations

$$iH = i(J_1 + J_3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad iD = iJ_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad iK = i(J_1 - J_3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfy the $(0+1)$-dimensional conformal algebra

$$[H, D] = iH, \quad [D, K] = iK, \quad [K, H] = 2iD$$

- These generators generate the following one-parameter subgroups:

$$\exp(i\epsilon H) = \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix} \in E(1)$$

$$\exp(i\epsilon D) = \begin{pmatrix} e^{\epsilon/2} & 0 \\ 0 & e^{-\epsilon/2} \end{pmatrix} \in SO(1,1)$$

$$\exp(i\epsilon K) = \begin{pmatrix} 1 & 0 \\ \epsilon & 1 \end{pmatrix} \in E(1)$$

## $SO(2)$, $E(1)$ and $SO(1,1)$ ①

- It is known that the Lie group $SO(2,1)$ contains only three distinct one-parameter subgroups: compact rotation group $SO(2)$, noncompact Euclidean group $E(1)$ and noncompact Lorentz group $SO(1,1)$.
- Correspondingly, there exist three distinct classes of coordinates patches on $AdS_2$ where time-translation generators generate the one-parameter subgroups $SO(2)$, $E(1)$ and $SO(1,1)$.
- I shall show that the time-translation generator of $AdS_2$ black hole generates the one-parameter subgroup $SO(1,1) \subset SO(2,1)$, which, after Wick rotation, becomes the compact rotation group $SO(2)$ and hence leads to quantized Matsubara frequencies conjugate to the imaginary time.

Here are the three distinct coordinate patches on the 2d hyperboloid $-(X^1)^2-(X^2)^2+(X^3)^2=-1$:

1. **Global coordinates**: $SO(2)$ diagonal basis[1]

$$(X^1,X^2,X^3)=\left(\frac{\sin\tau}{\sin\sigma},\frac{\cos\tau}{\sin\sigma},\frac{\cos\sigma}{\sin\sigma}\right),\quad ds^2_{\text{global}}=\frac{-d\tau^2+d\sigma^2}{\sin^2\sigma}$$

2. **Poincaré coordinates**: $E(1)$ diagonal basis

$$(X^1,X^2,X^3)=\left(\frac{t}{x},\frac{1-t^2+x^2}{2x},\frac{1+t^2-x^2}{2x}\right),\quad ds^2_{\text{Poincaré}}=\frac{-dt^2+dx^2}{x^2}$$

