ブラックホール地平面における粒子と弦の運動のカオス

棚橋典大 [阪大理]

based on

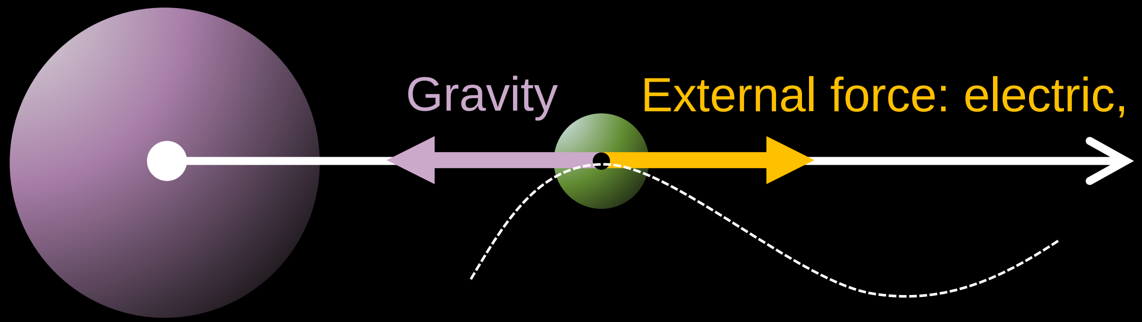
Universality in Chaos of Particle Motion near Black Hole Horizon

橋本幸士、棚橋典大 [arXiv:1610.06070]

Work in progress (橋本幸士、村田佳樹、棚橋典大)

Universality in Chaos of Particle Motion near Black Hole Horizon

- Set-up: **Classical particle** moving **near BH horizon**



- We find that...
  - ✓**Particle motion becomes chaotic** due to BH gravity
  - ✓**For any force, Lyapunov exponent** λ for particle trajectories obeys

$$\lambda \leq \kappa = 2\pi T/\hbar$$

(κ : surface gravity of BH T : Hawking temperature)

Universality in Chaos of Particle Motion near Black Hole Horizon

- Background story:

  A bound on chaos in QFT at temperature T :

$$\lambda \leq 2\pi T/\hbar$$

[Maldacena-Shenker-Stanford '15]

- They focused on effect of temperature to chaos in QFT.

- We want to study effect of temperature to chaos in classical gravity.

  Use BH surface gravity $\kappa = 2\pi T/\hbar$ instead.

- To probe effect of κ, we look at trajectories very close to BH horizon.

CONTENTS

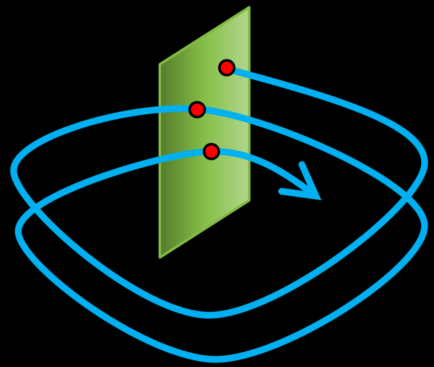
CHAOS

Classic chaos in deterministic dynamical systems

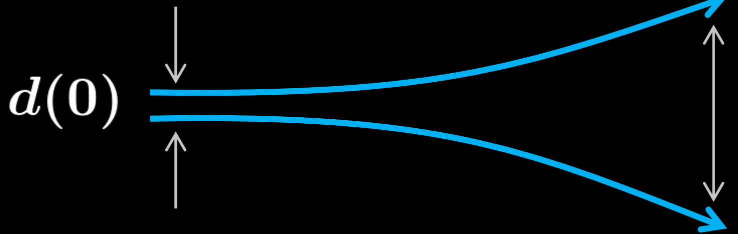
= Non-periodic bounded orbits sensitive to initial conditions

Diagnostics of chaos

Poincaré plot = Section of orbits in phase space



Lyapunov exponent λ = Separation growth rate of nearby orbits



$$d(t) \sim d(0)e^{\lambda t}$$

QUANTUM CHAOS?

Definition of Quantum Chaos?

Chaos arises from nonlinear dynamics,
but Schrodinger eq. = Linear eq. of wave function.

Quantum effect washes out small scale $\Delta x\, \Delta p \lesssim \hbar$.

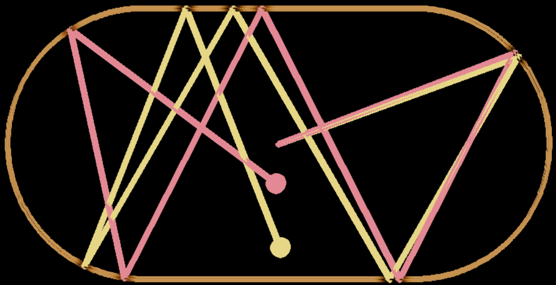
→ No chaos?

Other probes of "quantum chaos"

Chaos in semi-classical regime: Quantum billiard

Energy spectrum of excited atoms

Chaos in QFT and AdS/CFT



QUANTUM CHAOS

Chaos in QFT and AdS/CFT

Look at Out-of-Time-Ordered Correlator [Kitaev '14] [Maldacena-Shenker-Stanford '15]

$$\langle Q(t)P(0)Q(t)P(0)\rangle \sim f_0 - f_1 e^{\lambda t}$$

$$\langle [q(t),p(0)]^2\rangle \sim \left(\frac{\partial q(t)}{\partial q(0)}\right)^2$$

: Dependence of $q(t)$ on initial condition $q(0)$

[Larkin, Ovchinnikov '69]

Conjecture [Maldacena-Shenker-Stanford '15]

In a QFT with temperature T, Lyapunov exponent λ obeys

$$\lambda \leq 2\pi T/\hbar$$

A BOUND ON CHAOS

- Background story:

  A bound on chaos in QFT at temperature T :

$$\lambda \leq 2\pi T/\hbar$$

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  Use BH surface gravity $\kappa = 2\pi T/\hbar$ instead.

- To probe effect of κ, we look at trajectories very close to BH horizon.

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DERIVE THE BOUND

- Background story:

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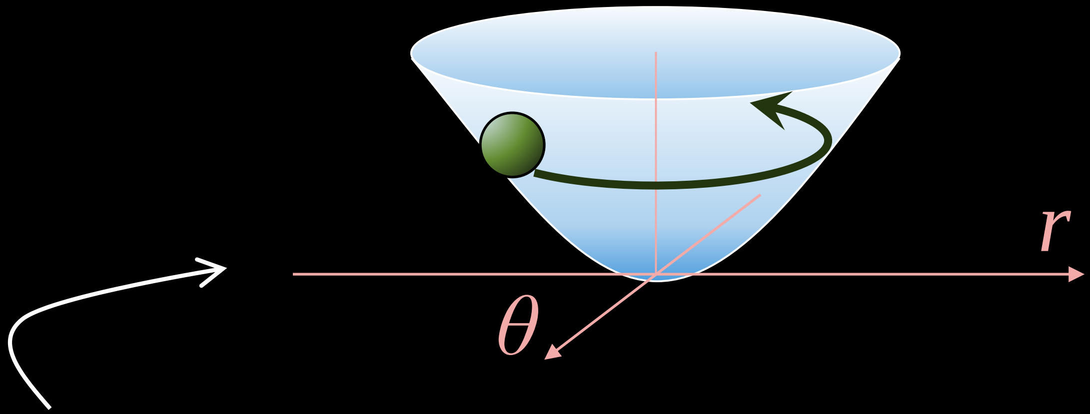
DERIVE THE BOUND

- To realize a particle moving very close to BH horizon,

  1. put a particle in a trapping harmonic potential



◆ Integrable system:

- 2 degrees of freedom
- Conserved quantities: Energy, angular momentum

→ No Chaos: Trajectories fully specified by conserved quantities

DERIVE THE BOUND

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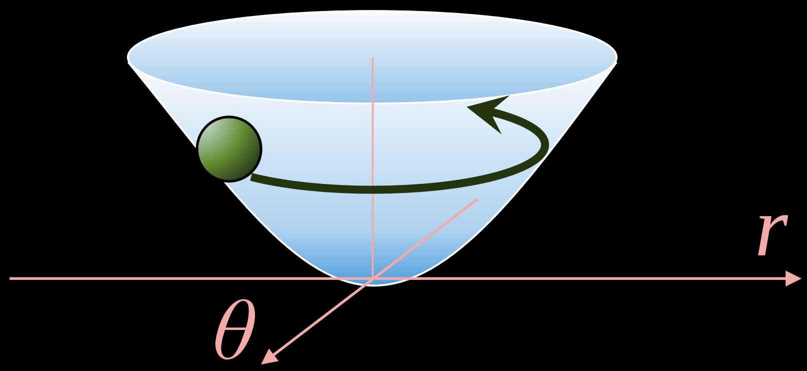
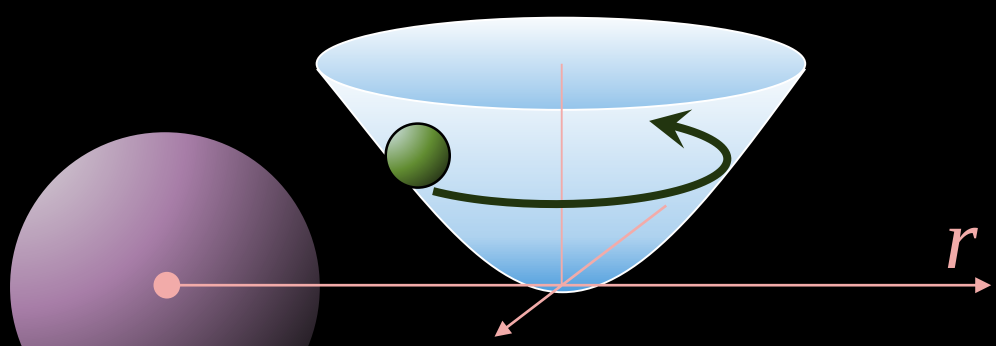


(← no chaos)