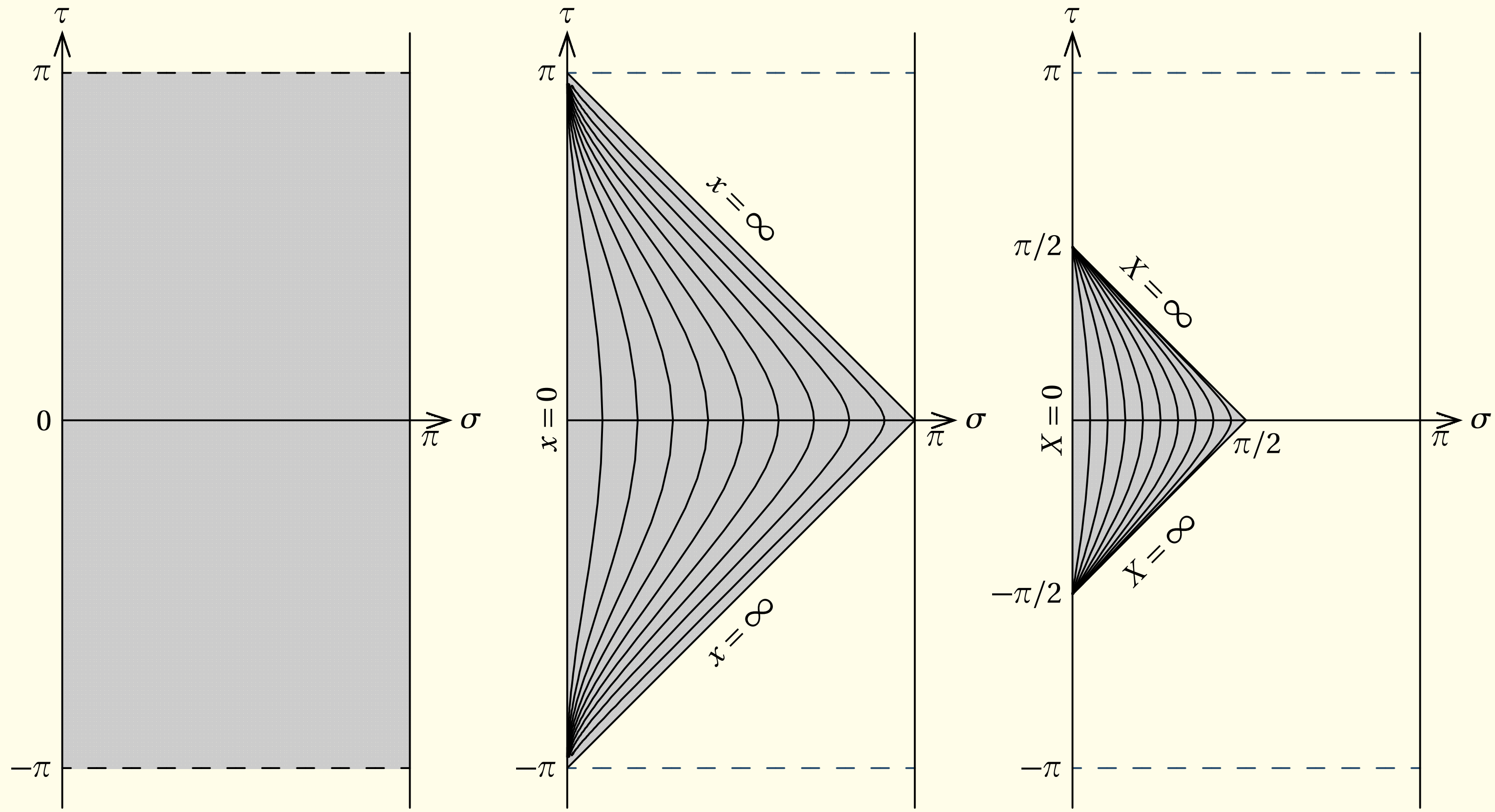
3. **Schwarzschild coordinates**: $SO(1,1)$ diagonal basis

$$(X^1,X^2,X^3)=\left(\frac{\sinh T}{\sinh X},\frac{\cosh X}{\sinh X},\frac{\cosh T}{\sinh X}\right),\quad ds^2_{\text{Schwarzschild}}=\frac{-dT^2+dX^2}{\sinh^2 X}$$

---

[1] $\tau$ ranges from $-\infty$ to $+\infty$ for the covering space of $AdS_2$ without identification $\tau\sim\tau+2\pi$.

- The global coordinate patch covers the whole hyperboloid, whereas the Poincaré and Schwarzschild coordinate patches cover only 1/2 and 1/8 of the hyperboloid, respectively:




(a) Global (b) Poincaré (c) Schwarzschild

1. **Global coordinates**. The line element

$$ds^2_{\text{global}} = \frac{-d\tau^2 + d\sigma^2}{\sin^2\sigma}$$

is invariant under the following $SL(2,\mathbb{R})/\mathbb{Z}_2 \cong SO(2,1)$ transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \sigma^{\pm} \mapsto \sigma'^{\pm} = 2\arctan\left(\frac{a\tan\frac{\sigma^{\pm}}{2} + b}{c\tan\frac{\sigma^{\pm}}{2} + d}\right), \quad \sigma^{\pm} := \tau \pm \sigma$$