  2. take it close to a BH horizon



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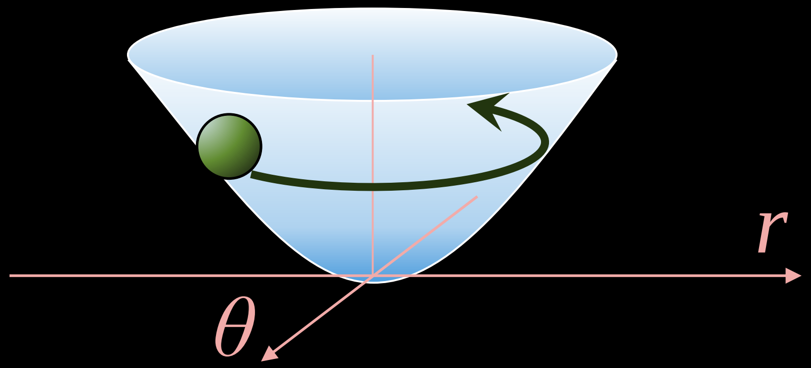
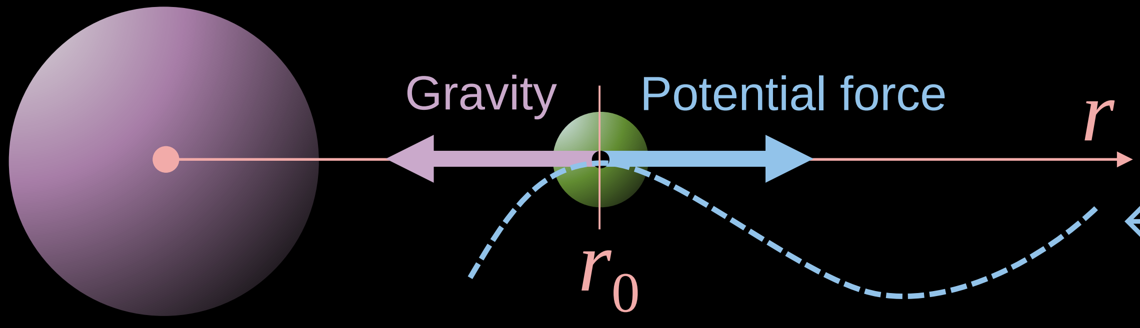


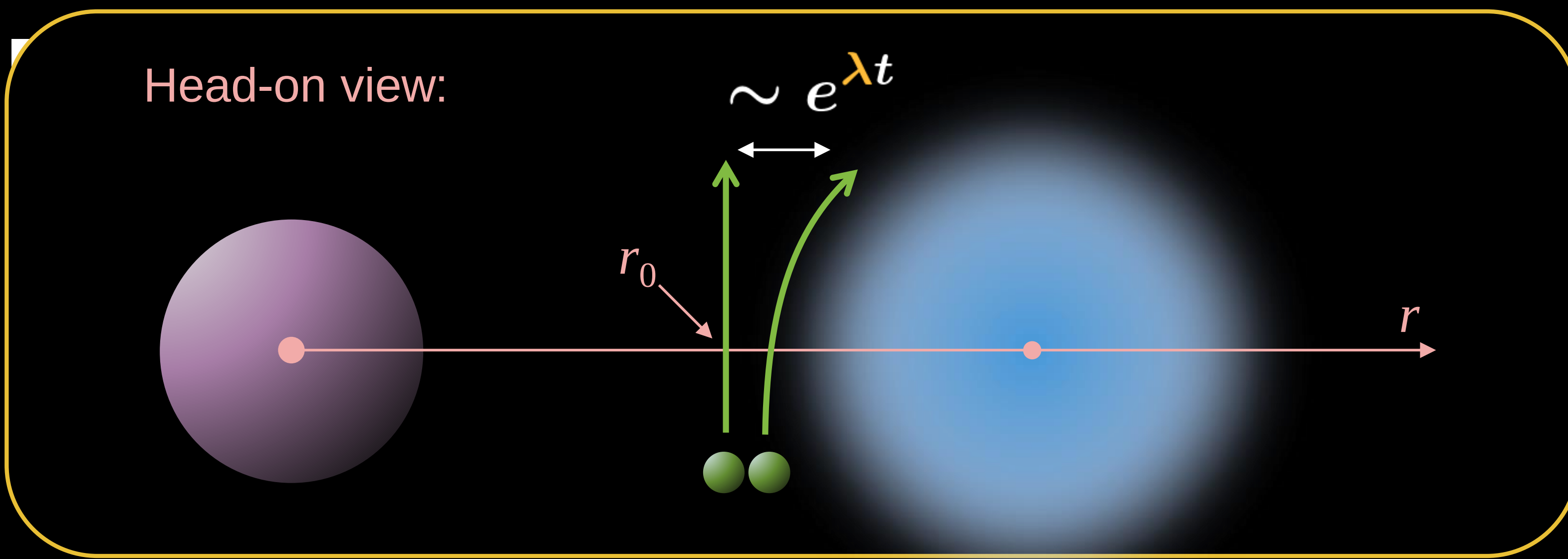
(← no chaos)

  2. take it close to a BH horizon & look at the separatrix



← Effective potential for particle motion

DERIVE THE BOUND



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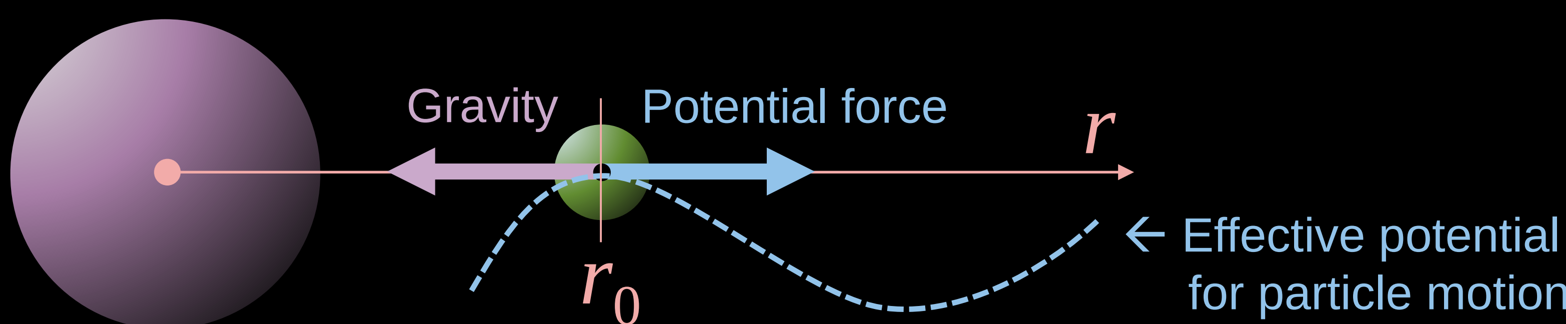

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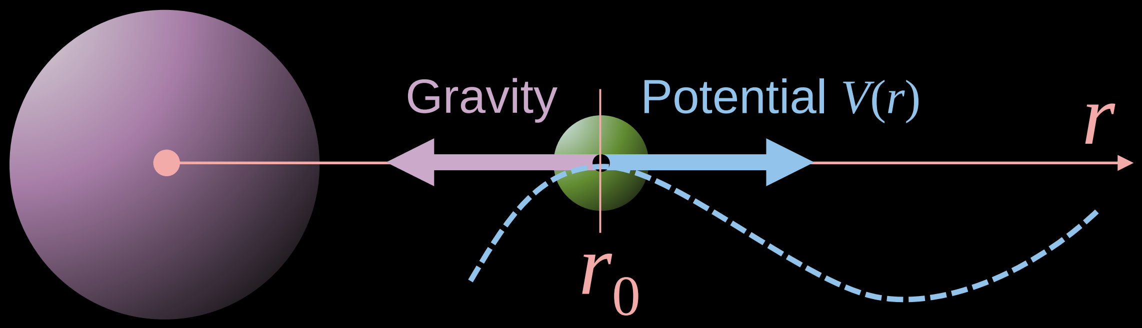


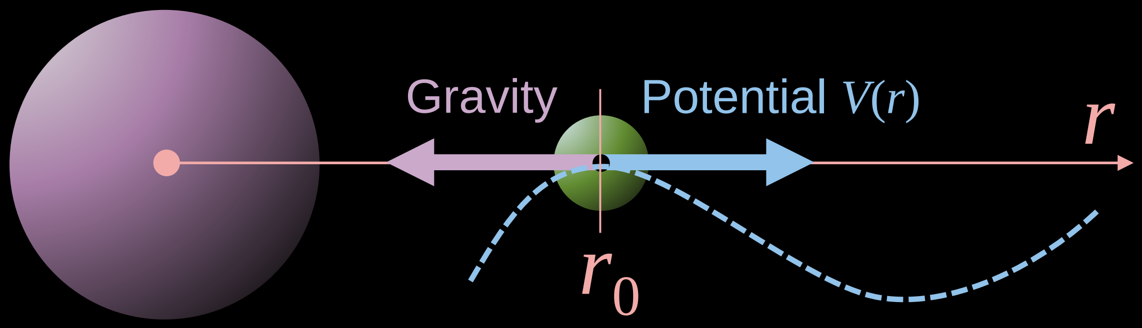
$$\mathcal{L} = -m\sqrt{-g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}} - V(X) = -m\sqrt{f(r) - \frac{\dot{r}^2}{f(r)}} - V(r)$$

$$\left(ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \qquad f(r) = 1 - \frac{r_{\mathrm{hor}}}{r} \right)$$

3. Focus on slow radial motion near potential maximum

$$\mathcal{L} \simeq -m\sqrt{f(r)}\left[1 - \frac{\dot{r}^2}{2f^2(r)}\right] - V(r)$$

DERIVE THE BOUND



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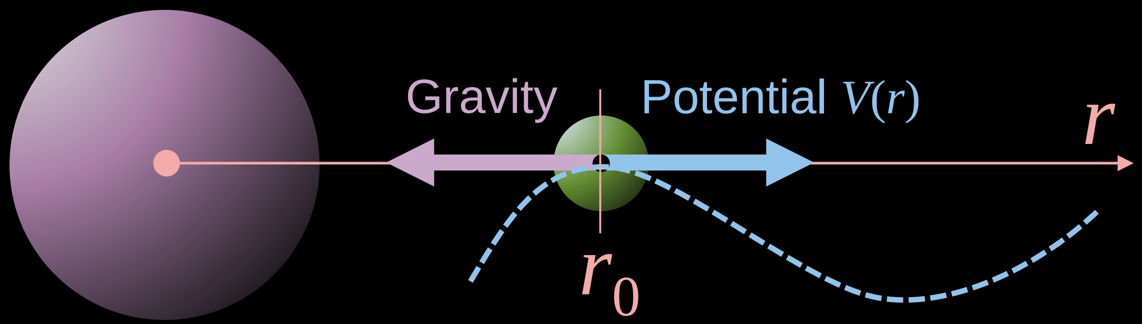
4. Taylor expand $V(r)$ near the potential maximum

$$\mathcal{L} \simeq -m\sqrt{f(r)}\left[1 - \frac{\dot{r}^2}{2f^2(r)}\right] - c \times (r - r_0)$$

DERIVE THE BOUND



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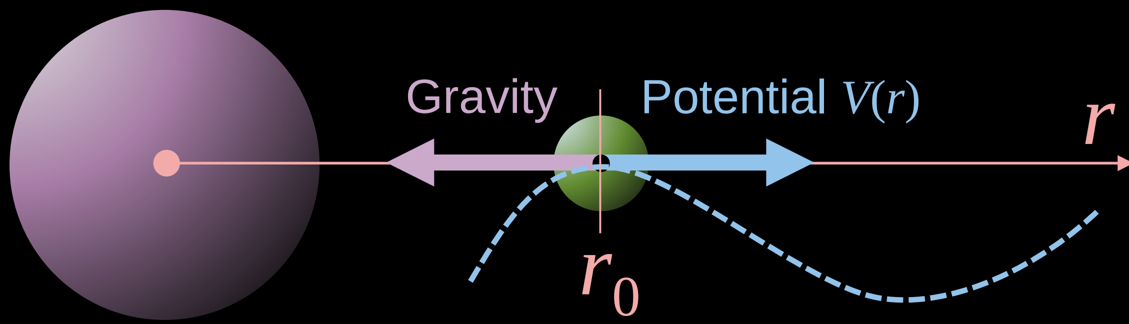
$$\mathcal{L} \simeq -m\sqrt{f(r)}\left[1 - \frac{\dot{r}^2}{2f^2(r)}\right] - c \times (r - r_0)$$

$$\equiv \frac{m}{2f^{3/2}(r)}\dot{r}^2 - V_{\rm eff}(r)$$

DERIVE THE BOUND



$$\mathcal{L} = -m\sqrt{-g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} - V(X) = -m\sqrt{f(r) - \frac{\dot{r}^2}{f(r)}} - V(r)$$

$$\left(ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \qquad f(r) = 1 - \frac{r_{\rm hor}}{r} \right)$$

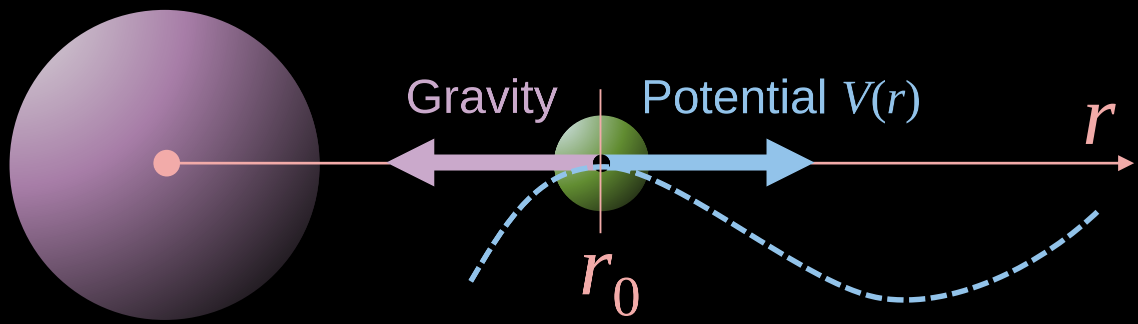
5. Move the potential maximum $r = r_0$ toward BH horizon

$$\mathcal{L} \simeq -m\sqrt{f(r)}\left[1 - \frac{\dot{r}^2}{2f^2(r)}\right] - c \times (r - r_0)$$

$$\equiv \frac{m}{2f^{3/2}(r)}\dot{r}^2 - V_{\rm eff}(r)$$

$$f(r) \simeq 2\kappa \times (r - r_{\rm hor})$$

DERIVE THE BOUND



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$$\equiv \frac{m}{2f^{3/2}(r)}\dot{r}^2 - V_{\rm eff}(r)$$

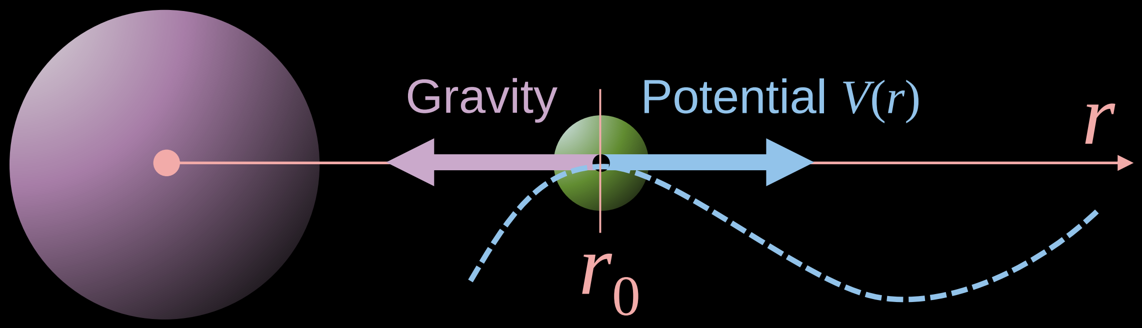
$$\left(f(r) \simeq 2\kappa \times (r - r_{\rm hor})\right)$$

6. Expanding $\mathcal{L}$ around $r = r_0$ for small distance, we get

$$\mathcal{L} \simeq -\frac{mc^3}{2\kappa^3}\left[\dot{r}^2 + \kappa^2 (r - r_0)^2\right]$$

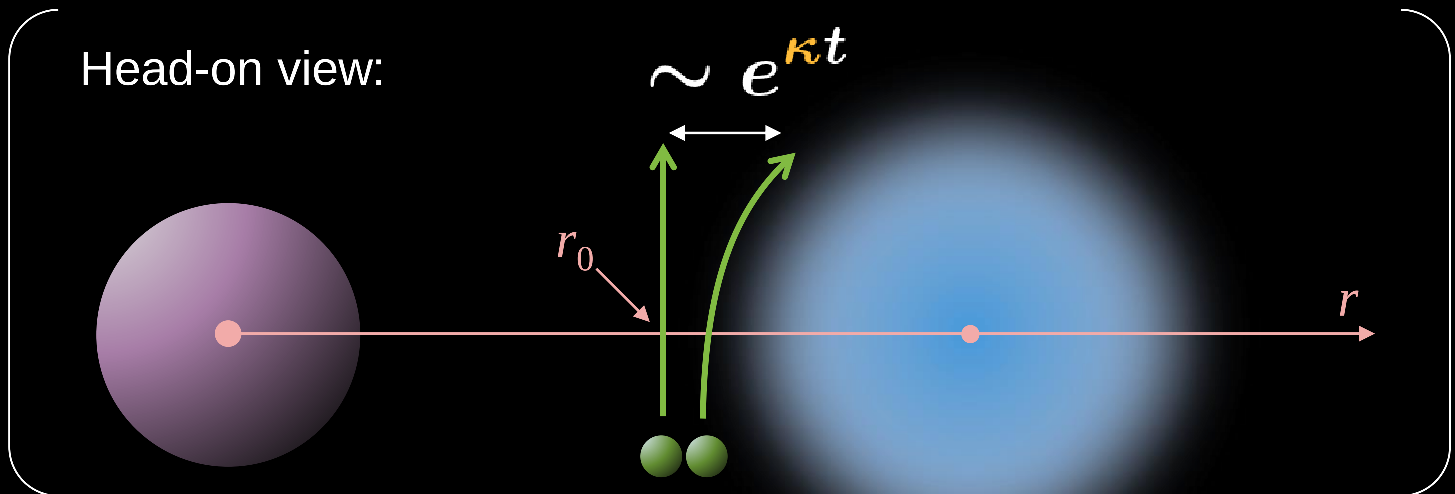
$$\left(\begin{aligned} r_0 &= r_{\rm hor} + \frac{\kappa}{2c^2} \\ V_{\rm eff} &= -\frac{\kappa}{2c} - \frac{\kappa^{1/2}(r - r_0)^2}{4\sqrt{2}(r_0 - r_{\rm hor})^{3/2}} + \cdots \end{aligned}\right)$$

DERIVE THE BOUND



$$\mathcal{L} \simeq -\frac{mc^3}{2\kappa^3}\left[\dot{r}^2 + \kappa^2 (r - r_0)^2\right] \Rightarrow r(t) - r_0 \propto e^{\kappa t}$$

$\Rightarrow$ Lyapunov exponent $\lambda = \kappa$



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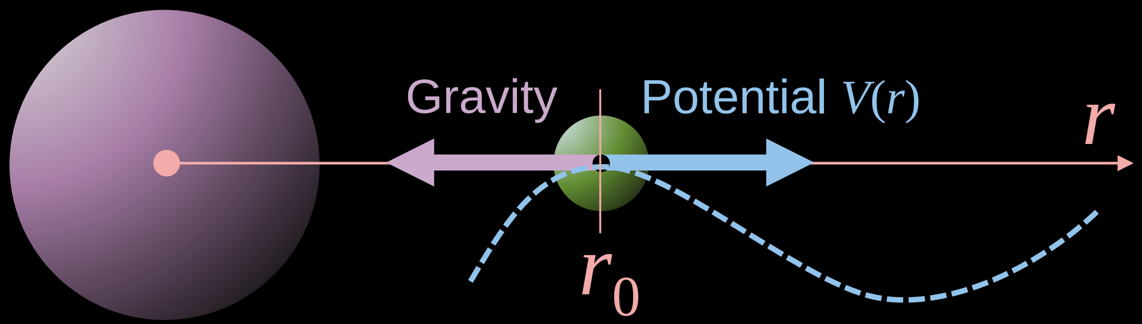