where $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$ and $(a, b, c, d) \sim (-a, -b, -c, -d)$. It is easy to check that the $SO(2)$ transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\frac{\epsilon}{2} & \sin\frac{\epsilon}{2} \\ -\sin\frac{\epsilon}{2} & \cos\frac{\epsilon}{2} \end{pmatrix} \in SO(2)$$

induces the translation $\sigma'^{\pm} = \sigma^{\pm} + \epsilon$, which in the original coordinates reads

$$\tau' = \tau + \epsilon, \quad \sigma' = \sigma$$

**time translation = $SO(2)$ transformation**

2. **Poincaré coordinates**. The line element

$$ds^2_{\text{Poincaré}} = \frac{-dt^2 + dx^2}{x^2}$$

is invariant under the following $SL(2,\mathbb{R})/\mathbb{Z}_2 \cong SO(2,1)$ transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : x^{\pm} \mapsto x'^{\pm} = \frac{a x^{\pm} + b}{c x^{\pm} + d}, \quad x^{\pm} := t \pm x$$

It is easy to check that the $E(1)$ transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix} \in E(1)$$

induces the translation $x'^{\pm} = x^{\pm} + \epsilon$, which in the original coordinates reads

$$t' = t + \epsilon, \quad x' = x$$

**time translation = $E(1)$ transformation**

3. **Schwarzschild coordinates**. The line element

$$ds^2_{\text{Schwarzschild}} = \frac{-dT^2 + dX^2}{\sinh^2 X}$$

is invariant under the following $SL(2,\mathbb{R})/\mathbb{Z}_2 \cong SO(2,1)$ transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : X^\pm \mapsto X'^\pm = 2\,\text{arctanh}\left(\frac{a\tanh\frac{X^\pm}{2} + b}{c\tanh\frac{X^\pm}{2} + d}\right), \quad X^\pm := T \pm X$$

It is easy to check that the $SO(1,1)$ transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cosh\frac{\epsilon}{2} & \sinh\frac{\epsilon}{2} \\ \sinh\frac{\epsilon}{2} & \cosh\frac{\epsilon}{2} \end{pmatrix} \in SO(1,1)$$

induces the translation $X'^\pm = X^\pm + \epsilon$, which in the original coordinates reads

$$T' = T + \epsilon, \quad X' = X$$

**time translation = $SO(1,1)$ transformation**

- Let us consider infinitesimal forms of $SO(2,1)$ coordinate transformations

$$\exp(i\epsilon J_a): x^\mu \longmapsto x'^\mu = x^\mu + \delta_a x^\mu + O(\epsilon^2)$$

- **Background gauge field**. Let $A_\mu$ be a background gauge field that satisfies $dA = E\sqrt{|g|}dx^0 \wedge dx^1$, where $E$ is a constant background electric field. Under the $SO(2,1)$ coordinate transformations, $A_\mu$ transforms as a vector:

$$\exp(i\epsilon J_a): A_\mu(x) \mapsto A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$$

  Interestingly, the infinitesimal field variations $\delta_a A_\mu$ turn out to be of the forms of *gauge transformations*:

$$\begin{aligned}
\delta_a A_\mu(x) &= A'_\mu(x) - A_\mu(x) \\
&= -\delta_a x^\nu \partial_\nu A_\mu(x) - A_\nu(x)\partial_\mu \delta_a x^\nu + O(\epsilon^2) \\
&= \partial_\mu \Lambda_a(x) + O(\epsilon^2)
\end{aligned}$$

  where $\Lambda_a$ are some scalar functions.

- **Scalar field**. Let $\Phi$ be a charged scalar field that couples to the background gauge field via the covariant derivative $D_\mu = \partial_\mu - iqA_\mu$. Since the $SO(2,1)$ coordinate transformations act on $A_\mu$ as gauge transformations, transformation law of $\Phi$ must be accompanied with the $U(1)$ gauge transformation. Hence

$$\exp(i\epsilon J_a) : \Phi(x) \mapsto \Phi'(x') = \mathrm{e}^{iq\Lambda_a(x)}\Phi(x)$$

from which we find

$$\begin{aligned}\delta_a \Phi(x) &= \Phi'(x) - \Phi(x)\\ &= -\delta_a x^\mu \partial_\mu \Phi(x) + iq\Lambda_a(x)\Phi(x) + O(\epsilon^2)\\ &= i\epsilon J_a \Phi(x) + O(\epsilon^2)\end{aligned}$$

where $J_a$ are the coordinate representations of $SO(2,1)$ generators given by

$$\boxed{i\epsilon J_a = -\delta_a x^\mu \partial_\mu + iq\Lambda_a(x)}$$

1. **Global coordinates**. The $SO(2,1)$ generators are turned out to be of the forms

$$J_3 = i\partial_\tau$$
$$J_\pm = -J_1 \pm i J_2 = e^{\mp i\tau} \sin\sigma \left[\mp\partial_\sigma - \cot\sigma (i\partial_\tau) - \alpha\right]$$

where $\alpha := qE = qER^2$. The quadratic Casimir $C = -J_1^2 - J_2^2 + J_3^2$ is given by

$$C = \sin^2\sigma \left[-\partial_\tau^2 + \partial_\sigma^2 - \alpha^2 - 2\alpha\cot\sigma(i\partial_\tau)\right]$$

We are interested in the basis in which the time-translation generator becomes diagonal. Let $|\Delta,\omega\rangle$ be a simultaneous eigenstate of $J_3$ and $C$ that satisfies the eigenvalue equations

$$J_3|\Delta,\omega\rangle = \omega|\Delta,\omega\rangle \quad \text{and} \quad C|\Delta,\omega\rangle = \Delta(\Delta-1)|\Delta,\omega\rangle$$

which in the coordinate space become the following differential equations:

$$i\partial_\tau \Phi_{\Delta,\omega}(\tau,\sigma) = \omega\Phi_{\Delta,\omega}(\tau,\sigma)$$
$$\left[-\partial_\sigma^2 + \frac{\Delta(\Delta-1)}{\sin^2\sigma} + 2\omega\alpha\cot\sigma\right]\Phi_{\Delta,\omega}(\tau,\sigma) = (\omega^2-\alpha^2)\Phi_{\Delta,\omega}(\tau,\sigma)$$

2. **Poincaré coordinates**. The $SO(2,1)$ generators are given by

$$H = J_1 + J_3 = i\partial_t$$
$$D = J_2 = i(t\partial_t + x\partial_x)$$
$$K = J_1 - J_3 = -i(t^2 + x^2)\partial_t - 2itx\partial_x + 2\alpha x$$

The quadratic Casimir $C = -\frac{1}{2}(HK + KH) - D^2$ is

$$C = x^2(-\partial_t^2 + \partial_x^2) - 2i\alpha x\partial_t$$

Let $|\Delta, \omega\rangle$ be a simultaneous eigenstate of $H$ and $C$ that satisfies the eigenvalue equations

$$H|\Delta,\omega\rangle = \omega|\Delta,\omega\rangle \quad \text{and} \quad C|\Delta,\omega\rangle = \Delta(\Delta-1)|\Delta,\omega\rangle$$

which in the coordinate space become the following differential equations:

$$i\partial_t\Phi_{\Delta,\omega}(t,x) = \omega\Phi_{\Delta,\omega}(t,x)$$
$$\left[-\partial_x^2 + \frac{\Delta(\Delta-1)}{x^2} + \frac{2\omega\alpha}{x}\right]\Phi_{\Delta,\omega}(t,x) = \omega^2\Phi_{\Delta,\omega}(t,x)$$

3. **Schwarzschild coordinates**. The $SO(2,1)$ generators are turned out to be of the forms

$$L_1 = J_1 = i\partial_T + \alpha$$
$$L_\pm = J_2 \pm J_3 = \pm e^{\pm T} \sinh X \left[\pm i\partial_X + \coth X(i\partial_T + \alpha) - \alpha\right]$$