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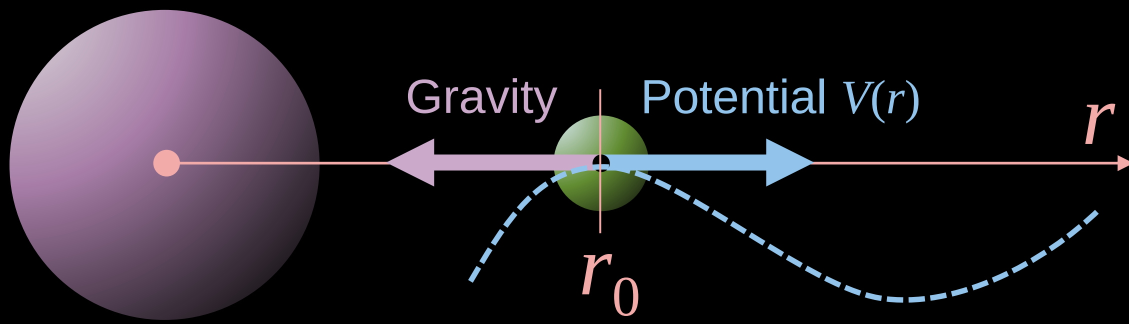
$\Rightarrow$ Lyapunov exponent $\lambda = \kappa$

- ✓ This λ is independent of particle mass, strength & species of potential force, metric form, cosmological constant and dimensions

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega_n^2 \quad \text{with} \quad \begin{cases} f(r) = \alpha_f (r - r_{\rm hor})^{\beta_f} \\ g(r) = \alpha_g (r - r_{\rm hor})^{\beta_g} \end{cases}$$

- ✓ λ averaged over trajectory $\Rightarrow$ generic trajectory will obey $\lambda \leq \kappa = 2\pi T/\hbar$

SUMMARY: DERIVE THE BOUND



$$\mathcal{L} = -m\sqrt{-g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}} - V(X) \simeq -m\sqrt{f(r) - \frac{\dot{r}^2}{f(r)}} - V(r)$$

- Slow motion near unstable maximum $r = r_0$
- Near-horizon limit $r_0 \longrightarrow r_{\text{horizon}}$
- Linear approximation for $V(r)$ ~ (slope) × $(r - r_{\text{horizon}})$

$$\mathcal{L} \simeq C(m, \kappa, \text{slope of } V) \times \left[\dot{r}^2 + \kappa^2 (r - r_0)^2\right]$$

⇒ A generic trajectory would obey $\lambda \leq \kappa = 2\pi T/\hbar$

REALIZATIONS

Electric force:

$$\mathcal{L} = -m\sqrt{-g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu} - V(X) \quad \text{with} \quad V(X) = e\frac{dX^0}{dt}A_0(X)$$

$$\partial_r\left(\sqrt{-\det g}\, g^{rr} g^{00} \partial_r A_0\right) = 0 \quad \rightarrow \quad V \sim c \times r$$

Scalar force:

$$\mathcal{L} = -\sqrt{-g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu}\,(m + \phi(X))$$

$$\partial_r\left(\sqrt{-\det g}\, g^{rr} \partial_r \phi\right) = 0 \quad \rightarrow \quad \phi \sim c \times \log r$$

These two examples gives $\lambda = \kappa$ for any m and c.

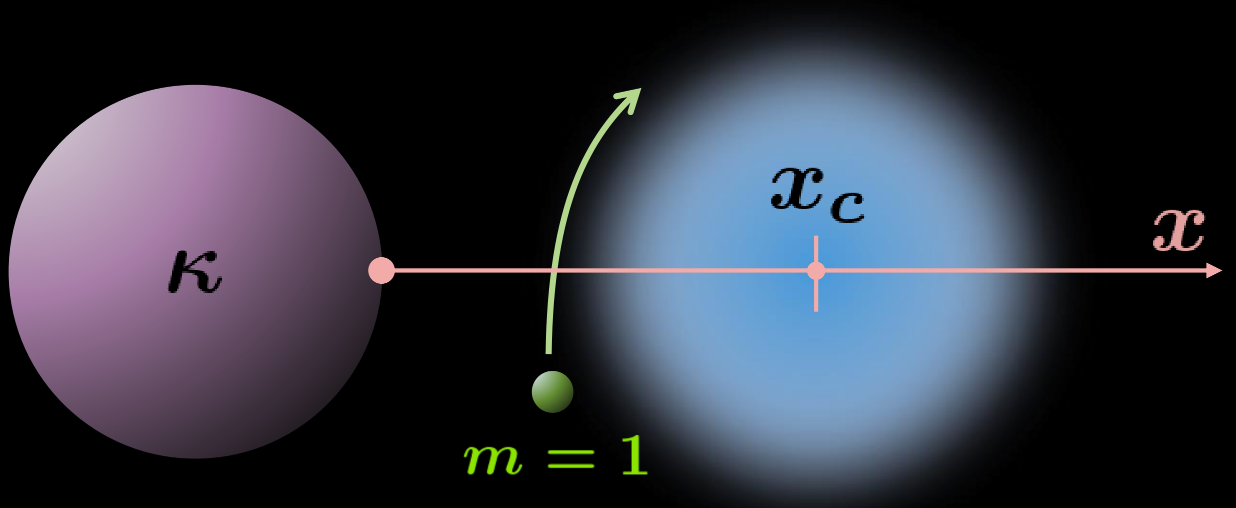
CONTENTS

NUMERICAL CHECK

- Setup: Particle in harmonic potential near BH

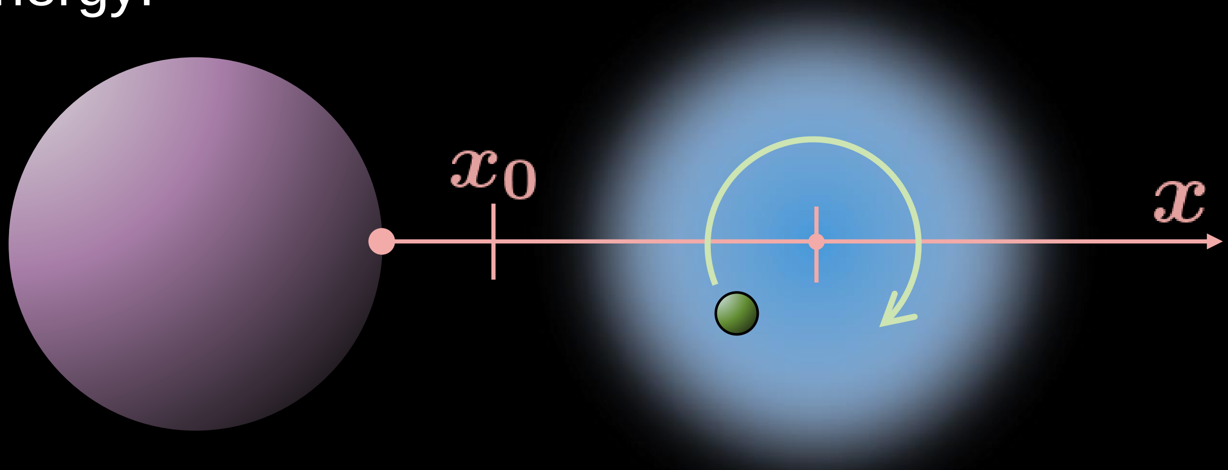
$$\mathcal{L} = -\sqrt{f(x) - \frac{\dot{x}^2}{f(x)} - \dot{y}^2} - \frac{\omega^2}{2}\left[(x - x_c)^2 + y^2\right]$$

$$\left(f(x) \equiv 2\kappa x \right)$$

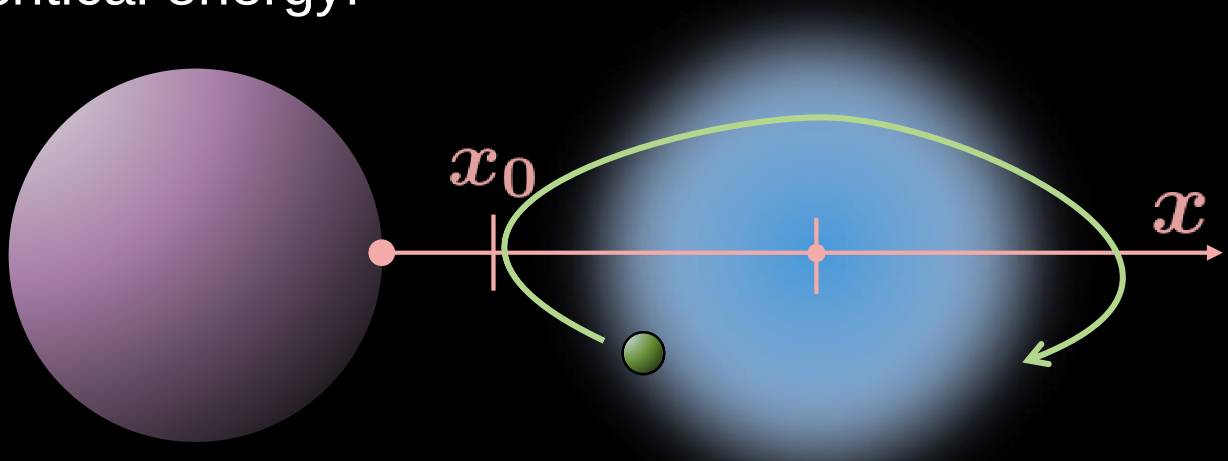


NUMERICAL CHECK

- Low energy:



- Near-critical energy:



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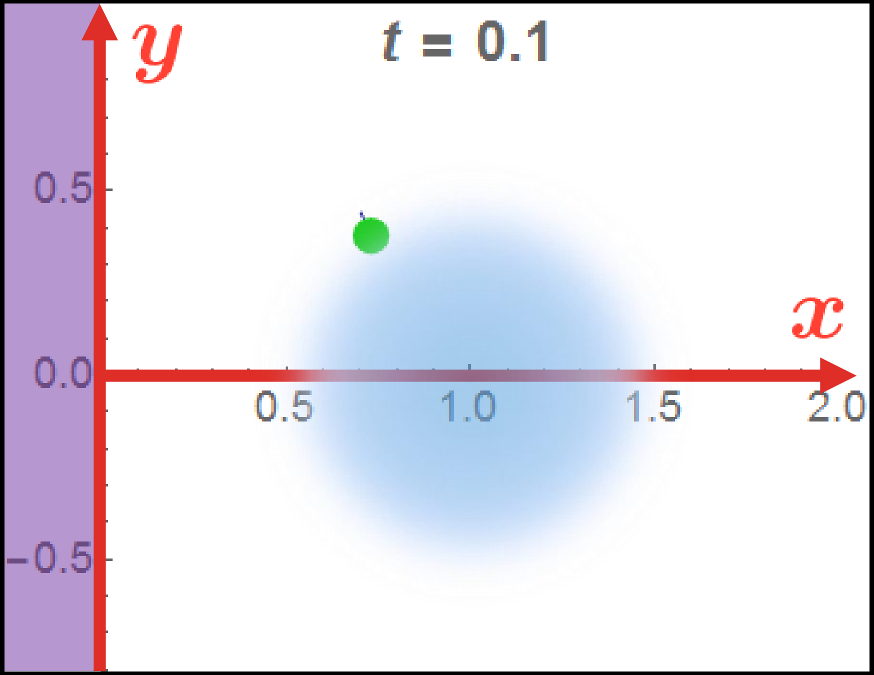

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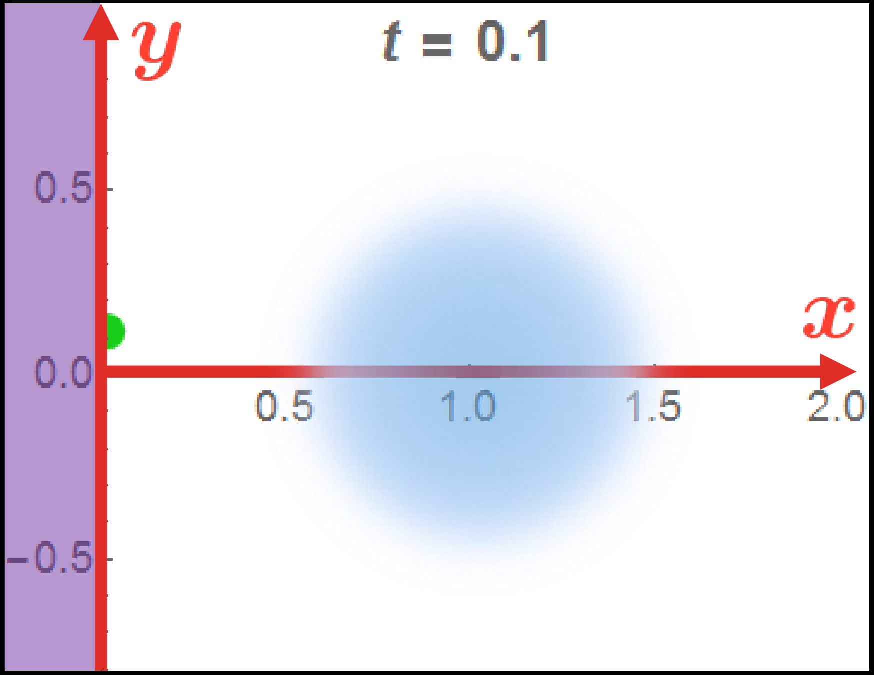

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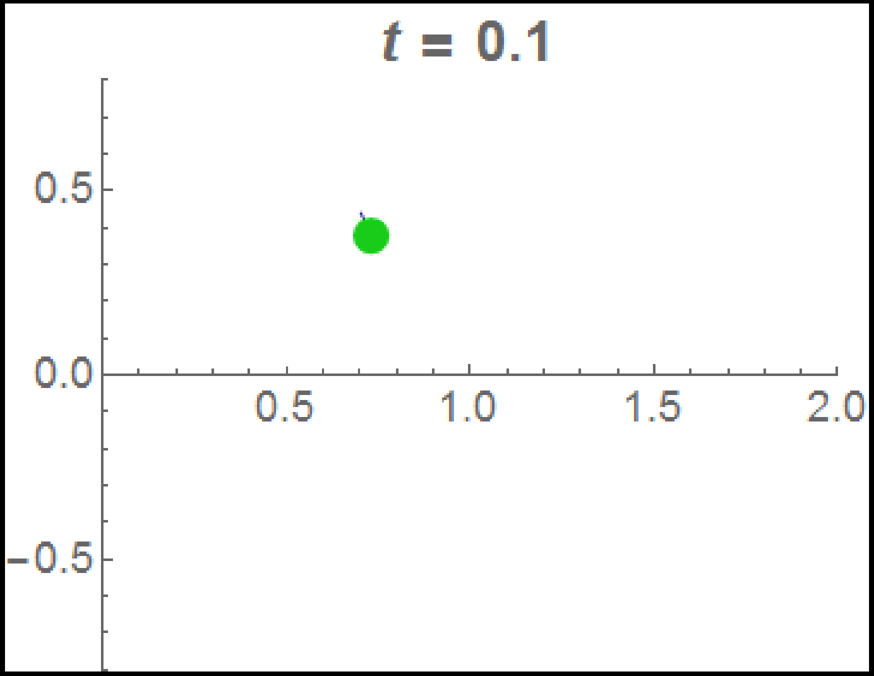

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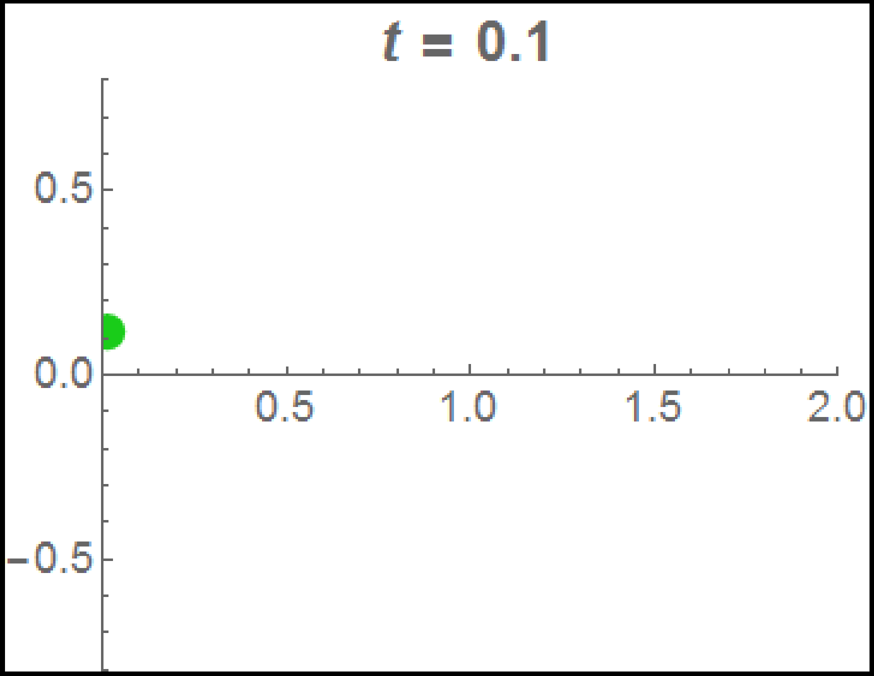

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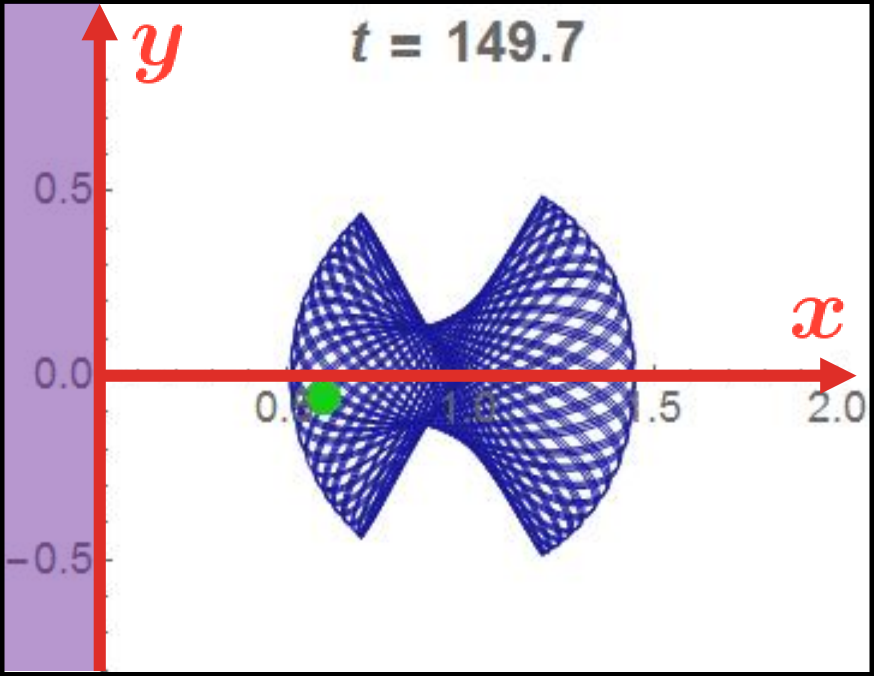