The quadratic Casimir $C = -L_1(L_1 \pm i) - L_\mp L_\pm$ is given by

$$C = \sinh^2 X\left[(i\partial_T + \alpha)^2 + \partial_X^2 + \alpha^2 - 2\alpha\coth X(i\partial_T + \alpha)\right]$$

Let $|\Delta, \omega\rangle$ be a simultaneous eigenstate of $L_1$ and $C$ that satisfies the eigenvalue equations

$$L_1|\Delta,\omega\rangle = \omega|\Delta,\omega\rangle \quad \text{and} \quad C|\Delta,\omega\rangle = \Delta(\Delta-1)|\Delta,\omega\rangle$$

which in the coordinate space become the following differential equations:

$$i\partial_T\Phi_{\Delta,\omega}(T,X) = (\omega-\alpha)\Phi_{\Delta,\omega}(T,X)$$
$$\left[-\partial_X^2 + \frac{\Delta(\Delta-1)}{\sinh^2 X} + 2\omega\alpha\coth X\right]\Phi_{\Delta,\omega}(T,X) = (\omega^2+\alpha^2)\Phi_{\Delta,\omega}(T,X)$$

- $(0+1)$-dimensional conformal group $SO(2,1) \cong SL(2,\mathbb{R})/\mathbb{Z}_2$ contains three distinct one-parameter subgroups:
  - compact rotation group $SO(2)$
  - noncompact Euclidean group $E(1)$
  - noncompact Lorentz group $SO(1,1)$
- Correspondingly, there exist three distinct classes of static $AdS_2$ coordinate patches in which time-translation generators generate the one-parameter subgroups $SO(2)$, $E(1)$ and $SO(1,1)$.
- In Lorentzian signature, these coordinate patches are given by the so-called global, Poincaré and Schwarzschild coordinates, respectively.

| coordinate patch | time-translation group | | spectrum | |
|---|---|---|---|---|
| | Lorentzian | Euclidean | Lorentzian | Euclidean |
| global | $SO(2)$ | $SO(1,1)$ | discrete | continous |
| Poincaré | $E(1)$ | $E(1)$ | continuous | continous |
| Schwarzschild | $SO(1,1)$ | $SO(2)$ | continuous | discrete (Matsubara frequency) |

- The Klein-Gordon equation $(\Box_{\mathrm{AdS}_2} - m^2)\Phi = 0$ on charged $\mathrm{AdS}_2$ black hole reduces to the following Schrödinger equations:

  1. **Global coordinates**: Rosen-Morse potential

$$\left[-\partial_\sigma^2 + \frac{\Delta(\Delta-1)}{\sin^2\sigma} + 2\omega\alpha\cot\sigma\right]\Phi_{\Delta,\omega} = (\omega^2 - \alpha^2)\Phi_{\Delta,\omega}$$

  2. **Poincaré coordinates**: Coulomb potential

$$\left[-\partial_x^2 + \frac{\Delta(\Delta-1)}{x^2} + \frac{2\omega\alpha}{x}\right]\Phi_{\Delta,\omega} = \omega^2\Phi_{\Delta,\omega}$$

  3. **Schwarzschild coordinates**: Eckart potential

$$\left[-\partial_X^2 + \frac{\Delta(\Delta-1)}{\sinh^2 X} + 2\omega\alpha\coth X\right]\Phi_{\Delta,\omega} = (\omega^2 + \alpha^2)\Phi_{\Delta,\omega}$$

# Correlator Recurrence Relations

- The Lie algebra $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$ is spanned by the three generators $\{J_1, J_2, J_3\}$ that satisfy the commutation relations

$$[J_1, J_2] = i J_3, \quad [J_2, J_3] = -i J_1, \quad [J_3, J_1] = -i J_2$$

- In the Cartan-Weyl basis $\{J_3, J_\pm := -J_1 \pm i J_2\}$ the commutation relations become

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2 J_3$$

- The quadratic Casimir of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is

$$C = -J_1^2 - J_2^2 + J_3^2 = J_3(J_3 \pm 1) - J_\mp J_\pm$$

- Let $|\Delta, \omega\rangle$ be a simultaneous eigenstate of $C$ and $J_3$ that satisfies

$$C|\Delta, \omega\rangle = \Delta(\Delta - 1)|\Delta, \omega\rangle \quad \text{and} \quad J_3|\Delta, \omega\rangle = \omega|\Delta, \omega\rangle$$

  Then the state $J_\pm|\Delta, \omega\rangle$ satisfies $J_3 J_\pm|\Delta, \omega\rangle = (\omega \pm 1) J_\pm|\Delta, \omega\rangle$, which implies the ladder equations

$$\boxed{\boldsymbol{J_\pm|\Delta, \omega\rangle \propto |\Delta, \omega \pm 1\rangle}}$$

- Let us next consider the following linear combinations

$$L_1 = J_1, \quad L_\pm = J_2 \pm J_3$$

which satisfy the commutation relations

$$[L_1, L_\pm] = \pm i L_\pm, \quad [L_+, L_-] = 2i L_1$$

- The quadratic Casimir of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is

$$C = -J_1^2 - J_2^2 + J_3^2 = -L_1(L_1 \pm i) - L_\mp L_\pm$$

- Let $|\Delta, \omega\rangle$ be a simultaneous eigenstate of $C$ and $L_1$ that satisfies

$$C|\Delta,\omega\rangle = \Delta(\Delta-1)|\Delta,\omega\rangle \quad \text{and} \quad L_1|\Delta,\omega\rangle = \omega|\Delta,\omega\rangle$$

Then the state $L_\pm|\Delta,\omega\rangle$ satisfies $L_1 L_\pm|\Delta,\omega\rangle = (\omega \pm i)L_\pm|\Delta,\omega\rangle$, which implies the ladder equations

$$\boxed{\boldsymbol{L_\pm|\Delta,\omega\rangle \propto |\Delta,\omega \pm i\rangle}}$$

- In the $AdS_2$ black hole problem, the generators $\{L_1, L_\pm\}$ are given by the following first-order differential operators:

$$L_1 = i\partial_T + \alpha$$
$$L_\pm = \pm e^{\pm T} \sinh x \left[\pm i\partial_X + \coth x(i\partial_T + \alpha) - \alpha\right]$$

- The quadratic Casimir gives the d'Alembertian on $AdS_2$ black hole:

$$C := L_1(L_1 \pm i) - L_\mp L_\pm = \sinh^2 x\left[(i\partial_T^2 + \alpha) + \partial_X^2 + \alpha^2 - 2\alpha \coth X(i\partial_T + \alpha)\right]$$

- The eigenvalue equations reduce to the Schrödinger equation

$$i\partial_T \Phi_{\Delta,\omega}(T,X) = (\omega - \alpha)\Phi_{\Delta,\omega}(T,X)$$
$$\left(-\partial_X^2 + \frac{\Delta(\Delta-1)}{\sinh^2 X} + 2\alpha\omega \coth X\right)\Phi_{\Delta,\omega}(T,X) = (\omega^2 + \alpha^2)\Phi_{\Delta,\omega}(T,X)$$

- The ladder equations are

$$\boxed{\boldsymbol{L_\pm \Phi_{\Delta,\omega} \propto \Phi_{\Delta,\omega\pm i}}}$$

- In the asymptotic near-boundary limit $X \to 0$ the generators behave as

$$L_1^0 := \lim_{X\to 0} L_1 = i\partial_T + \alpha$$
$$L_\pm^0 := \lim_{X\to 0} L_\pm = \mathrm{e}^{\pm T}\left[iX\partial_X \pm (i\partial_T + \alpha)\right]$$