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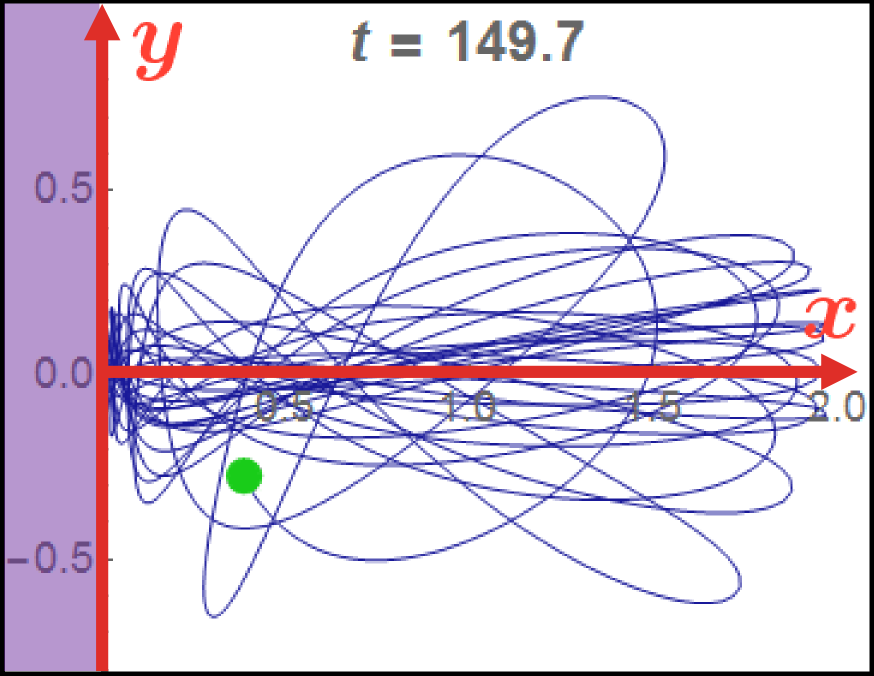
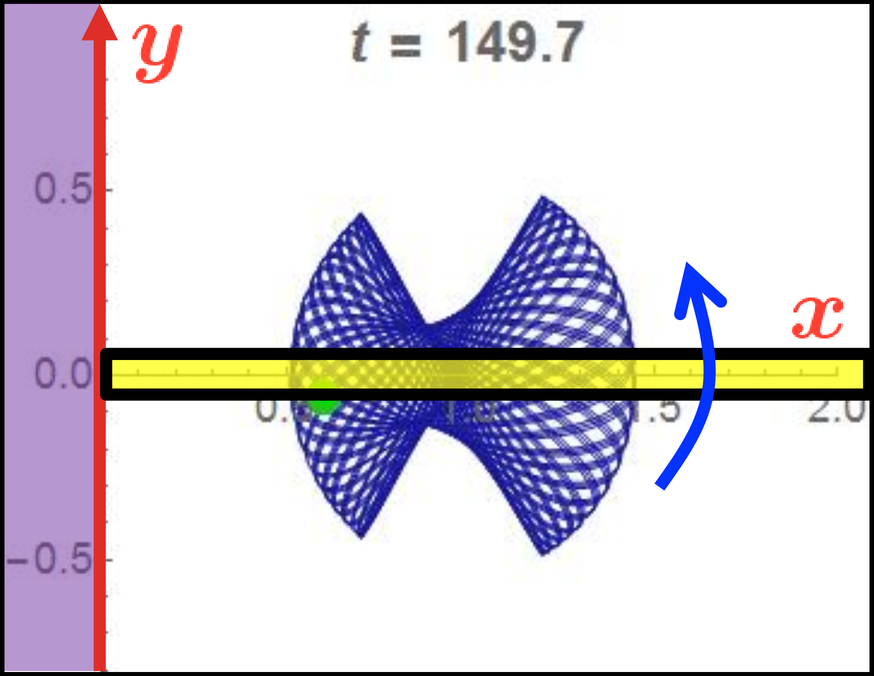


Poincaré plot at $y = 0 \ \& \ \dot{y} > 0$

- Low energy:



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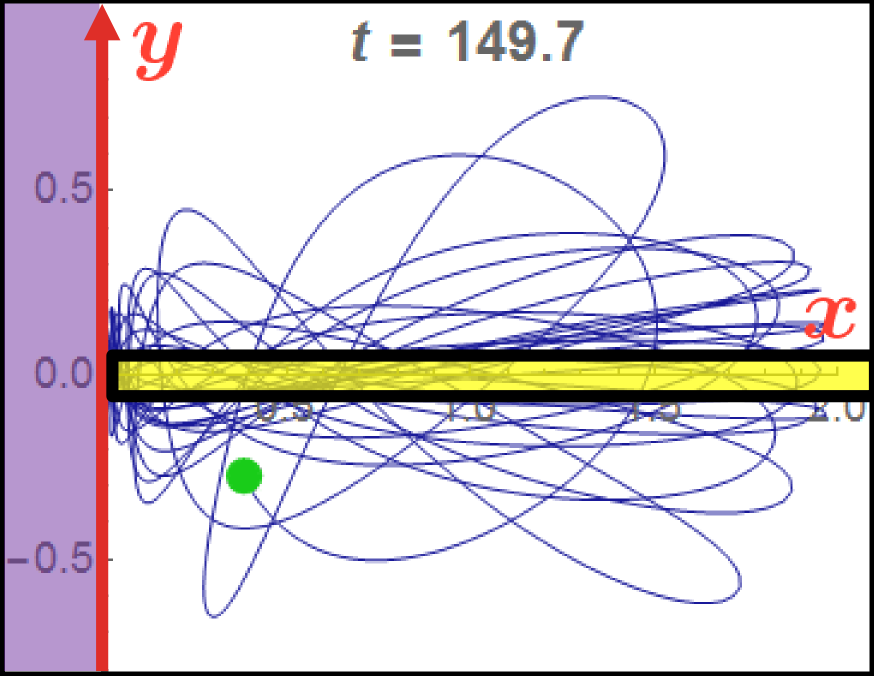
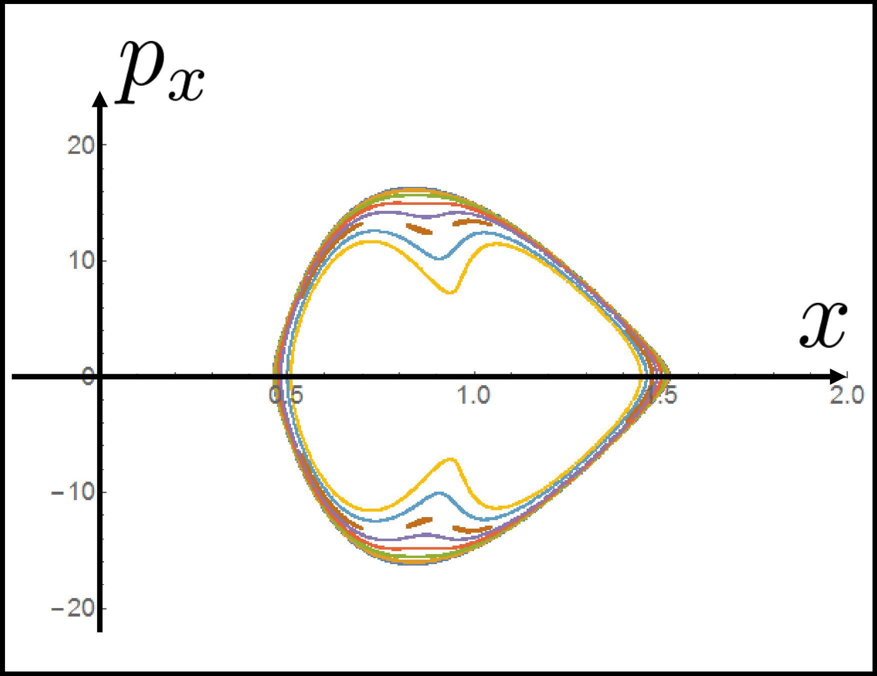


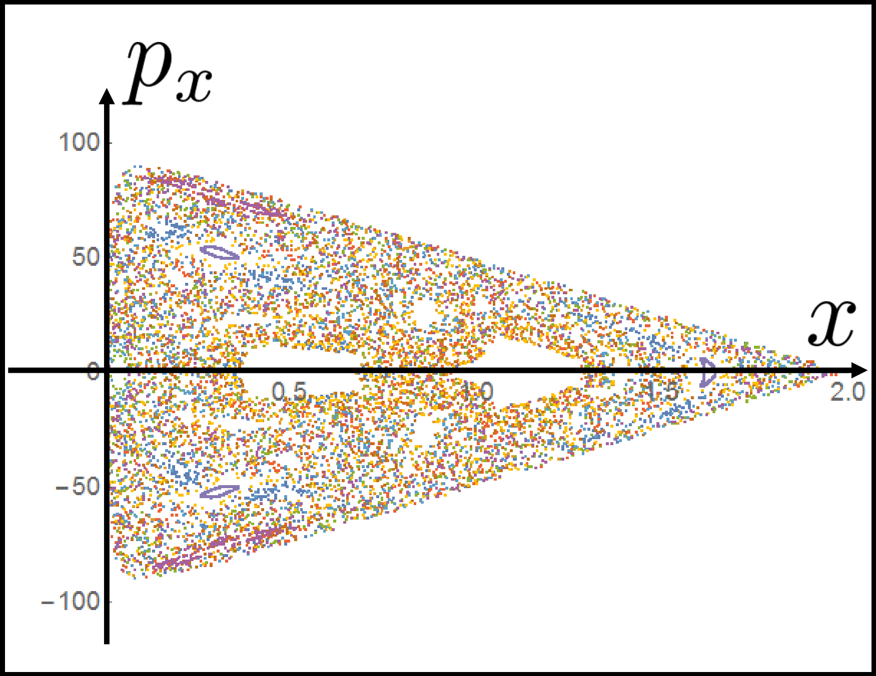
Poincaré plot at $y = 0$ & $\dot{y} > 0$

- Low energy:



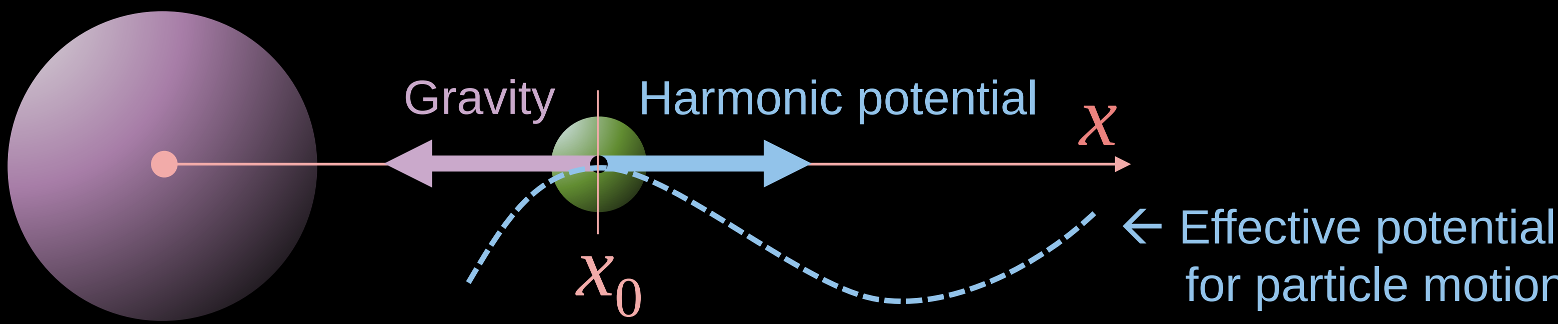
- Regular KAM tori, no chaos

- Near-critical energy:



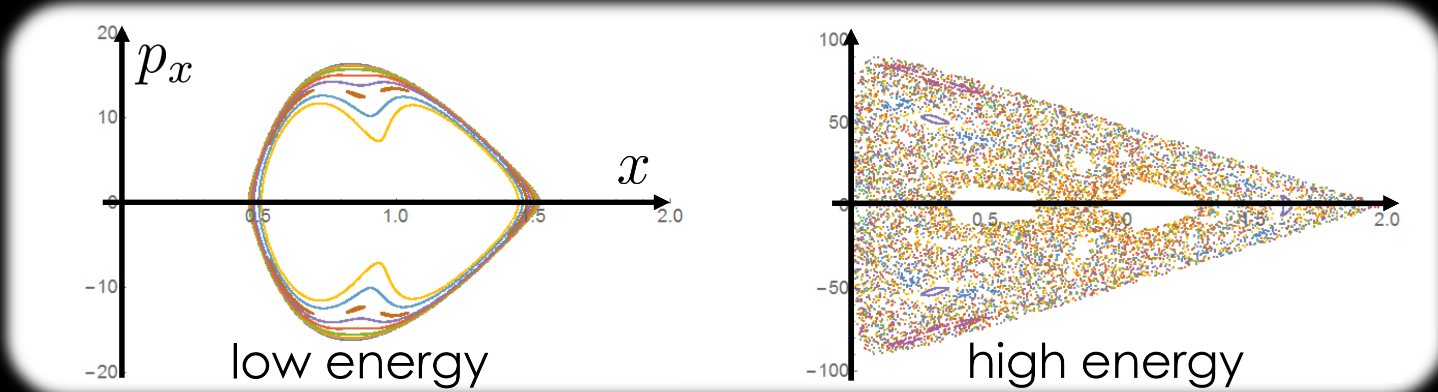
- Lyapunov exponent $\lambda \sim 0.2\,\kappa$, satisfying the bound $\lambda \leq \kappa$.

SUMMARY: NUMERICAL CHECK



$$\mathcal{L} = -\sqrt{f - \frac{\dot{x}^2}{f} - \dot{y}^2} - \frac{\omega^2}{2}\left[(x - x_c)^2 + y^2\right] \qquad (f \equiv 2\kappa x)$$

✓ Poincaré plot at $y = 0$: Chaotic when particle approach BH horizon



✓ Lyapunov exponent $\lambda \sim 0.2\,\kappa$, which satisfies the bound $\lambda \leq \kappa$.

SUMMARY

- We got a bound on chaos from classical BH-particle system

$$\lambda \leq \kappa = 2\pi T/\hbar$$

which coincides with the bound by Maldacena-Shenker-Stanford.

- Independent of particle mass, external force & metric form.

◆ If the force is generated by field with higher spin s,

$$\lambda \leq \sqrt{2s-1}\,\kappa$$

(CFT result: $\lambda \leq (s-1)\kappa$ [Roberts & Stanford '14, Perlmutter '16])

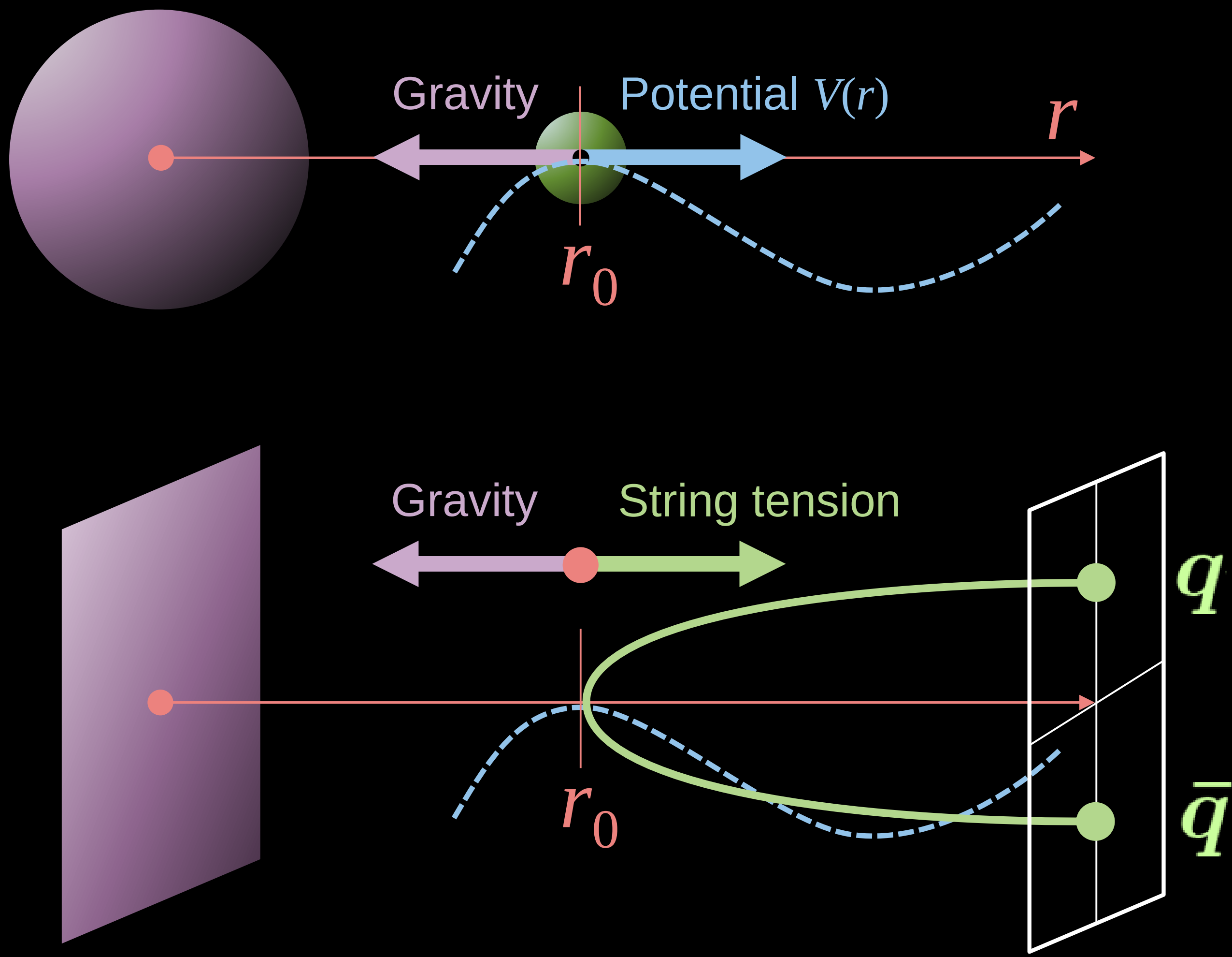
◆ Extensions to string & branes in AdS to get insights from AdS/CFT

Chaotic motion of string & membrane in AdS BH spacetime

→ Bound on chaos in holographic QCD-like setup?