- The quadratic Casimir is

$$C^0 := L_1^0(L_1^0 \pm i) - L_\mp^0 L_\pm^0 = X^2\partial_X^2$$

- The eigenvalue equations are

$$i\partial_T \Phi^0_{\Delta,\omega}(T,X) = (\omega - \alpha)\Phi^0_{\Delta,\omega}(T,X)$$
$$\left(-\partial_X^2 + \frac{\Delta(\Delta-1)}{X^2}\right)\Phi^0_{\Delta,\omega}(T,X) = 0$$

which are easily solved with the result

$$\Phi^0_{\Delta,\omega}(T,X) = A_\Delta(\omega)X^\Delta \mathrm{e}^{-i(\omega-\alpha)T} + B_\Delta(\omega)X^{1-\Delta}\mathrm{e}^{-i(\omega-\alpha)T}$$

where $A_\Delta(\omega)$ and $B_\Delta(\omega)$ are integration constants that depend on $\Delta$ and $\omega$.

- The ladder equations $L^0_\pm \Phi^0_{\Delta,\omega} \propto \Phi^0_{\Delta,\omega\pm i}$ become

$$
\begin{aligned}
&(i\Delta \pm \omega)A_\Delta(\omega)X^\Delta \mathrm{e}^{-i(\omega\pm i-\alpha)T} + (i(1-\Delta)\pm\omega)B_\Delta(\omega)X^{1-\Delta}\mathrm{e}^{-i(\omega\pm i-\alpha)T} \\
\propto \quad & A_\Delta(\omega\pm i)X^\Delta \mathrm{e}^{-i(\omega\pm i-\alpha)T} + B_\Delta(\omega\pm i)X^{1-\Delta}\mathrm{e}^{-i(\omega\pm i-\alpha)T}
\end{aligned}
$$

  from which we get

$$
\begin{aligned}
(i\Delta \pm \omega)A_\Delta(\omega) &\propto A_\Delta(\omega\pm i) \\
(i(1-\Delta)\pm\omega)B_\Delta(\omega) &\propto B_\Delta(\omega\pm i)
\end{aligned}
$$

- Hence the Green function

$$G_\Delta(\omega) = (2\Delta-1)\frac{A_\Delta(\omega)}{B_\Delta(\omega)}$$

  satisfies the recurrence relations

$$\boxed{\boldsymbol{G_\Delta(\omega) = \frac{-1+\Delta\pm i\omega}{-\Delta\pm i\omega}G_\Delta(\omega\pm i)}}$$

# Correlator recurrence relations ②

- The recurrence relations

$$G_\Delta(\omega) = \frac{-1+\Delta \pm i\omega}{-\Delta \pm i\omega} G_\Delta(\omega \pm i)$$

are easily solved by iteration:

$$G_\Delta^{A/R}(\omega) = \frac{\Gamma(\Delta \pm i\omega)}{\Gamma(1-\Delta \pm i\omega)} g^{A/R}(\Delta)$$

where $g^{A/R}(\Delta)$ are $\omega$-independent normalization factors.

- Shifting the frequency $\omega \longrightarrow \omega + \alpha$ and restoring $R$ via $\omega \longrightarrow \omega R$, we get the advanced/retarded Green functions

$$G_\Delta^{A/R}(\omega) = \frac{\Gamma(\Delta \pm \frac{i\omega}{2\pi T} \pm i\alpha)}{\Gamma(1-\Delta \pm \frac{i\omega}{2\pi T} \pm i\alpha)} g^{A/R}(\Delta)$$

where $T$ is the Hawking temperature given by

$$T = \frac{1}{2\pi R}$$

# Summary and Perspective

## Summary

- $SO(2,1)$ symmetry of $\mathrm{AdS}_2$ black hole induces the recurrence relations for two-point functions:

$$G_\Delta(\omega) = \frac{-1+\Delta \pm i\omega}{-\Delta \pm i\omega} G_\Delta(\omega \pm i)$$

- The recurrence relations are exactly solvable and completely determine the frequency dependence of advanced/retarded two-point functions.

## Perspective

- Generalizations to $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ for $d > 2$.
(The case $d = 2$ has been done in my previous work `arXiv:1312.7348`.)

**Thank you for your attention!**

\(^o^)/