CONTENTS

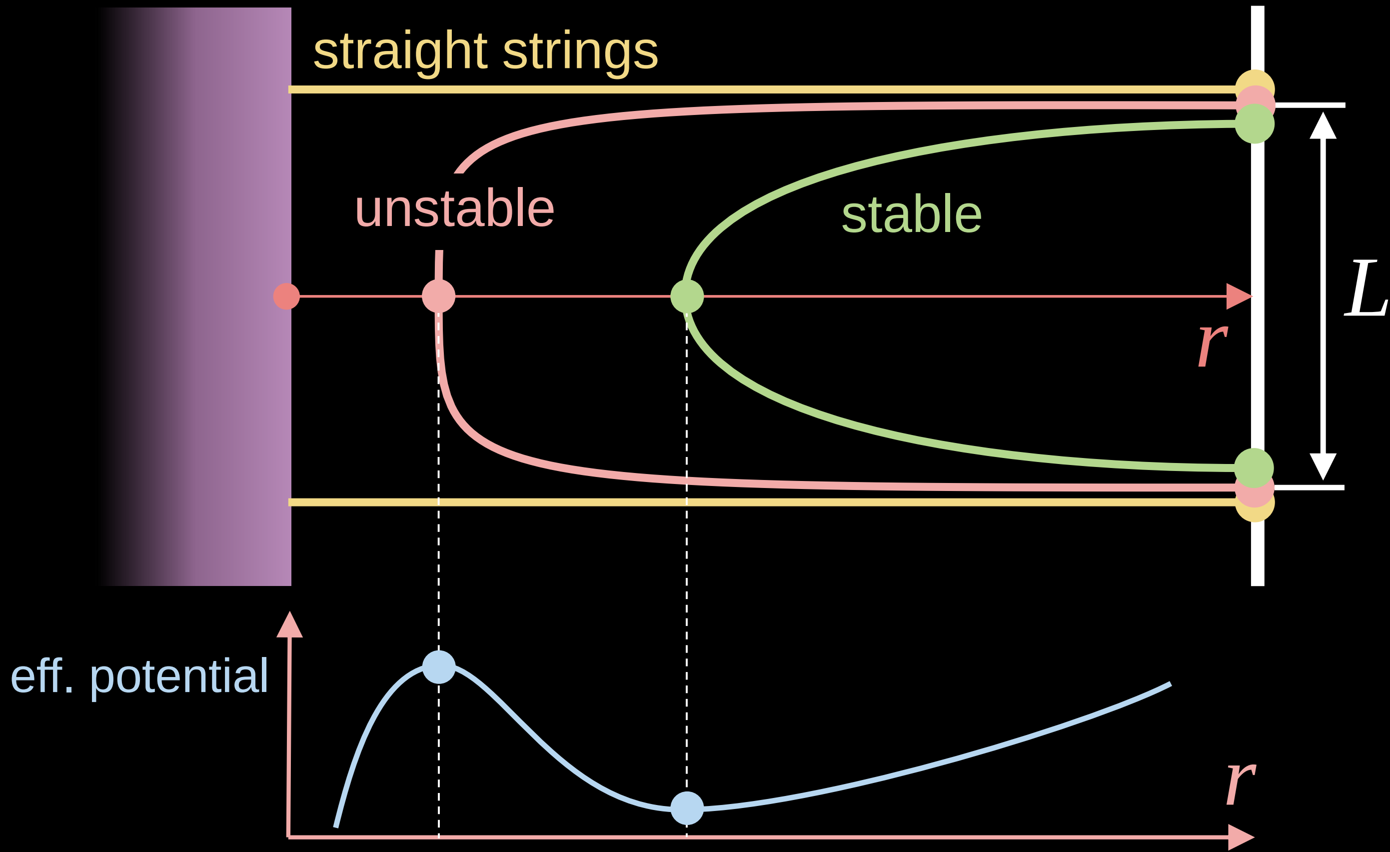
STRING IN ADS



String hanging from AdS boundary = "quark-anti quark pair"

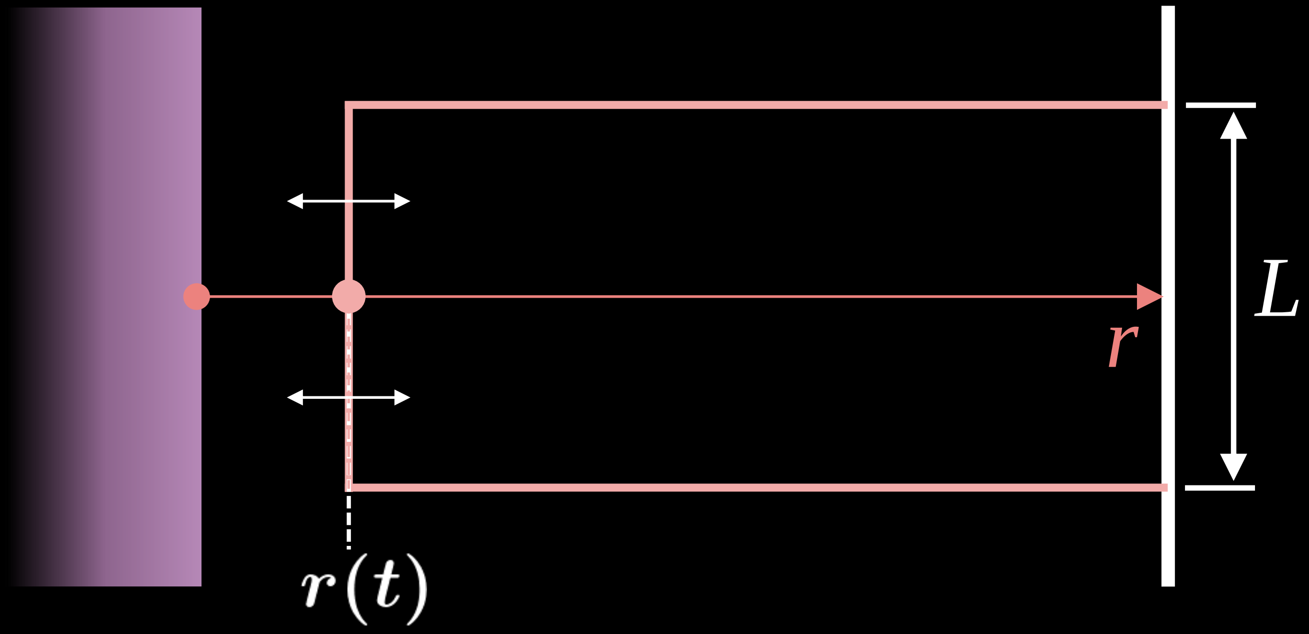
STRING IN ADS

Three shapes of static Nambu-Goto string in AdS



STRING IN ADS

Square-shape approximation for string in AdS



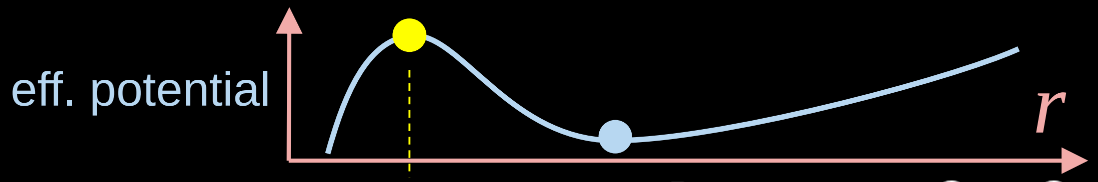
$$\mathcal{L} = -T\int d\lambda \simeq -L\sqrt{r^4(t)f\big(r(t)\big) - \frac{\dot{r}^2(t)}{f\big(r(t)\big)}} + 2\big(r(t) - r_H\big) \quad \left(f(r) = 1 - \frac{r_H^4}{r^4}\right)$$

STRING IN ADS

Square-shape approximation for string in AdS

$$\mathcal{L} \simeq -L\sqrt{r^4(t) f\big(r(t)\big) - \frac{\dot{r}^2(t)}{f\big(r(t)\big)}} + 2\big(r(t) - r_H\big)$$

$$\simeq \frac{L}{2r^2 f^{3/2}(r)}\dot{r}^2(t) - \left[L\sqrt{r^4(t) f\big(r(t)\big)} - 2\big(r(t) - r_H\big)\right]$$



$$r_* = r_H\left[1 + \mathcal{O}(r_H^2 L^2)\right]$$

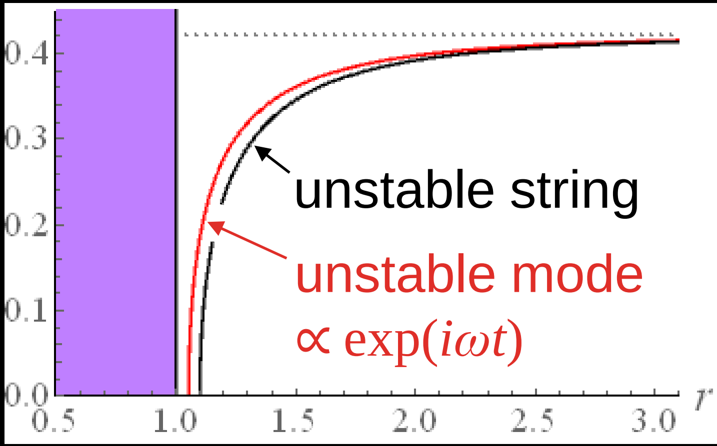
In the near horizon limit ($r_* \to r_H$),

$$\mathcal{L} \simeq \frac{1}{2r_H^5 L^2}\left[\dot{r}^2 + \lambda^2\big(r(t) - r_*\big)^2\right] \qquad \left(\begin{array}{l} \lambda = 2\pi T_H\left[1 + \mathcal{O}(r_H^2 L^2)\right] \\ T_H = r_H/\pi \end{array}\right)$$

STRING IN ADS

Numerical check of square-shape approx. & $\lambda = 2\pi T_H$

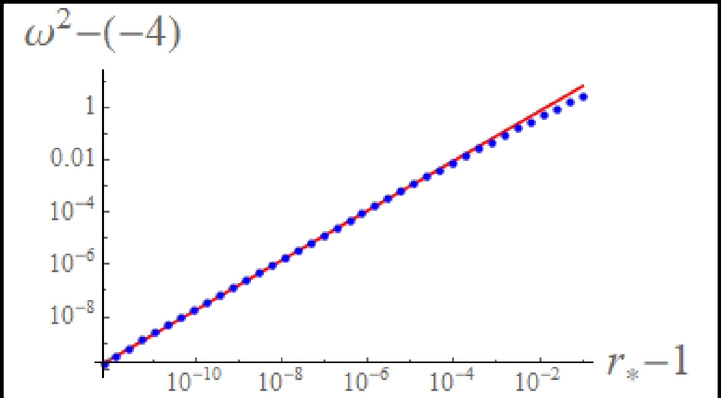
- ✓ String shape
  ($r_H = 1.0,\ r_* = 1.1$)



- ✓ Instability growth rate:

$$\omega^2 \sim -\left(2 - 3.8 \times L^{1.9}\right)^2$$

Consistent with
$\lambda \sim \kappa = 2 r_H$



SUMMARY

- We got a bound on chaos from classical BH-particle system

$$\lambda \leq \kappa = 2\pi T/\hbar$$

which coincides with the bound by Maldacena-Shenker-Stanford.

- Independent of particle mass, external force & metric form.

◆ Extension to string in AdS

✓ Unstable mode similar to the BH-particle system

✓ Instability growth rate: $\lambda \lesssim \kappa = 2\pi T/\hbar$

?: Does this govern chaotic motion of string in AdS?